Journal of Computational Mathematics, Vol.26, No.2, 2008, 250–258.

THE SENSITIVITY OF THE EXPONENTIAL OF AN ESSENTIALLY NONNEGATIVE MATRIX*

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Abstract

This paper performs perturbation analysis for the exponential of an essentially nonnegative matrix which is perturbed in the way that each entry has a small relative perturbation. For a general essentially nonnegative matrix, we obtain an upper bound for the relative error in 2-norm, which is sharper than the existing perturbation results. For a triangular essentially nonnegative matrix, we obtain an upper bound for the relative error in entrywise sense. This bound indicates that, if the spectral radius of an essentially nonnegative matrix is not large, then small entrywise relative perturbations cause small relative error in each entry of its exponential. Finally, we apply our perturbation results to the sensitivity analysis of RC networks and complementary distribution functions of phase-type distributions.

Mathematics subject classification: 65F35, 15A42.
Key words: Essentially nonnegative matrix, Matrix exponential, Entrywise perturbation theory, RC network, Phase-type distribution.

1. Introduction

The matrix exponential is an important matrix function and receives extensive attention in the literatures. Matrix exponential e^{At}, which is defined as

$$e^{At} = \sum_{k=0}^{\infty} (At)^k / k!,$$

is the unique solution to the initial value problem

$$\frac{d}{dt} X(t) = AX(t), \qquad X(0) = I.$$

Many methods have been developed to compute matrix exponentials, see [7, 9, 15, 16, 20, 22, 23, 26] and references therein. Also, much perturbation analysis has been performed, see [10, 12, 13, 24] and references therein, to assess the algorithms and estimate error bounds. However, the existing error bounds are obtained for general matrices under general perturbations, with no regard to the structures of the matrices and their perturbations. In this paper, we consider the entrywise perturbation theory for essentially nonnegative matrices.

A matrix $A = (a_{ij})_{i,j=1}^{n}$ is said to be an essentially nonnegative matrix if its off-diagonal entries are non-negative, i.e., $a_{ij} \geq 0$ for $i \neq j$, see [25]. The exponentials of essentially nonnegative matrices frequently arise in many research areas, such as Markov chains, queuing systems and RC networks. In this paper, for a general essentially nonnegative matrix A, we

* Received September 11, 2006 / Revised version received June 1, 2007 / Accepted September 19, 2007 /

consider how the exponential e^{At} for $t \geq 0$ is perturbed when each entry of A gets a small relative perturbation. More specifically, let $E = (e_{ij})_{i,j=1}^{n}$ be a small perturbation to A in entrywise sense, i.e., for all i and j, and some $\epsilon > 0$,

$$|e_{ij}| \leq \epsilon |a_{ij}|, \tag{1.1}$$

we present an upper bound for $\phi(t)$, which is defined as

$$\phi(t) = \frac{\|e^{(A+E)t} - e^{At}\|}{\|e^{At}\|}. \tag{1.2}$$

Here, $\|\cdot\|$ denotes 2-norm. Compared to the upper bounds obtained by applying the perturbation results of Van Loan [24] directly to the problem under consideration, our bound is tighter.

In some applications, the entries of A and its exponential e^{At} have some physical meaning. It is of great interest to study how the individual entries of e^{At} are perturbed when each entry of A gets a small relative perturbation. The exponentials of triangular essentially nonnegative matrices play an important role in representation of phase-type distributions. In this paper, for a triangular essentially nonnegative matrix A, we present an upper bound for $\psi(t)$, which is defined as

$$\psi(t) = \max_{(e^{At})_{ij} \neq 0} \frac{|(e^{(A+E)t})_{ij} - (e^{At})_{ij}|}{|(e^{At})_{ij}|}. \tag{1.3}$$

The error bound indicates that if $\rho(A)t$, where $\rho(A)$ is the spectral radius of A, is not large, then small entrywise relative perturbation in A only causes small relative error in each entry of e^{At}.

Finally we apply our perturbation results to the sensitivity analysis of RC networks and complementary distribution functions of phase-type distributions with triangular representation.

