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# A NEW ALGORITHM FOR COMPUTING THE INVERSE AND GENERALIZED INVERSE OF THE SCALED FACTOR CIRCULANT MATRIX*

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**Abstract**

A new algorithm for finding the inverse of a nonsingular scaled factor circulant matrix is presented by the Euclid's algorithm. Extension is made to compute the group inverse and the Moore-Penrose inverse of the singular scaled factor circulant matrix. Numerical examples are presented to demonstrate the implementation of the proposed algorithm.

*Mathematics subject classification:* 15A21, 65F15.
*Key words:* Scaled factor circulant matrix, Inverse, Group inverse, Moore-Penrose inverse.

## 1. Introduction

Circulant matrices, as an important class of special matrices, have a wide range of interesting applications [12–19]. They have in recent years been applied in many areas, see, e.g., [2, 3, 6, 10, 11, 15, 17]. Scaled circulant permutation matrices and the matrices that commute with them are natural extensions of this well-studied class, see, e.g., [1, 20–23]. In particular, it will be seen that $r$-circulant matrices [10, 11] are precisely those matrices commuting with the scaled circulant permutation matrix.

This paper presents an efficient algorithm to compute the inverse of a nonsingular scaled factor circulant matrix or to compute the group inverse and Moore-Penrose inverse of the circulant matrix when it is singular. The algorithm has small computational complexity. It is a notable character of the algorithm that the singularity of the scaled factor circulant matrix need not be priori known.

We define $\mathcal{R}$ as the scaled circulant permutation matrix, that is,

$$\mathcal{R} = \begin{pmatrix} 0 & d_1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & d_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & 0 & d_{n-1} \\ d_n & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}_{n\times n}. \tag{1.1}$$

This paper deals with the case where $\mathcal{R}$ is nonsingular ($d_i \neq 0$ and fixed).

It is easily verified that the polynomial $g(x) = x^n - d_1 d_2 \ldots d_n$ is both the minimal polynomial and the characteristic polynomial of the matrix $\mathcal{R}$. In addition, $\mathcal{R}$ is nondergatory.

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Moreover, $\mathcal{R}$ is normal if and only if $|d_1| = |d_2| = \cdots = |d_n|$, where $|d_i|, i = 1, \cdots, n$ denote the modulus of the complex number $d_i, i = 1, \cdots, n$.

**Definition 1.1.** *An $n \times n$ matrix $A$ over $\mathbb{C}$ is called a scaled factor circulant matrix if $A$ commutes with $\mathcal{R}$, that is,*

$$A\mathcal{R} = \mathcal{R}A, \tag{1.2}$$

*where $\mathcal{R}$ is given in* (1.1).

Let $\mathcal{R}SFCM_n$ be the set of all complex $n \times n$ matrices which commute with $\mathcal{R}$. In the following, with $A = \mathrm{scacirc}_{\mathcal{R}}(a_0, a_1, \cdots, a_{n-1})$ we denote the scaled factor circulant matrix $A$ whose first row is $(a_0, a_1, \ldots, a_{n-1})$. Remark that the first row of $A$ completely defines the matrix. Indeed, since $\mathcal{R}$ is nonderogatory, Eq. (1.2) is fulfilled if and only if $A = f(\mathcal{R})$ for some polynomial $f$. Furthermore, $\mathcal{R}SFCM_n$ is a vector space of dimension $n$, and there is a clear one-to-one correspondence between the polynomials of degree at most $n-1$ and the numbers $a_0, \cdots, a_{n-1}$.

For an $m \times n$ matrix $A$, any solution to the matrix equation $AXA = A$ is called a *generalized inverse* of $A$. In addition, if $X$ satisfies $X = XAX$, then $A$ and $X$ are said to be semi-inverses, see, e.g., [2].

In this paper we only consider square matrices $A$. In [8, p.51] the smallest positive integer $k$ for which $\mathrm{rank}(A^{k+1})=\mathrm{rank}(A^k)$ holds is called the *index* of $A$. If $A$ has index 1, the generalized inverse $X$ of $A$ is called the *group inverse* $A^{\#}$ of $A$. Clearly, $A$ and $X$ are group inverses if and only if they are semi-inverses and $AX = XA$.

