A PRIORI ERROR ESTIMATE AND SUPERCONVERGENCE ANALYSIS FOR AN OPTIMAL CONTROL PROBLEM OF BILINEAR TYPE[*]

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Abstract

In this paper, we investigate a priori error estimates and superconvergence properties for a model optimal control problem of bilinear type, which includes some parameter estimation application. The state and co-state are discretized by piecewise linear functions and control is approximated by piecewise constant functions. We derive a priori error estimates and superconvergence analysis for both the control and the state approximations. We also give the optimal L^2-norm error estimates and the almost optimal L^∞-norm estimates about the state and co-state. The results can be readily used for constructing a posteriori error estimators in adaptive finite element approximation of such optimal control problems.

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Key words: Bilinear control problem, Finite element approximation, Superconvergence, A priori error estimate, A posteriori error estimator.

1. Introduction

The finite element approximation of optimal control problems has been extensively studied in the literature. There have been extensive studies in convergence of the standard finite element approximation of optimal control problems, see, some examples in [2, 3, 9, 10, 18, 20, 23], although it is impossible to give even a very brief review here. For optimal control problems governed by linear state equations, a priori error estimates of the finite element approximation were established long ago; see, e.g., [9, 10]. But it is more difficult to obtain such error estimates for nonlinear control problems. For some classes of nonlinear optimal control problems, a priori error estimates were established in [4, 11, 17]. The optimal control problem of bilinear type considered in this paper includes a useful model problem of parameter estimation, and there does not seem to exist systematical studies in the literature on its finite element approximation and error analysis, except [14] where a posteriori error estimates were presented.

Furthermore superconvergence analysis is an important topic for finite element approximation of PDEs. Due to the lower regularity of the constrained optimal control (normally only $H^1(\Omega) \cap W^{1,\infty}(\Omega)$), only a half order of convergence rate can be expected to gain by using

the standard recovery techniques, see [26]. Very recently Meyer and Rösch in [19] showed that in fact one order can be gained via using a special projection, which was unique to the linear optimal control problem they studied. This is a quite interesting result considering the low regularity of the optimal control. It is useful to establish such a superconvergence property for our model control problem, which is normally difficult to compute with higher accuracy.

In this paper we firstly study a priori error estimates, superconvergence analysis of the control problem with a projection and interpolator different from [19].

The plan of the paper is as follows. In Sections 2-3, we describe the control problem and give the finite element approximation. In Section 4, we derive a priori error estimates. In Section 5, superconvergence analysis is carried out. Then some applications and super-convergent result are discussed in Section 6.

2. Optimal Control Problem

In this section, we formulate the bilinear optimal control problem. Let Ω be a bounded convex polygon in R^2 with boundary $\partial\Omega$. We adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the norm $\|\cdot\|_{m,q,\Omega}$ and the seminorm $|\cdot|_{m,q,\Omega}$. We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$ and denote $W^{m,2}(\Omega)$ $(W_0^{m,2}(\Omega))$ by $H^m(\Omega)$ $(H_0^m(\Omega))$ with the norm $\|\cdot\|_{m,\Omega}$ and the seminorm $|\cdot|_{m,\Omega}$. We shall take the state space $V = H_0^1(\Omega)$, the control space $U = L^2(\Omega)$, and the observation space $Y = L^2(\Omega)$. Define the control constraint set $K \subset U$:

$$K = \{v \in U : \quad v \geq 0\}.$$

We are interested in the following optimal control problem of bilinear type :

$$
\begin{aligned}
(a) &\quad \min_{v \in K}\{\frac{1}{2}\|y - y_d\|_{0,\Omega}^2 + \frac{\alpha}{2}\|v\|_{0,\Omega}^2\}, \\
(b) &\quad -\operatorname{div}(A\nabla y) + vy = f \text{ in } \Omega , \quad y|_{\partial\Omega} = 0,
\end{aligned}
\tag{2.1}
$$

where α is a positive constant. $f \in L^2(\Omega)$ and $A(\cdot) = (a_{ij}(\cdot))_{2\times 2} \in [W^{1,\infty}(\Omega)]^{2\times 2}$ is a symmetric positive definite matrix. This problem can be interpreted as a model of estimating the true parameter u via the measured data y_d using the least square formulation.