Throughout this paper, $\|\cdot\|$ denotes 2-norm. For a matrix $X = (x_{ij})$, we denote by $|X|$ the matrix of entries $|x_{ij}|$ and by $X \geq Y$, where $Y = (y_{ij})$ is of identical dimension as X, if $x_{ij} \geq y_{ij}$ for all i and j. Especially, $X \geq 0$ means that every entry of X is nonnegative. These symbols are also applicable to row and column vectors. In accordance with this convention, Eq. (1.1) can be written as

$$|E| \leq \epsilon |A|. \tag{1.4}$$

2. Perturbation Bound in 2-norm

In this section, we will obtain an upper bound for $\phi(t)$ in (1.2). To this end, we need the following identity which appeared in [1]:

$$e^{(A+E)t} = e^{At} + \int_0^t e^{A(t-s)} E e^{(A+E)s} ds. \tag{2.1}$$

We first obtain the upper bound for the case that A is nonnegative.

Lemma 2.1. *Let A be an $n \times n$ nonnegative matrix and E a perturbation matrix to A satisfying $|E| \leq \epsilon A$. Then*

$$\phi(t) \leq \epsilon \|A\| t e^{\epsilon \|A\| t}.$$

Proof. Using (2.1) and $|E| \leq \epsilon A$, we have

$$\begin{aligned}
&|e^{(A+E)t} - e^{At}| \\
&= \left| \int_0^t e^{A(t-s)} E e^{(A+E)s} ds \right| \leq \int_0^t e^{A(t-s)} |E| e^{(A+E)s} ds \\
&\leq \int_0^t e^{A(t-s)} \epsilon A e^{(A+\epsilon A)s} ds = e^{At}(e^{\epsilon At} - I).
\end{aligned}$$

Note that $\|X\| \geq \|Y\|$ for $X \geq Y \geq 0$. Then

$$\begin{aligned}
\phi(t) &\leq \|e^{\epsilon At} - I\| = \left\| \sum_{k=1}^{\infty} \frac{(\epsilon At)^k}{k!} \right\| \\
&\leq \sum_{k=1}^{\infty} \frac{(\epsilon t\|A\|)^k}{k!} = \epsilon\|A\|t \sum_{k=0}^{\infty} \frac{(\epsilon\|A\|t)^k}{(k+1)!} \\
&\leq \epsilon\|A\|te^{\epsilon\|A\|t}. \quad \square
\end{aligned}$$

With Lemma 2.1, we now get the error bound for essentially nonnegative matrices.

Theorem 2.1. *Let A be an $n \times n$ essentially nonnegative matrix and E a perturbation matrix to A satisfying $|E| \leq \epsilon|A|$. Then*

$$\phi(t) \leq 3\epsilon\|A\|te^{3\epsilon\|A\|t}. \tag{2.2}$$

Proof. Let $B = 2\alpha I + A$ with $\alpha = \|A\|$. Clearly, B is a nonnegative matrix with

$$\begin{aligned}
&b_{ii} = 2\alpha + a_{ii} \geq |a_{ii}| \geq 0, \\
&b_{ij} = a_{ij} \geq 0, \quad \forall\ 0 \leq i \neq j \leq n.
\end{aligned}$$

It follows from $|E| \leq \epsilon|A|$ that $|E| \leq \epsilon B$. Applying Lemma 2.1, we have

$$\phi(t) = \frac{\|e^{(A+E)t} - e^{At}\|}{\|e^{At}\|} = \frac{\|e^{(B+E)t} - e^{Bt}\|}{\|e^{Bt}\|} \leq \epsilon\|B\|te^{\epsilon\|B\|t}.$$

The theorem follows since $\|B\| \leq 3\|A\|$. $\square$

Remark 2.1. Write $A = D + N$, where D is a diagonal matrix consisting of diagonal entries of A and N is obtained from A by setting the diagonal entries zero. If A is an essentially nonnegative matrix and E satisfies $|E| \leq \epsilon|A|$, then

$$\|E\| \leq \epsilon(\|D\| + \|N\|) \leq \epsilon(\|D\| + \|D + N\| + \|D\|) \leq 3\epsilon\|A\|.$$

Denote

$$\begin{aligned}
&\lambda(A) = \{\lambda \mid \det(A - \lambda I) = 0\}, \\
&\alpha(A) = \max\{\mathrm{Re}(\lambda) \mid \lambda \in \lambda(A)\}, \\
&\mu(A) = \max\{\mu \mid\ \mu \in \lambda((A^* + A)/2)\}, \\
&\rho(A) = \max\{|\lambda| \mid \lambda \in \lambda(A)\}.
\end{aligned}$$