In [4, 5] a semi-inverse $X$ of $A$ was considered in which the nonzero eigenvalues of $X$ are the reciprocals of the nonzero eigenvalue of $A$. These matrices were called *spectral inverses*. It was shown in [5] that a nonzero matrix $A$ has a unique spectral inverse, $A^s$, if and only if $A$ has index 1: when $A^s$ is the group inverse $A^{\#}$ of $A$.

## 2. The Properties of the Scaled Factor Circulant Matrix

**Lemma 2.1.** ([1]) *If $\mathcal{R}$ is a scaled circulant permutation matrix, and if $k$ is a positive integer, then $\mathcal{R}^k = D^{(k)}C^k$, where $D^{(k)}$ is the diagonal matrix whose $(j, j)$ entry is $\prod_{t=j}^{j+k-1} d_t$ for $1 \le j \le n$ and $C = circ(0, 1, 0, \cdots, 0)$ is the circulant permutation. Furthermore,*

$$\mathcal{R}^n = (\prod_{j=1}^{n} d_j)I_n, \quad \det \mathcal{R} = (-1)^{n-1} \prod_{j=1}^{n} d_j.$$

Let $\omega = \exp(\frac{2\pi i}{n})$ be a primitive $n$th root of unity. Then $\omega_j = d\omega^j, j = 0, 1, \cdots, n-1$ are the distinct roots of $g(x)$, where $g(x) = x^n - d_1 d_2 \cdots d_n$, and

$$d = (\prod_{t=1}^{n} d_t)^{\frac{1}{n}} \neq 0. \tag{2.1}$$

Let $F$ be the $n \times n$ unitary Fourier matrix such that

$$F_{ij} = \frac{1}{\sqrt{n}}\, \omega^{(i-1)(j-1)} \quad \text{for } 1 \le i,\ j \le n. \tag{2.2}$$

Let

$$\Delta = \mathrm{diag}(\delta_1, \delta_2, \cdots, \delta_n), \tag{2.3}$$

where the elements $\delta_j$ of $\Delta$ are computed by the recursion formula

$$\delta_{j+1} = \frac{d}{d_j}\delta_j, \quad 1 \le j \le n, \quad \delta_{n+1} = \delta_1 = 1.$$

**Lemma 2.2.** ([1]) *Let* $A = \mathrm{scacirc}_{\mathcal{R}}(a_0, a_1, \cdots, a_{n-1})$ *be a scaled factor circulant matrix over the complex field* $\mathbb{C}$*. Then*

$$\sigma(A) = \{\lambda_j | \lambda_j = f(d\omega^j) = a_0 + \sum_{i=1}^{n-1} a_i (\prod_{t=1}^{i} d_t)^{-1} (d\omega^j)^i | 0 \le j \le n-1\} \tag{2.4}$$

*is the spectrum of* $A$ *and*

$$A = f(\mathcal{R}) = a_0 I + \sum_{i=1}^{n-1} a_i (\prod_{t=1}^{i} d_t)^{-1} \mathcal{R}^i, \tag{2.5}$$

*where*

$$f(x) = a_0 + \sum_{i=1}^{n-1} a_i (\prod_{t=1}^{i} d_t)^{-1} x^i. \tag{2.6}$$

The polynomial (2.6) will be called the representor of the scaled factor circulant matrix $A$.

**Lemma 2.3.** ([1]) *Let* $A$=$\mathrm{scacirc}_{\mathcal{R}}(a_0, a_1, \cdots, a_{n-1})$ *be a scaled factor circulant matrix over the complex field* $\mathbb{C}$*. If* $F$ *is the Fourier matrix, then*

$$A = (\Delta F)\mathrm{diag}(\lambda_0, \cdots, \lambda_i, \cdots, \lambda_{n-1})(\Delta F)^{-1}, \tag{2.7}$$

*where* $\Delta$ *is given by* (2.3) *and* $\lambda_j, j = 0, 1, \cdots, n-1$ *are the eigenvalues of* $A$ *given by* (2.4).

Let $D_n$ denote the multiplicative semigroup of all $n \times n$ diagonal complex matrices. By Lemma 1 in [2, p.27] the mapping

$$A \to (\triangle F)^{-1} A (\triangle F)$$

is a semigroup isomorphism of $\mathcal{R}SFCM_n$ onto $D_n$, where $F$ and $\triangle$ are defined by (2.2) and (2.3), respectively.