To consider the finite element approximation of the above optimal control problem, we have to give a weak formula for the state equation. Let

$$
\begin{aligned}
a(y, w) &= \int_\Omega (A\nabla y) \cdot \nabla w \quad \forall\, y, w \in V, \\
(v, w) &= \int_\Omega vw \quad \forall\, v, w \in U.
\end{aligned}
$$

We assume that there are constants $a_0 > 0$ and $C_0 > 0$ such that

$$a_0\|y\|_V^2 \leq a(y, y), \quad |a(y, w)| \leq C_0\|y\|_V\|w\|_V, \quad \forall\, y, w \in V. \tag{2.2}$$

Then the standard weak formula for the state equation reads as follows: find $y(v) \in V$ such that

$$a(y(v), w) + (vy(v), w) = (f, w) \quad \forall\, w \in V. \tag{2.3}$$

Introduce a cost function

$$J(v) = \frac{1}{2}\|y(v) - y_d\|_{0,\Omega}^2 + \frac{\alpha}{2}\|v\|_{0,\Omega}^2. \tag{2.4}$$

Then the above control problem can be restated as follows, which we shall label OCP: find $u \in K$ such that

$$J(u) = \min_{v \in K} J(v), \tag{2.5}$$

associated with

$$a(y(v), w) + (vy(v), w) = (f, w), \qquad \forall \ w \in V. \tag{2.6}$$

It is well known (see, for example [16]) that the control problem OCP has at least one solution (y, u), and that if the pair (y, u) is a solution of OCP then there is a co-state $p \in V$ such that the triplet (y, p, u) satisfies the following optimality conditions, which we shall label OCP-OPT:

$$
\begin{aligned}
&(a) \ \ a(y, w) + (uy, w) = (f, w), \quad \forall \ w \in V, \\
&(b) \ \ a(q, p) + (up, q) = (y - y_d, q), \quad \forall \ q \in V, \\
&(c) \ \ (\alpha u - yp, v - u) \geq 0, \quad \forall \ v \in K.
\end{aligned}
\tag{2.7}
$$

The inequality (2.7)(c) is equivalent to

$$u = \max\{0, \frac{1}{\alpha} yp\}. \tag{2.8}$$

3. Finite Element Approximation

In this section, we study the finite element approximation of the bilinear optimal control problem OCP. Here we consider only n-simplex elements, as they are among the most widely used ones. Also we consider only conforming finite elements.

Let Ω^h be a polygonal approximation to Ω. Then we know that $\Omega^h = \Omega$ in this paper. Let T^h be a partitioning of Ω^h into disjoint regular n-simplices τ, such that $\bar{\Omega}^h = \bigcup_{\tau \in T^h} \bar{\tau}$. Each element has at most one face on $\partial \Omega^h$, and $\bar{\tau}$ and $\bar{\tau}'$ have either only one common vertex or a whole edge or face if τ and $\tau' \in T^h$. Associated with T^h is a finite-dimensional subspace S^h of $C(\bar{\Omega}^h)$, such that $\chi|_\tau$ are polynomials of m-order $(m \geq 1)$ for each $\chi \in S^h$ and $\tau \in T^h$. Let $V^h = S^h \cap H_0^1(\Omega)$. It is easy to see that $V^h \subset V$.