In [24], Van Loan obtained some bounds for $\phi(t)$ for general matrices under small perturbations in normwise sense. If we don't take into consideration the fact that A is essentially nonnegative and E is a small perturbation to A in entrywise sense, from Van Loan's perturbation results we obtain the following error bounds:

1. If A and E commute, then
$$\phi(t) \le 3\epsilon\|A\|te^{3\epsilon\|A\|t}. \tag{2.3}$$
2. Power series bound
$$\phi(t) \le 3\epsilon\|A\|te^{3\epsilon\|A\|t}e^{(\|A\|-\alpha(A))t}. \tag{2.4}$$
3. Log norms bound
$$\phi(t) \le 3\epsilon\|A\|te^{3\epsilon\|A\|t}e^{(\mu(A)-\alpha(A))t}. \tag{2.5}$$
4. Schur decomposition bound. Let $Q^*AQ = \widehat{D} + \widehat{N}$ be the Schur decomposition form of A, where $Q \in \mathbb{C}^{n\times n}$ is a unitary matrix, $\widehat{D} = \mathrm{diag}(\lambda_i)$, and $\widehat{N}$ is a strictly upper-triangular matrix. Then
$$\phi(t) \le 3\epsilon\|A\|te^{3\epsilon M_S(t)\|A\|t}M_S(t)^2, \quad \text{where} \quad M_S(t) = \sum_{k=0}^{n-1} \frac{\|\widehat{N}t\|^k}{k!}. \tag{2.6}$$

Error bound (2.2) is the same as (2.3), but we obtain it without the assumption that A and E commute. It holds that
$$\|A\| \ge \alpha(A), \quad \mu(A) \ge \alpha(A), \quad M_S(t) \ge 1,$$
and generally these inequalities strictly hold. Thus error bound (2.2) is tighter than (2.4)-(2.6). The advantage of bound (2.2) is more pronouncing as t grows.

3. Entrywise Relative Error Bounds for Triangular Essentially Nonnegative Matrices

In some applications the entries of an essentially nonnegative matrix A and its exponential have some physical meaning. For example, suppose A is a generator matrix of a finite continuous-time Markov chain. Then A is an essentially nonnegative matrix with zero row sums. The off-diagonal entries of A describe the transition rates among states and the entry $(e^{At})_{ij}$ is the transition probability from state i to state j during the time period $[0, t]$. In such cases, it is of interest to estimate how the individual entries of e^{At} are perturbed when each entry of A gets a small relative perturbation. More precisely, under the assumption that $|E| \le \epsilon|A|$ we are interested in obtaining an upper bound for $\psi(t)$ in (1.3). We note that $(e^{At})_{ij} = 0$ if and only if $(e^{(A+E)t})_{ij} = 0$.

From the relation,
$$\frac{|(e^{(A+E)t})_{ij} - (e^{At})_{ij}|}{|(e^{At})_{ij}|} \le \frac{\|e^{(A+E)t} - e^{At}\|}{\|e^{At}\|}\frac{\|e^{At}\|}{|(e^{At})_{ij}|}, \quad \text{where} \quad (e^{At})_{ij} \ne 0,$$
we can get an immediate upper bound for $\psi(t)$ in terms of $\phi(t)$,
$$\psi(t) \le \phi(t) \min_{(e^{At})_{ij}\ne 0} \frac{\|e^{At}\|}{|(e^{At})_{ij}|}. \tag{3.1}$$
This bound implies that, even if $\phi(t)$ is not large, $\psi(t)$ could be large if there exists some nonzero entry of e^{At} that is tiny compared to $\|e^{At}\|$. The exponentials of triangular essentially nonnegative matrices play an important role in representation of phase-type distributions. For the case that A is triangular, we present in this section an upper bound for $\psi(t)$, which is far better than what (3.1) implies. In the rest of this section, we only deal with upper-triangular essentially nonnegative matrices. The result also holds for lower-triangular matrices.