Let $A = \mathrm{scacirc}_{\mathcal{R}}(a_0, a_1, \cdots, a_{n-1}) \in \mathcal{R}SFCM_n$ be a scaled factor circulant matrix. Then $\sigma(A) = \{\lambda_i | i = 0, 1, \cdots, n-1\}$ by (2.4). Let

$$T_i = \begin{cases} 0, & \text{if } \lambda_i = 0, \\ 1/\lambda_i, & \text{if } \lambda_i \ne 0, \end{cases}$$

for $i = 0, 1, \cdots, n-1$. If

$$B = (\triangle F)\mathrm{diag}(T_0, \cdots, T_i, \cdots, T_{n-1})(\triangle F)^{-1},$$

then by Theorem 1 of [2], $B = A^s$, the spectral inverse of $A$.

Since each $A$ in $\mathcal{R}SFCM_n$ has index 1, $A^s$ is also the group inverse $A^{\#}$ of $A$. Moreover, if $\mathcal{R}$ is normal, then by Theorem 1 of [2], $A^s = A^+$, where $A^+$ denotes the Moore-Penrose inverse of A.

We summarize the above discussions in the following theorems.

**Theorem 2.1.** *Let $A \in M_n$. Then $A \in \mathcal{R}SFCM_n$ if and only if $(\triangle F)^{-1}A(\triangle F)$ is a diagonal matrix. Let $A \in \mathcal{R}SFCM_n$. If $A$ is a singular matrix, then $A^s = A^{\#} \in \mathcal{R}SFCM_n$. If $\mathcal{R}$ is normal, then $A^+ \in \mathcal{R}SFCM_n$ and $A^+ = A^{\#}$.*

**Theorem 2.2.** *If $A = \text{scacirc}_{\mathcal{R}}(a_0, a_1, \cdots, a_{n-1})$ is nonsingular, then $f(\omega_j) \neq 0$, where*

$$f(x) = a_0 + \sum_{i=1}^{n-1} a_i(\prod_{t=1}^{i} d_t)^{-1}x^i, \quad \omega_j = d\omega^j, \quad j = 0, 1, \cdots, n-1$$

*are the distinct roots of $g(x)$. If $A$ is singular and has $k$ zero eigenvalues, then there are $\omega_{i_0}, \omega_{i_1}, \cdots, \omega_{i_{k-1}}$, such that $f(\omega_{i_j}) = 0$, for $j = 0, 1, \cdots, k-1$. Conversely, if there exists $\omega_k$ satisfying $f(\omega_k) = 0$, then the scaled factor circulant matrix $A$ is singular.*

*Proof.* According to Theorem 2.1, we know that $(\triangle F)^{-1}A(\triangle F) = D$, where

$$D = \text{diag}(f(\omega_0), f(\omega_1), \cdots, f(\omega_{n-1})),$$

and $\omega_j = d\omega^j, j = 0, 1, \cdots, n-1$ are the distinct roots of $g(x)$. Thus $A\triangle F = \triangle FD$. Since $\triangle F$ is a nonsingular matrix, then

$$\text{rank}A = \text{rank}A\triangle F = \text{rank}\triangle FD = \text{rank}D.$$

If there exist $\omega_{i_0}, \omega_{i_1}, \cdots, \omega_{i_{k-1}}$ such that $f(\omega_{i_j}) = 0$, for $j = 0, 1, \cdots, k-1$, then there are $\omega_{i_k}, \omega_{i_{k+1}}, \cdots, \omega_{i_{n-1}}$ such that $f(\omega_{i_j}) \neq 0$, for $j = k, k+1, \cdots, n-1$. Thus $\text{rank}A = n - k$.

Conversely, if $\text{rank}A = n - k$, then there exist $\omega_{i_k}, \omega_{i_{k+1}}, \cdots, \omega_{i_{n-1}}$ such that $f(\omega_{i_j}) \neq 0$, for $j = k, k+1, \cdots, n-1$. Therefore, there are $\omega_{i_0}, \omega_{i_1}, \cdots, \omega_{i_{k-1}}$ such that $f(\omega_{i_j}) = 0$, for $j = 0, 1, \cdots, k-1$.

In addition, let $A, B \in \mathcal{R}SFCM_n$. Then $AB = BA \in \mathcal{R}SFCM_n$. If $A$ is a nonsingular matrix, then $A^{-1} \in \mathcal{R}SFCM_n$. Thus $\mathcal{R}SFCM_n$ is a ring.