Let T_U^h be another partitioning of Ω^h into disjoint regular n-simplices τ_U, such that $\bar{\Omega}^h = \bigcup_{\tau_U \in T_U^h} \bar{\tau}_U$. Assume that $\bar{\tau}_U$ and $\bar{\tau}'_U$ have either only one common vertex or a whole face or are disjoint if τ_U and $\tau'_U \in T_U^h$. Associated with T_U^h is another finite-dimensional subspace W_U^h of $L^2(\Omega)$, such that $\chi|_{\tau_U}$ are polynomials of order m $(m \geq 0)$ for each $\chi \in W^h$ and $\tau_U \in T_U^h$.

In this paper, we will only consider the simplest finite element spaces, i.e., $m = 1$ for V^h and $m = 0$ for U^h. Let h_τ (h_{τ_U}) denote the maximum diameter of the element τ (τ_U) in T^h (T_U^h), let $h = \max_{\tau \in T^h}\{h_\tau\}$, $h_U = \max_{\tau_U \in T_U^h}\{h_{\tau_U}\}$.

Then a possible finite element approximation of OCP, which we shall label OCP^h, is: find $u_h \in K^h$ such that

$$J(u_h) = \min_{v_h \in K^h} J(v_h), \tag{3.1}$$

associated with

$$a(y_h, w_h) + (v_h y_h, w_h) = (f, w_h), \quad \forall \ w_h \in V^h, \tag{3.2}$$

where K^h is a closed convex set in U^h. This is a finite-dimensional optimization problem and may be solved by existing mathematical programming methods such as the steepest descent method, conjugate gradient method, trust domain method, and SQP.

It follows that the control problem OCP^h has at least one solution (y_h, u_h) and that if the pair $(y_h, u_h) \in V^h \times K^h$ is a solution of OCP^h then there is a co-state $p_h \in V^h$ such that the triplet $(y_h, p_h, u_h) \in V^h \times V^h \times K^h$ satisfies the following optimality conditions, which we shall label $OCP - OPT^h$:

$$
\begin{aligned}
&(a) \quad a(y_h, w_h) + (u_h y_h, w_h) = (f, w_h), \quad \forall\, w_h \in V^h, \\
&(b) \quad a(q_h, p_h) + (u_h p_h, q_h) = (y_h - y_d, q_h), \quad \forall\, q_h \in V^h, \\
&(c) \quad (\alpha u_h - y_h p_h, v_h - u_h) \geq 0, \quad \forall\, v_h \in K^h.
\end{aligned}
\tag{3.3}
$$

Introduce an averaging operator π_h^c from U onto U^h such that

$$
(\pi_h^c v)|_\tau = \frac{1}{|\tau|} \int_\tau v, \quad \forall\, \tau \in T_U^h,
\tag{3.4}
$$

where $|\tau|$ is the measure of τ. The inequality (3.3)(c) is equivalent to

$$
u_h = \max\{0, \frac{1}{\alpha} \pi_h^c(y_h p_h)\}.
\tag{3.5}
$$

In the next section, we will analyze convergence and convergent rate of the finite element approximation.

4. Convergence and a Priori Error Estimates

In this section, we are interested in deriving some error estimates for the finite element approximation of the control problem. To this end, we firstly need to show the strong convergence of the approximation. We will proceed in two steps: first to show the weak convergence and then the strong convergence.

4.1. Weak convergence

Theorem 4.1. *Let (y_h, p_h, u_h) be the solutions of (3.3). Then there exists a subsequence $(y_{h_k}, p_{h_k}, u_{h_k})$ such that $(y_{h_k}, p_{h_k}, u_{h_k})$ weakly converge to a solution (y, p, u) of the system (2.7) in $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ as k tends to infinity such that h_k tends to zero.*

Proof. The proof is divided into two steps. In the first step, we prove there exists one weakly convergent subsequence. In the second step, we prove the limit of the subsequence is a solution of the problem (2.7).