Theorem 3.1. *Let A be an upper triangular essentially nonnegative matrix. Write $A = D+N$, where D is an $n \times n$ diagonal matrix and N a nonnegative strictly upper-triangular matrix. Let E be a perturbation matrix to A satisfying $|E| \le \epsilon|A|$. Then, $(e^{(A+E)t})_{ij} = 0$ if and only if $(e^{At})_{ij} = 0$ and*

$$\psi(t) \le \max\left\{(1+\epsilon)^{n-1}e^{\epsilon\rho(A)t} - 1, 1-(1-\epsilon)^{n-1}e^{-\epsilon\rho(A)t}\right\}. \tag{3.2}$$

Proof. Write $A + E = \widetilde{D} + \widetilde{N}$, where $\widetilde{D}$ is a diagonal matrix and $\widetilde{N}$ a nonnegative strictly upper-triangular matrix. Obviously,

$$|\widetilde{D} - D| \le \epsilon|D|, \quad (1-\epsilon)N \le \widetilde{N} \le (1+\epsilon)N.$$

It is shown in [24] that

$$\begin{aligned} e^{At} &= e^{Dt} + \sum_{k=1}^{n-1}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}} e^{D(t-t_1)}Ne^{D(t_1-t_2)}N\cdots Ne^{Dt_k}dt_k\cdots dt_1, \\ e^{(A+E)t} &= e^{\widetilde{D}t} + \sum_{k=1}^{n-1}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}} e^{\widetilde{D}(t-t_1)}\widetilde{N}e^{\widetilde{D}(t_1-t_2)}\widetilde{N}\cdots\widetilde{N}e^{\widetilde{D}t_k}dt_k\cdots dt_1. \end{aligned}$$

Note $e^{-\epsilon\rho(A)s}e^{Ds} \le e^{\widetilde{D}s} \le e^{\epsilon\rho(A)}e^{Ds}$ for $s \ge 0$. Then

$$\begin{aligned} e^{(A+E)t} &\le e^{\epsilon\rho(A)t}e^{Dt} + \sum_{k=1}^{n-1}(1+\epsilon)^k e^{\epsilon\rho(A)t}\int_0^t\cdots\int_0^{t_{k-1}} e^{D(t-t_1)}N\cdots Ne^{Dt_k}dt_k\cdots dt_1 \\ &\le (1+\epsilon)^{n-1}e^{\epsilon\rho(A)t}e^{At}, \end{aligned}$$

and, similarly,

$$e^{(A+E)t} \ge (1-\epsilon)^{n-1}e^{-\epsilon\rho(A)t}e^{At}. \quad \square$$

Theorem 3.1 gives an upper bound for the entrywise relative error between e^{At} and $e^{(A+E)t}$. With error bound (3.2), we can straightforwardly get a new bound for $\phi(t)$ for the case that A is triangular.

Corollary 3.1. *Let A and E be as in Theorem 3.1. Then,*

$$\phi(t) \le \max\left\{(1+\epsilon)^{n-1}e^{\epsilon\rho(A)t} - 1, 1-(1-\epsilon)^{n-1}e^{-\epsilon\rho(A)t}\right\}. \tag{3.3}$$

Proof. The result follows from

$$|e^{(A+E)t} - e^{At}| \le \max\left\{(1+\epsilon)^{n-1}e^{\epsilon\rho(A)t} - 1, 1-(1-\epsilon)^{n-1}e^{-\epsilon\rho(A)t}\right\}e^{At}$$

and the fact that e^{At} is nonnegative. $\square$

Remark 3.1. 1. If ϵ is sufficiently small, writing out the first order terms in the bounds (3.2) and (3.3), we have

$$\psi(t) \le (n-1+\rho(A)t)\epsilon + \mathcal{O}(\epsilon^2)$$

and

$$\phi(t) \le (n-1+\rho(A)t)\epsilon + \mathcal{O}(\epsilon^2).$$

These bounds indicate that the sensitivity of e^{At} to small entrywise relative perturbation depends on $\rho(A)t$, the spectral radius of At. If $\rho(A)t$ is not large, no matter how tiny it is, any entry of e^{At} gets a small relative error if A is perturbed in the way that each entry of it gets a small relative perturbation.

2. It holds that $\|A\| \geq \rho(A)$ and generally this inequality strictly holds. Thus bound (3.3) is tighter than (2.2) when t is sufficiently large. The advantage of bound (3.3) over (2.2) becomes clear in the case that $\|A\| \gg \rho(A)$. The following example demonstrates the tightness of bounds (3.2) and (3.3).