Polynomial ring has an intimate relation to the scaled factor circulant matrix ring. Let $P(x)$ be the polynomial ring. For all $f(x)$ in $P(x)$, the degree of $f(x)$ is denoted by $\deg(f(x))$. Let $P_{n-1}(x)$ be the quotient ring $P(x)/\langle x^n - d_1d_2\cdots d_n\rangle$, where $\langle x^n - d_1d_2\cdots d_n\rangle$ is an ideal. Define $\varphi$ as a function which maps scaled factor circulant matrix ring onto the polynomial ring by

$$\varphi(A) \mapsto f(x) = a_0 + \sum_{i=1}^{n-1} a_i(\prod_{t=1}^{i} d_t)^{-1}x^i,$$

where $A = \text{scacirc}_{\mathcal{R}}(a_0, a_1, \cdots, a_{n-1})$.

Then, we can conclude that $\varphi$ is a ring isomorphism. The scaled factor circulant matrix ring and the polynomial quotient ring $P_{n-1}(x)$ are isomorphic. So, if $A$ is nonsingular, then $\varphi$ maps the inverse of $A$ onto the inverse of the representor $f(x)$ of $A$.

## 3. Main Results

**Theorem 3.1.** *Let $A = \text{scacirc}_{\mathcal{R}}(a_0, a_1, \cdots, a_{n-1})$ be a scaled factor circulant matrix which is nonsingular, with the representor of $A$ being*

$$f(x) = a_0 + \sum_{i=1}^{n-1} a_i(\prod_{t=1}^{i} d_t)^{-1}x^i.$$

*Then there exists a polynomial*

$$u(x) = b_0 + \sum_{i=1}^{n-1} b_i(\prod_{t=1}^{i} d_t)^{-1}x^i$$

*such that* $u(\omega_j) = 1/f(\omega_j)$, *where* $\omega_j, j = 0, 1, \cdots, n-1$, *are the roots of* $g(x) = x^n - d_1 d_2 \cdots d_n$ *and the inverse of* $A$ *is given by*

$$B = \text{scacirc}_{\mathcal{R}}(b_0, b_1, \cdots, b_{n-1}).$$

*Proof.* From Theorem 2.2, we know that $f(x) = f(\omega_j) \neq 0, j = 0, 1, \cdots, n-1$. Let

$$g(x) = \prod_{j=0}^{n-1}(x - \omega_j) = x^n - d_1 d_2 \cdots d_n.$$

Then $f(x)$ and $g(x)$ are coprime. Hence there exist $u'(x)$ and $v(x)$ satisfying

$$f(x)u'(x) + g(x)v(x) = 1.$$

When $x = \omega_j$, $j = 0, 1, \cdots, n-1$, then $g(x) = 0$. Consequently, $f(\omega_j)u'(\omega_j) = 1$. Let

$$u(x) = u'(x)\text{mod}(x^n - d_1 d_2 \cdots d_n).$$

Then $\deg(u(x)) < n$. Since $\omega_j^n - d_1 d_2 \cdots d_n = 0$, and $u(\omega_j) = u'(\omega_j), j = 0, 1, \cdots, n-1$, the existence of $u(x)$ in Theorem 3.1 is then proved.

For the scaled factor circulant matrix $B$ we have

$$\begin{aligned} B &= \text{scacirc}_{\mathcal{R}}(b_0, b_1, \cdots, b_{n-1}) \\ &= \triangle F \text{diag}(u(\omega_0), u(\omega_1), \cdots, u(\omega_{n-1}))(\triangle F)^{-1} \\ &= \triangle F \text{diag}(1/f(\omega_0), 1/f(\omega_1), \cdots, 1/f(\omega_{n-1}))(\triangle F)^{-1}. \end{aligned}$$

Consequently, $BA = I$. Therefore, $u(x)$ is the inverse of $f(x)$ in the quotient ring $P_{n-1}(x)$. The polynomial $u'(x)$ can be obtained by Euclid's Algorithm. This is the main idea of the algorithm for computing the inverse of the scaled factor circulant matrix.

To reduce the computation, suppose $a$ is the leading coefficient of $f(x)$ and $a \neq 0$, let $f'(x) = f(x)/a$. Then $f(x) = af'(x)$. The leading coefficient of $f'(x)$ is 1.