`First step of proof.` Taking $w_h = y_h$ and $q_h = p_h$ in (3.3), we have

$$
\|\nabla y_h\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}, \quad \|\nabla p_h\|_{L^2(\Omega)} \leq C \|y_h - y_d\|_{L^2(\Omega)}.
\tag{4.1}
$$

These imply that y_h and p_h are bounded in $H_0^1(\Omega)$. Furthermore, we see that

$$
\begin{aligned}
\|u_h\|_{L^2(\Omega)}^2 &= \int_\Omega |u_h|^2 = \sum_{\tau \in T_U^h} \int_\tau |u_h|^2 \\
&\leq C \sum_{\tau \in T_U^h} h^{-d} (\int_\tau |y_h p_h|)^2 \leq C \sum_{\tau \in T_U^h} (\int_\tau |y_h|^4)^{1/2} (\int_\tau |p_h|^4)^{1/2} \\
&\leq C [\int_\Omega |y_h|^4]^{1/2} [\int_\Omega |p_h|^4]^{1/2} \leq C \|y_h\|_{H^1(\Omega)}^2 \|p_h\|_{H^1(\Omega)}^2,
\end{aligned}
\tag{4.2}
$$

where we have used the fact that $\|w\|_{L^4(\Omega)} \leq C\|w\|_{H^1(\Omega)}$ for each $w \in H^1(\Omega)$. This means that u_h is uniformly bounded in $L^2(\Omega)$. It follows from the embedding theorems that there exists a subsequence $(y_{h_k}, p_{h_k}, u_{h_k})$ such that (y_{h_k}, p_{h_k}) is weakly convergent in $H_0^1(\Omega) \times H_0^1(\Omega)$ and strongly convergent in $H^s(\Omega)$ for $0 \leq s < 1$, and u_{h_k} is weakly convergent in $L^2(\Omega)$. Let the limit of $(y_{h_k}, p_{h_k}, u_{h_k})$ be (y, p, u).

Second step of proof. We prove (y, p, u) is a solution of (2.7). Let $w_{I_h} \in V_h$ and $q_{I_h} \in V_h$ be the interpolations of $w \in V$ and $q \in V$. Note that

$$
\begin{aligned}
a(y, w) + (uy, w) =& (f, w) + a(y - y_{h_k}, w) + (u(y - y_{h_k}), w) \\
&+ (u - u_{h_k}, (y_{h_k} - y)w) + (u - u_{h_k}, yw) \\
&+ a(y_{h_k}, w - w_{I_{h_k}}) + (u_{h_k} y_{h_k}, w - w_{I_{h_k}}) + (f, w_{I_{h_k}} - w) \quad (4.3)
\end{aligned}
$$

and

$$
\begin{aligned}
a(q, p) + (up, q) =& (y - y_d, q) + a(q, p - p_{h_k}) + (u(p - p_{h_k}), q) \\
&+ (u - u_{h_k}, (p_{h_k} - p)q) + (u - u_{h_k}, pq) \\
&+ a(q - q_{I_{h_k}}, p_{h_k}) + (u_{h_k} p_{h_k}, q - q_{I_{h_k}}) + (y - y_d, q_{I_{h_k}} - q). \quad (4.4)
\end{aligned}
$$

By use of weak convergence of (y_{h_k}, p_{h_k}) in $H^1(\Omega) \times H^1(\Omega)$ and weak convergence of u_{h_k} in $L^2(\Omega)$, we have

$$
\lim_{h_k \to 0} [\, a(y - y_{h_k}, w) + a(q, p - p_{h_k}) + ((u - u_{h_k}), yw) + ((u - u_{h_k}), pq)\,] = 0.
$$

By use of strong convergence of (y_{h_k}, p_{h_k}) in $H^s(\Omega)$ for $0 \leq s < 1$, we obtain

$$
\begin{aligned}
\lim_{h_k \to 0} \big[\, &|(u(y - y_{h_k}), w)| + |(u - u_{h_k}, (y - y_{h_k})w)| \\
&+ |(u(p - p_{h_k}), q)| + |(u - u_{h_k}, (p - p_{h_k})q)|\, \big] = 0.
\end{aligned}
$$