Example 3.1. Let

$$A = \begin{pmatrix} -1.2132 & 0 & 0 & 0 & 0 \\ 0 & -1.3194 & 0 & 10^5 & 0 \\ 0 & 0 & 0.9312 & 0 & 0 \\ 0 & 0 & 0 & 0.0112 & 0 \\ 0 & 0 & 0 & 0 & -0.6451 \end{pmatrix},$$

$$E = 10^{-3} \times \begin{pmatrix} 0.0115 & 0 & 0 & 0 & 0 \\ 0 & 0.0060 & 0 & -935.5000 & 0 \\ 0 & 0 & -0.0086 & 0 & 0 \\ 0 & 0 & 0 & -0.0000 & 0 \\ 0 & 0 & 0 & 0 & 0.0009 \end{pmatrix}.$$

Note that $|E| \leq \epsilon|A|$ with $\epsilon \approx 10^{-5}$. Using MATLAB's **expm**, we get

$$\frac{\|e^{A+E} - e^{A}\|}{\|e^{A}\|} = 7.0015 \times 10^{-6},$$

$$\max_{i,j} \frac{|(e^{A+E})_{ij} - (e^{A})_{ij}|}{(e^{A})_{ij}} = 1.1500 \times 10^{-5}.$$

Bounds (3.3) and (3.2) give upper bounds

$$\frac{\|e^{A+E} - e^{A}\|}{\|e^{A}\|} \leq 5.3195 \times 10^{-5},$$

$$\max_{i,j} \frac{|(e^{A+E})_{ij} - (e^{A})_{ij}|}{(e^{A})_{ij}} \leq 5.3195 \times 10^{-5}.$$

However, the upper bounds from (2.2) and (3.1) are

$$\frac{\|e^{A+E} - e^{A}\|}{\|e^{A}\|} \leq 60.2566,$$

$$\max_{i,j} \frac{|(e^{A+E})_{ij} - (e^{A})_{ij}|}{(e^{A})_{ij}} \leq 1.2604 \times 10^{7},$$

which are not satisfactory.

4. Applications

In this section, we apply our entrywise perturbation results to sensitivity analysis of RC networks and the complementary distribution functions of phase-type distributions with triangular representation.

4.1. RC networks

Consider a digital network with n nodes, in which the ith node is capacitively grounded by $c_i > 0$ and resistively grounded by a conductance $g_i \geq 0$, and also connected by a conductance $g_{ij} \geq 0$ with the node j. If we denote by $v(t) = [v_1(t), v_2(t), \cdots, v_n(t)]^T$ the vector of node voltages and by $v^\infty \in \mathbb{R}^n$ the stationary voltage vector, the transient evolution of this circuit is described by the equation [21]

$$C\frac{dx}{dt} = Gx,$$

where $x(t) = v(t) - v^\infty$, $C = \mathrm{diag}(c_1, c_2, \cdots, c_n)$ and $G = (g_{ij})$ with its diagonal entries defined by the given physical parameters as follows

$$g_{ii} = -\left(g_i + \sum_{j\neq i} g_{ij}\right).$$

Apparently, the matrix $A = C^{-1}G$ is an essentially nonnegative matrix and $x(t) = e^{At}x_0$, where x_0 is the value of $x(t)$ when $t = 0$.

Next, we perturb c_i, g_i, g_{ij} and x_0 by small entrywise relative errors not exceeding ϵ and correspondingly denote by $\widetilde{c}_i$, $\widetilde{g}_{i_0}$, $\widetilde{g}_{ij}$, $\widetilde{x}_0$ these perturbed parameters. Write $\widetilde{C} = \mathrm{diag}(\widetilde{c}_1, \widetilde{c}_2, \cdots, \widetilde{c}_n)$, $\widetilde{G} = (\widetilde{g}_{ij})$ and $\widetilde{A} = \widetilde{C}^{-1}\widetilde{G}$. Easily we can get

$$|\widetilde{A} - A| \leq \widetilde{\epsilon}|A|, \quad \text{where} \quad \widetilde{\epsilon} = \frac{2\epsilon}{1-\epsilon}.$$

Let $\widetilde{x}(t)$ denote the solution of the perturbed circuit. Since

$$\|\widetilde{x}(t) - x(t)\| \leq \|e^{\widetilde{A}t} - e^{At}\|\|\widetilde{x}_0\| + \|e^{At}\|\|\widetilde{x}_0 - x_0\|,$$

by Theorem 2.2 we have

$$\frac{\|\widetilde{x}(t) - x(t)\|}{\|x(t)\|} \leq [(1 + 3\|A\|t)\widetilde{\epsilon} + \mathcal{O}(\widetilde{\epsilon}^2)]\frac{\|e^{At}\|\|x_0\|}{\|e^{At}x_0\|},$$

which implies that if $\|e^{At}\|\|x_0\|/\|e^{At}x_0\|$ is not large, the solution $x(t)$ is not sensitive to small relative perturbations in the physical parameters of the circuit.