**Theorem 3.2.** *Let* $A = \text{scacirc}_{\mathcal{R}}(a_0, a_1, \cdots, a_{n-1})$ *be a singular scaled factor circulant matrix with the representor*

$$f(x) = a_0 + \sum_{i=1}^{n-1} a_i(\prod_{t=1}^{i} d_t)^{-1}x^i.$$

*Suppose* $A$ *has* $m$ *nonzero eigenvalues. Without loss of generality, suppose* $f(\omega_j) = 0$, *for* $j = m, m+1, \cdots, n-1$, *where* $\omega_j, j = 0, 1, \cdots, n-1$, *are roots of* $g(x) = x^n - d_1 d_2 \cdots d_n$.

Let

$$g_1(x) = \prod_{j=0}^{m-1}(x - \omega_j), \quad g_2(x) = \prod_{j=m}^{n-1}(x - \omega_j), \quad f_1(x) = f(x)g_2(x).$$

Then there exists a polynomial

$$u_1(x) = b'_0 + \sum_{i=1}^{n-1} b'_i(\prod_{t=1}^{i} d_t)^{-1}x^i$$

such that $u_1(\omega_j) = 1/f_1(\omega_j), j = 0, 1, \cdots, m-1$.

Let

$$u(x) = u_1(x)g_2(x) = b_0 + \sum_{i=1}^{n-1} b_i(\prod_{t=1}^{i} d_t)^{-1}x^i.$$

Then $B = \text{scacirc}_{\mathcal{R}}(b_0, b_1, \cdots, b_{n-1})$ is the group inverse $A^{\#}$ of $A$. If $\mathcal{R}$ is normal, then $B = \text{scacirc}_{\mathcal{R}}(b_0, b_1, \cdots, b_{n-1})$ is the Moore-Penrose inverse $A^+$ of $A$.

*Proof.* Since

$$x^n - d_1 d_2 \cdots d_n = \prod_{j=0}^{n-1}(x - \omega_j),$$

it follows that $g_1(x)$ and $g_2(x)$ are coprime. From the condition of Theorem 3.2, we know that $g_1(x)$ and $f(x)$ are coprime. So $f_1(x)$ and $g_1(x)$ are coprime, and there exist $u_2(x)$ and $v(x)$ satisfying

$$f_1(x)u_2(x) + g_1(x)v(x) = 1.$$

When $x = \omega_j, j = 0, 1, \cdots, m-1, g_1(x) = 0$, thus $f_1(\omega_j)u_2(\omega_j) = 1$. Let $u_1(x) = u_2(x)\text{mod}(g_1(x))$. Then the existence of the $u_1(x)$ in Theorem 3.2 has been proved.

Since $u(x) = u_1(x)g_2(x)$, when $j = m, m+1, \cdots, n-1$, $u(\omega_j) = 0$, when $j = 0, 1, \cdots, m-1$,

$$u(\omega_j) = u_1(\omega_j)g_2(\omega_j) = g_2(\omega_j)/f_1(\omega_j) = 1/f(\omega_j).$$

The scaled factor circulant matrix $B$ is given by

$$\begin{aligned} B &= \text{scacirc}_{\mathcal{R}}(b_0, b_1, \cdots, b_{n-1}) \\ &= \triangle F \text{diag}(u(\omega_0), u(\omega_1), \cdots, u(\omega_{n-1}))(\triangle F)^{-1} \\ &= \triangle F \text{diag}(1/f(\omega_0), 1/f(\omega_1), \cdots, 1/f(\omega_{m-1}), 0, \cdots, 0)(\triangle F)^{-1}. \end{aligned}$$

It follows from Theorem 2.1 that $B$ is the group inverse $A^{\#}$ of $A$. If $\mathcal{R}$ is normal, then $B$ is the Moore-Penrose inverse $A^+$ of $A$.

Theorem 3.2 implies that for computing the group inverse $A^{\#}$ and the Moore-Penrose inverse $A^+$ of the singular scaled factor circulant matrix $A$, we only need to invert $f(x)g_2(x)$ in the quotient ring $P_{n-1}(x)/\langle g_1(x)\rangle$.

It can be verified that $g_2(x)$ is the largest common factor of $f(x)$ and $g(x) = x^n - d_1 d_2 \cdots d_n$. In our computations, if $\deg(f_1(x)) > \deg(g_1(x))$, we can do polynomial division $f_1(x) = g_1(x)s(x) + f_{12}(x)$. As $f_{12}(\omega_j) = f_1(\omega_j)$, $j = 0, 1, \cdots, m-1$, $f_1(x)$ can be taken the place by $f_{12}(x)$.