Since

$$
\lim_{h_k \to 0} [\|w - w_{I_{h_k}}\|_{H^1(\Omega)} + \|q - q_{I_{h_k}}\|_{H^1(\Omega)}] = 0,
$$

we have

$$
\begin{aligned}
&|a(y_{h_k}, w - w_{I_{h_k}})| + |(f, w_{I_{h_k}} - w)| \\
&\leq \|\nabla y_{h_k}\|_{L^2(\Omega)} \|\nabla(w - w_{I_{h_k}})\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \|w_{I_{h_k}} - w\|_{L^2(\Omega)} \to 0, \quad as \ h_k \to 0
\end{aligned}
$$

and

$$
\begin{aligned}
&|a(q - q_{I_{h_k}}, p_{h_k})| + |(y - y_d, q_{I_{h_k}} - q)| \\
&\leq \|\nabla(q - q_{I_{h_k}})\|_{L^2(\Omega)} \|\nabla p_{h_k}\|_{L^2(\Omega)} + \|y - y_d\|_{L^2(\Omega)} \|q_{I_{h_k}} - q\|_{L^2(\Omega)} \to 0, \quad as \ h_k \to 0.
\end{aligned}
$$

Noting that $(u_{h_k} y_{h_k}, w - w_{I_{h_k}}) \leq Ch\|u_{h_k} y_{h_k}\|_{L^2(\Omega)} \|w\|_{H^1(\Omega)}$ and

$$
\begin{aligned}
\|u_{h_k} y_{h_k}\|_{L^2(\Omega)}^2 &= \int_\Omega |u_{h_k}|^2 |y_{h_k}|^2 \leq [\int_\Omega |y_{h_k}|^4]^{1/2} [\int_\Omega |u_{h_k}|^4]^{1/2} \\
&\leq [\int_\Omega |y_{h_k}|^4]^{1/2} [h^d \sum_{\tau \in T_U^h} (h^{-d} \int_\tau |y_{h_k} p_{h_k}|)^4]^{1/2} \\
&\leq [\int_\Omega |y_{h_k}|^4]^{1/2} [h^d \sum_{\tau \in T_U^h} (h^{-d/2} (\int_\tau |y_{h_k}|^4)^{1/4} (\int_\tau |p_{h_k}|^4)^{1/4})^4]^{1/2} \\
&\leq h^{-d/2} [\int_\Omega |y_{h_k}|^4]^{1/2} [\sum_{\tau \in T_U^h} [\int_\tau |y_{h_k}|^4 \int_\tau |p_{h_k}|^4]^{1/2}
\end{aligned}
$$

and using $\|v\|_{L^4(\Omega)} \leq C\|v\|_{H^1(\Omega)}$ for each $v \in H^1(\Omega)$, we infer

$$|(u_{h_k} y_{h_k}, w - w_{I_{h_k}})| \leq Ch^{1-d/4} \to 0.$$

Similarly, we have

$$|(u_{h_k} p_{h_k}, q - q_{I_{h_k}})| \leq Ch^{1-d/4} \to 0.$$

Substituting above estimates into (4.3) and (4.4) leads to

$$(a) \quad a(y, w) + (uy, w) = (f, w), \quad \forall\, w \in V,$$
$$(b) \quad a(q, p) + (up, q) = (y - y_d, q), \quad \forall\, q \in V.$$

Finally, we see that for each $v \in K$,

$$(\alpha u - yp, v - u_{h_k}) = (\alpha(u - u_{h_k}) - (yp - y_{h_k} p_{h_k}), v - u_{h_k})$$
$$+ (\alpha u_{h_k} - y_{h_k} p_{h_k}, (v - \pi_{h_k}^c v)) + (\alpha u_{h_k} - y_{h_k} p_{h_k}, \pi_{h_k}^c v - u_{h_k})$$
$$\geq (\alpha(u - u_{h_k}), v - u_{h_k}) - (yp - y_{h_k} p_{h_k}, v - u_{h_k})$$
$$+ (\alpha u_{h_k} - y_{h_k} p_{h_k}, (v - \pi_{h_k}^c v)).$$