4.2. Phase-type distributions with triangular representation

We consider a Markov chain on the state-space $\{1, \cdots, n, n+1\}$ for which $\{1, 2, \cdots, n\}$ is a transient set of states, so that the generator matrix of the chain takes the form

$$Q = \begin{bmatrix} T & -T\mathbf{1} \\ 0 & 0 \end{bmatrix}, \tag{4.1}$$

where T is an $n \times n$ invertible matrix which has nonnegative off-diagonal entries and non-positive row-totals, and $\mathbf{1}$ denotes the column vector of ones of the appropriate dimensions. Let $(\alpha_1, \alpha_2, \cdots, \alpha_{n+1})$ denote the initial distribution of the chain, and write $\boldsymbol{\alpha}$ for the vector $(\alpha_1, \alpha_2, \cdots, \alpha_n)$. Then the distribution of time to absorption of the chain is said to be a phase-type distribution with representation $(\boldsymbol{\alpha}, T)$, which is denoted by $\mathrm{PH}(\boldsymbol{\alpha}, T)$. Matrix T is essentially nonnegative and is called the PH-generator of $\mathrm{PH}(\boldsymbol{\alpha}, T)$. Clearly, the distribution of $\mathrm{PH}(\boldsymbol{\alpha}, T)$ is given as $1 - \boldsymbol{\alpha}e^{Tx}\mathbf{1}$ for $x \geq 0$. Phase-type distributions can approximate any probability distribution on $[0, \infty)$ and has been widely used in stochastic modeling. We refer to [11, 17] for a detailed description and applications of phase-type distribution.

A phase-type distribution has many distinct representations. A lot of studies have been carried out to find specially structured representations for phase-type distributions in order to simplify computation and application [2–6, 8, 14, 19]. Among the structures triangular structure, where the PH-generator is triangular, is a very important orientation. The set of phase-type distributions with triangular representation, which includes generalized Erlang distribution and Coxian distribution as its special cases, is also dense in the set of probability distribution on $[0, \infty)$. We now perform sensitivity analysis for complementary distribution function of PH$(\boldsymbol{\alpha}, T)$, which is given by $\boldsymbol{\alpha} e^{Tx}\mathbf{1}$, for the case that T is triangular. Let $\boldsymbol{\alpha}$ and T be perturbed in a way that each entry has relative error no more than ϵ and denote by $\widetilde{\boldsymbol{\alpha}}$, $\widetilde{T}$ the perturbed counterparts. From Theorem 3.1, the relative error between $\boldsymbol{\alpha} e^{Tx}\mathbf{1}$ and $\widetilde{\boldsymbol{\alpha}} e^{\widetilde{T}x}\mathbf{1}$ is bounded as

$$\begin{aligned}\frac{|\boldsymbol{\alpha} e^{Tx}\mathbf{1} - \widetilde{\boldsymbol{\alpha}} e^{\widetilde{T}x}\mathbf{1}|}{\boldsymbol{\alpha} e^{Tx}\mathbf{1}} &\le \max\{1-(1-\epsilon)^n e^{-\epsilon\rho(T)x},\ (1+\epsilon)^n e^{\epsilon\rho(T)x} - 1\} \\ &\le (e^{\epsilon\rho(T)x} - 1) + n\epsilon + \mathcal{O}(\epsilon^2).\end{aligned}$$

As $x \to 0$, the relative error between $\boldsymbol{\alpha} e^{Tx}\mathbf{1}$ and $\widetilde{\boldsymbol{\alpha}} e^{\widetilde{T}x}\mathbf{1}$ grows at most as fast as the error between $e^{\epsilon\rho(T)x}$ and 1.

Acknowledgments. The research of Jungong Xue is supported by the National Science Foundation of China under grant number 10571031, the Program for New Century Excellent Talents in Universities of China and Shanghai Pujiang Program. The research of Weiguo Gao is supported by the Special Funds for Major State Basic Research Projects (2005CB321700) and the National Science Foundation of China under grant number 10571031. The authors would like to thank two anonymous referees and Dr. Qi-ming He for valuable comments and suggestions on the paper.

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