A similar device was used in [24] for computing the inverses and the group inverses of FLS $r$-circulant matrices.

## 4. Inverting the Scaled Factor Circulant Matrix

The problem becomes how to evaluate $u(x), v(x)$ when $f(x), g(x)$ are known and satisfy $f(x)u(x) + g(x)v(x) = 1$. Using Euclid's algorithm:

$$\begin{aligned}
&g(x) = q_0(x)f(x) + r_1(x),\\
&f(x) = q_1(x)r_1(x) + r_2(x),\\
&r_1(x) = q_2(x)r_2(x) + r_3(x),\\
&\cdots\cdots\cdots\\
&r_{i-1}(x) = q_i(x)r_i(x) + r_{i+1}(x),\\
&\cdots\cdots\cdots
\end{aligned}$$

Let $v_1(x) = 1$, $u_1(x) = -q_0(x)$, then $r_1(x) = f(x)u_1(x) + g(x)v_1(x)$. It is obvious that

$$\begin{aligned}
r_2(x) &= f(x) - q_1(x)[g(x) - q_0(x)f(x)]\\
&= [1 + q_0(x)q_1(x)]f(x) - g(x)q_1(x).
\end{aligned}$$

Let $v_2(x) = -q_1(x), u_2(x) = 1 + q_0(x)q_1(x)$. We then have

$$r_2(x) = f(x)u_2(x) + g(x)v_2(x).$$

Suppose

$$r_j(x) = f(x)u_j(x) + g(x)v_j(x),$$

and that $v_j(x), u_j(x)$ have been computed when $j = 0, 1, \cdots, i$. Then

$$\begin{aligned}
r_{i+1}(x) &= r_{i-1}(x) - q_i(x)r_i(x)\\
&= f(x)u_{i-1}(x) + g(x)v_{i-1}(x) - q_i(x)[f(x)u_i(x) + g(x)v_i(x)]\\
&= f(x)[u_{i-1}(x) - q_i(x)u_i(x)] + g(x)[v_{i-1}(x) - q_i(x)v_i(x)].
\end{aligned}$$

Let $r_{i+1}(x) = f(x)u_{i+1}(x) + g(x)v_{i+1}(x)$. Then

$$u_{i+1}(x) = u_{i-1}(x) - q_i(x)u_i(x).$$

Now, we have obtained the recurrence formula for $u_i(x)$. So, computing $u(x)$ is equivalent to computing a sequence of polynomials

$$q_0(x), r_1(x), u_1(x), \cdots, q_i(x), r_{i+1}(x), u_{i+1}(x), \cdots.$$

If $r_{j+1}$=const, then $u(x) = \frac{u_{j+1}(x)}{r_{j+1}(x)}$. We can improve the method. If the division of polynomial becomes

$$r_{i-1}(x) = q_i(x)r_i(x) + c_{i+1}r_{i+1}(x),$$

where $c_{i+1}$ is a nonzero number, then the recurrence formula becomes

$$u_{i+1}(x) = \frac{u_{i-1}(x) - q_i(x)u_i(x)}{c_{i+1}}.$$

Thus, we can make the leading coefficient of $r_i(x)$ equal to 1 by suitably choosing $c_i$.

**Algorithm 4.1.** Given a scaled factor circulant matrix

$$A = \text{scacirc}_{\mathcal{R}}(a_0, a_1, \cdots, a_{n-1}),$$

the algorithm computes a scaled factor circulant matrix

$$B = \text{scacirc}_{\mathcal{R}}(b_0, b_1, \cdots, b_{n-1}).$$

When $A$ is nonsingular, $B$ is the inverse of $A$. When $A$ is singular, $B$ is the group inverse of $A$. If $\mathcal{R}$ is normal, then $B$ is the Moore-Penrose inverse $A^+$ of the scaled factor circulant matrix $A$. Let

$$f(x) = a_0 + \sum_{i=1}^{n-1} a_i(\prod_{t=1}^{i} d_t)^{-1}x^i, \quad g(x) = x^n - d_1 d_2 \cdots d_n,$$
$$r_{-1}(x) = g(x), \quad r_0(x) = f(x), \quad u_{-1}(x) = 0, \quad u_0(x) = 1.$$