By using the weak convergence of u_{h_k} to u in $L^2(\Omega)$, the strong convergence of $y_{h_k} p_{h_k}$ to yp and the strong convergence of $\pi_h^c v$ to v, we infer

$$(\alpha u - yp, v - u) \geq 0, \quad \forall\, v \in K.$$

We thus have proved that (y, p, u) is a solution of (2.7). The proof of Theorem 4.1 is then completed. $\qquad\square$

4.2. Strong convergence

Next we prove a strong convergence result. To this end, introduce two auxiliary operators: for each $v \in U$, $(y(v), p(v)) \in V \times V$ is the solution of the equations:

$$(a) \quad a(y(v), w) + (vy(v), w) = (f, w), \quad \forall\, w \in V,$$
$$(b) \quad a(q, p(v)) + (vp(v), q) = (y(v) - y_d, q), \quad \forall\, q \in V, \tag{4.5}$$

and $(y_h(v), p_h(v)) \in V^h \times V^h$ is the solution of the equations:

$$(a) \quad a(y_h(v), w_h) + (vy_h(v), w_h) = (f, w_h), \quad \forall\, w_h \in V^h,$$
$$(b) \quad a(q_h, p_h(v)) + (vp_h(v), q_h) = (y_h(v) - y_d, q_h), \quad \forall\, q_h \in V^h. \tag{4.6}$$

Lemma 4.1. *For each $v \in U$, if there exists a constant $\alpha_0 > 0$ such that*

$$\alpha_0 \|w\|_{H^1(\Omega)}^2 \leq a(w, w) + (vw, w), \quad \forall\, w \in V, \tag{4.7}$$

then (4.5) and (4.6) have unique solutions respectively. Moreover, $y_h(v)$ and $p_h(v)$ strongly converge to $y(v)$ and $p(v)$ respectively. If the domain Ω is convex, then there hold a priori error estimates

$$(a) \quad \|y(v) - y_h(v)\|_{L^2(\Omega)} + h\|\nabla(y(v) - y_h(v))\|_{L^2(\Omega)} \leq Ch^2 \|y(v)\|_{H^2(\Omega)},$$
$$(b) \quad \|p(v) - p_h(v)\|_{L^2(\Omega)} + h\|\nabla(p(v) - p_h(v))\|_{L^2(\Omega)} \leq Ch^2 \|p(v)\|_{H^2(\Omega)}. \tag{4.8}$$

Proof.   It is clear that

$(a)\quad a(y(v) - y_h(v), w_h) + (u(y(v) - y_h(v)), w_h) = 0, \quad \forall\, w_h \in V^h,$

$(b)\quad a(q_h, p(v) - p_h(v)) + (u(p(v) - p_h(v)), q_h) = (y(v) - y_h(v), q_h), \quad \forall\, q_h \in V^h,$

$$(4.9)$$

such that

$(a)\quad \|\nabla(y(v) - y_h(v))\|_{L^2(\Omega)} \le C\|y(v) - y_{I_h}(v)\|_{H^1(\Omega)},$

$(b)\quad \|\nabla(p(v) - p_h(v))\|_{L^2(\Omega)} \le C\{\|p(v) - p_{I_h}(v)\|_{H^1(\Omega)} + \|y(v) - y_{I_h}(v)\|_{L^2(\Omega)}\}.$

$$(4.10)$$

These mean that $(y_h(v), p_h(v))$ strongly converges to $(y(v), p(v))$. If Ω is convex, then $y(v)$ and $p(v)$ are in $H^2(\Omega)$. Using the standard analysis of finite element methods, we can obtain the error estimate (4.8).   □

Lemma 4.2. *For $v, w \in U$, we have*

$$\|\nabla(y(v) - y(w))\|_{L^2(\Omega)} + \|\nabla(p(v) - p(w))\|_{L^2(\Omega)} \le C\|v - w\|_{H^{-1}(\Omega)}, \qquad (4.11)$$

$$\|\nabla(y_h(v) - y_h(w))\|_{L^2(\Omega)} + \|\nabla(p_h(v) - p_h(w))\|_{L^2(\Omega)} \le C\|v - w\|_{H^{-1}(\Omega)}. \qquad (4.12)$$

Furthermore, for the domain Ω is convex, then

$$\|y(v) - y(w)\|_{L^\infty(\Omega)} + \|p(v) - p(w)\|_{L^\infty(\Omega)} \le C\|v - w\|_{L^2(\Omega)}. \qquad (4.13)$$

Proof.   It is clear that for any $w, v \in V$,

$(a)\quad a(y(v) - y(w), w) + (v(y(v) - y(w)), w) = (w - v, wy(w)),$

$(b)\quad a(q, p(v) - p(w)) + (v(p(v) - p(w)), q) = (y(v) - y(w), q) + (w - v, qp(w)).$

$$(4.14)$$

As a result of (4.14), we obtain (4.11) and (4.12). On the other hand, we see that

$(a)\quad -div(A\nabla(y(v) - y(w))) + v(y(v) - y(u)) = (w - v)y(w),$

$(b)\quad -div(A\nabla(p(v) - p(w))) + v(p(v) - p(w)) = (w - v)p(w) + y(v) - y(w).$

$$(4.15)$$

For a convex domain, we have that $y(v) - y(w) \in H^2(\Omega)$ and $p(v) - p(w) \in H^2(\Omega)$ such that

$(a)\quad \|y(v) - y(w)\|_{H^2(\Omega)} \le \|v - w\|_{L^2(\Omega)},$

$(b)\quad \|p(v) - p(w)\|_{H^2(\Omega)} \le \|v - w\|_{L^2(\Omega)} + \|y(v) - y(w)\|_{L^2(\Omega)}.$

$$(4.16)$$

By virtue of the embedding theory, we know that $\|v\|_{L^\infty(\Omega)} \le C\|v\|_{H^2(\Omega)}$. So the estimate (4.13) is derived.   □

Theorem 4.2. *Let (y_h, p_h, u_h) be the sequence of solutions of (3.3) weakly converging to a solution (y, p, u) of the system (2.7) in $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$. Then (y_h, p_h, u_h) strongly converges to (y, p, u) in $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$.*

Proof. Using the embedding theory, we know that (y_h, p_h) is strongly convergent in $H^s(\Omega) \times H^s(\Omega)$ for $0 \le s < 1$. Furthermore, since

$$\|v\|_{L^4(\Omega)} \le C\|v\|_{H^{\frac{1}{2}}(\Omega)} \quad \forall\, v \in H^{\frac{1}{2}}(\Omega), \qquad (4.17)$$

hence (y_h, p_h) is strongly convergent in $L^4(\Omega) \times L^4(\Omega)$.

Firstly, we prove that u_h is strongly convergent in $L^2(\Omega)$. It is clear that

$$|u - u_h| = |\max\{0, \frac{1}{\alpha}yp\} - \max\{0, \frac{1}{\alpha}\pi_h^c(y_h p_h)\}|$$

$$\leq |\max\{0, \frac{1}{\alpha}yp\} - \max\{0, \frac{1}{\alpha}\pi_h^c(yp)\}| + \frac{1}{\alpha}|\pi_h^c(yp - y_h p_h)|$$

$$\leq \frac{1}{\alpha}|yp - \pi_h^c(yp)| + \frac{1}{\alpha}|\pi_h^c(yp - y_h p_h)|.$$