Perform the polynomial division with remainder

$$\text{do} \begin{cases} r_{i-1}(x) = q_i(x)r_i(x) + r_{i+1}(x), \\ (\text{let } c_{i+1} \text{ be the leading coefficient of } r_{i+1}(x)), \\ r_{i+1}(x) \leftarrow r_{i+1}(x)/c_{i+1}, \\ u_{i+1}(x) \leftarrow [u_{i-1}(x) - q_i(x)u_i(x)]/c_{i+1}, i = 0, 1, \cdots \end{cases} \tag{4.1}$$

until $r_m(x) = 1$ or $r_m(x) = 0$.
If $r_m(x) = 1$, then $u_m(x)$ is the representor of $B$. Then $B = u_m(\mathcal{R})$ is the inverse of $A$.
If $r_m(x) = 0$, then $r_{m-1}(x)$ is the largest common factor of $f(x)$ and $g(x)$.
Let $r(x) = r_{m-1}(x), r_{-1}(x) = g(x)/r_{m-1}(x), r_0(x) = f(x)r_{m-1}(x)\text{mod}(r_{-1}(x))$, $u_{-1}(x) = 0, u_0(x) = 1$, go to (4.1).
Now, if $r_{m'}(x) = 1$, then $u(x) = u_{m'}(x)r(x)\text{mod}(g(x))$ is the representor of $B$. Thus $B = u(\mathcal{R})$ is the group inverse $A^{\#}$ of $A$. Moreover, if $\mathcal{R}$ is normal, then $B = u(\mathcal{R})$ is the Moore-Penrose inverse $A^+$ of $A$.

## 5. Computational Complexity

If the matrix is nonsingular, the computational complexity is divided into two parts. Suppose that the order of the scaled factor circulant matrix $A$ is $n$, deg $(f(x)) = n - 1$. First we discuss the computational complexity on the division of polynomials.

$$r_{i-1}(x) = q_i(x)r_i(x) + r_{i+1}(x), r_{i+1}(x) \leftarrow r_{i+1}(x)/c_{i+1}, \quad \text{for } i = 0, 1, 2, \cdots.$$

It is obvious that the division of polynomials will be done for less than $n - 1$ times. If it is computed for $n - 1$ times, then $\deg(q_i(x)) = 1$, and the leading coefficient of $r_i(x), q_i(x)$ is 1. So the division of polynomials involves $2\sum_{i=1}^{n-1} i = n^2 + \mathcal{O}(n)$ flops. If it has been computed for less than $n - 1$ times, although deg $(q_i(x)) > 1$, the times of polynomial division need to be reduced. It can be deduced that this part involves less than $n^2$ flops.

Second we discuss the computational complexity on the multiplication of polynomials.

$$u_{i+1}(x) \leftarrow [u_{i-1}(x) - q_i(x)u_i(x)]/c_{i+1}, \quad i = 0, 1, 2, \cdots.$$

From Algorithm 4.1, we know that the multiplication of polynomials will be done for less than $n - 1$ times. If it has been done for $n - 1$ times, since $\deg(q_i(x)) = 1$, and the leading coefficient of $r_i(x), q_i(x)$ is 1, the multiplication of polynomials requires $2\sum_{i=1}^{n-1} i = n^2 + \mathcal{O}(n)$ flops. If it has been computed for less than $n-1$ times, the multiple of division involves less than $n^2$ flops. So, all the amount of work is $2n^2$ flops. Thus, when $\deg(f(x)) = m$, the algorithm involves $3nm$ flops. So when $m \ll n$, we only require $\mathcal{O}(n)$ flops, which indicates that the algorithm reduces the computational complexity greatly.

With the same reason, when the scaled factor circulant matrix is singular, the method requires $4n^2 + \mathcal{O}(n)$ flops to compute the group inverse or the Moore-Penrose inverse.

## 6. Numerical Examples

**Example 6.1.** Let

$$A = \begin{pmatrix} 1 & 3 & 2 & 8 \\ 16 & 1 & 6 & 8 \\ 8 & 8 & 1 & 12 \\ 6 & 2 & 4 & 1 \end{pmatrix}.$$

Is the matrix $A$ singular or nonsingular? If $A$ is nonsingular, find the inverse of $A$.

Since $A = \text{scacirc}_{\mathcal{R}}(1,3,2,8) = I + 3\mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 = f(\mathcal{R})$, where $f(x) = 1 + 3x + x^2 + x^3$ is the representor of $A$, and

$$\mathcal{R} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \\ 2 & 0 & 0 & 0 \end{pmatrix},$$

it is known that $A$ is a scaled factor circulant matrix. Let

$$\begin{aligned} r_{-1}(x) &= g(x) = x^4 - 16, \\ r_0(x) &= f(x) = 1 + 3x + x^2 + x^3, \\ u_{-1}(x) &= 0, \quad u_0(x) = 1. \end{aligned}$$

Using Algorithm 4.1, we have

$$\begin{aligned} & q_0(x) = x - 1, \quad r_1(x) = x^2 - x + \frac{15}{2}, \quad c_1 = -2, \quad u_1(x) = 0.5x - 0.5; \\ & q_1(x) = x + 2, \quad r_2(x) = x + \frac{28}{5}, \quad c_2 = -2.5, \quad u_2(x) = 0.2x^2 + 0.2x - 0.8; \\ & q_2(x) = x - \frac{33}{5}, \quad r_3(x) = 1, \quad c_3 = \frac{2223}{50}, \\ & u_3(x) = -\frac{289}{2223} + \frac{131}{2223}x + \frac{56}{2223}x^2 - \frac{10}{2223}x^3. \end{aligned}$$

Since $r_3(x) = 1$, $A$ is a nonsingular matrix and $u_3(x)$ is the representor of $A^{-1}$. Then

$$\begin{aligned} A^{-1} &= \text{scacirc}_{\mathcal{R}}\left(-\frac{289}{2223}, \frac{131}{2223}, \frac{112}{2223}, -\frac{80}{2223}\right) \\ &= \frac{1}{2223} \begin{pmatrix} -289 & 131 & 112 & -80 \\ -160 & -289 & 262 & 448 \\ 448 & -80 & -289 & 524 \\ 262 & 112 & -40 & -289 \end{pmatrix}. \end{aligned}$$

**Example 6.2.** Let

$$A = \begin{pmatrix} -4 & -3 & 2 \\ 64 & -4 & -6 \\ -96 & 32 & -4 \end{pmatrix}.$$

Is the matrix $A$ singular or nonsingular? If $A$ is singular, find the group inverse of $A$.

Since $A = \text{scacirc}_{\mathcal{R}}(-4, -3, 2) = -4I - 3\mathcal{R} + \mathcal{R}^2 = f(\mathcal{R})$, where $f(x) = -4 - 3x + x^2$ is the representor of $A$, and

$$\mathcal{R} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 32 & 0 & 0 \end{pmatrix},$$

it is known that $A$ is a scaled factor circulant matrix. Let $r'_{-1}(x) = g(x) = x^3 - 64, r'_0(x) = f(x) = -4 - 3x + x^2$. Using Algorithm 4.1, we have

$$q'_0(x) = x + 3, \quad r'_1(x) = x - 4, \quad c'_1 = 13, \quad q'_1(x) = x + 1, \quad r'_2(x) = 0.$$

Then the largest common factor of $f(x)$ and $g(x)$ is $r'_1(x) = x - 4$. Consequently, $A$ is a singular matrix. Let

$$r(x) = x - 4, \quad r_{-1}(x) = \frac{g(x)}{x-4} = x^2 + 4x + 16,$$
$$r_0(x) = f(x)(x-4)\text{mod}(x^2 + 4x + 16) = 36x + 192, \quad u_{-1}(x) = 0, \quad u_0(x) = 1.$$

Using Algorithm 4.1, we have

$$q_0(x) = \frac{1}{36}x - \frac{1}{27}, \quad r_1(x) = 1, \quad c_1 = \frac{208}{9}, \quad u_1(x) = -\frac{1}{832}x + \frac{1}{624}.$$

Consequently,

$$u_1(x)r(x) = -\frac{1}{156} + \frac{1}{156}x - \frac{1}{832}x^2$$

is the representor of $A^{\#}$. Then

$$A^{\#} = \text{scacirc}_{\mathcal{R}}(-\frac{1}{156}, \frac{1}{156}, -\frac{1}{416}) = \begin{pmatrix} -\frac{1}{156} & \frac{1}{156} & -\frac{1}{416} \\ -\frac{1}{13} & -\frac{1}{156} & \frac{2}{156} \\ \frac{32}{156} & -\frac{1}{26} & -\frac{1}{156} \end{pmatrix}.$$

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