Hence we have

$$\|u - u_h\|_{L^2(\Omega)} \leq C\{\|yp - \pi_h^c(yp)\|_{L^2(\Omega)} + \|\pi_h^c(yp - y_h p_h)\|_{L^2(\Omega)}\}.$$

Similarly to (4.2), we have

$$\|\pi_h^c(yp - y_h p_h)\|_{L^2(\Omega)}^2 \leq C[\int_\Omega |y_h - y|^4]^{1/2}[\int_\Omega |p_h - p|^4]^{1/2} \to 0, \quad as \ \ h \to 0. \qquad (4.18)$$

Hence,

$$\|u - u_h\|_{L^2(\Omega)} \to 0, \quad as \ \ h \to 0.$$

Then we prove that (y_h, p_h) is strongly convergent in $H^1(\Omega) \times H^1(\Omega)$. It follows from the Lemma 4.1 that $(y_h(u), p_h(u))$ strongly converges to $(y, p) = (y(u), p(u))$ in $H^1(\Omega) \times H^1(\Omega)$. On the other hand, from the Lemma 4.2, we know that $(y_h - y_h(u), p_h - p_h(u))$ strongly converges to $(0, 0)$ in $H^1(\Omega) \times H^1(\Omega)$, since $\|u - u_h\|_{L^2(\Omega)}$ tends to zero. We thus have proved that (y_h, p_h) strongly converges to (y, p) in $H^1(\Omega) \times H^1(\Omega)$. □

4.3. A priori error estimates

Now we are in the position of giving a priori error estimates. To this end, we need some facts as follows. Firstly, it is a matter of calculation to show that

$$(J'(v), w) = (\alpha v - y(v)p(v), w), \quad \forall \ w \in U. \qquad (4.19)$$

Similarly, for each $v_h \in U_h$, there holds

$$(J_h'(v_h), w_h) = (\alpha v_h - y_h(v_h)p_h(v_h), w_h), \quad \forall \ w_h \in U^h. \qquad (4.20)$$

Furthermore we have

Lemma 4.3. *Let u be a solution of (2.7). If there exist constants $\epsilon > 0$ and $c_0 > 0$ such that for all $v \in L^2(\Omega)$ and $w \in L^2(\Omega)$ satisfying $\|v - u\|_{L^2(\Omega)} \leq \epsilon$ and $\|w - u\|_{L^2(\Omega)} \leq \epsilon$, then the following inequality holds:*

$$c_0\|v - w\|_{L^2(\Omega)}^2 \leq (J'(v) - J'(w), v - w). \qquad (4.21)$$

The proof of Lemma 4.3 is referred to [5], [6] and [14]. Now we can derive a priori error estimates.

Theorem 4.3. *(Error Estimate) Let (y_h, p_h, u_h) be the sequence of solutions of (3.3) strongly converging to a solution (y, p, u) of the system (2.7) in $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$. Then, there holds the following a priori error estimate:*

$$\|u - u_h\|_{L^2(\Omega)} + \|y - y_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq C(h^2 + h_U). \qquad (4.22)$$

Proof. Since the domain Ω is convex, hence $y \in H^2(\Omega)$, $p \in H^2(\Omega)$ and $u \in H^1(\Omega) \cap L^\infty(\Omega)$. Note that u_h strongly converges to u in $L^2(\Omega)$. It follows from the definition of $J(u)$, Lemma 4.3 that we have

$$
\begin{aligned}
c_0 \|u - u_h\|_{L^2(\Omega)}^2 &\leq (J'(u) - J'(u_h), u - u_h) \\
&= (\alpha u, u - u_h) - (\alpha u_h, u - u_h) - (yp - y(u_h)p(u_h), u - u_h) \\
&\leq (yp, u - u_h) + (y_h p_h, u_h - u) + (y_h p_h, u - \pi_h^c u) \\
&\quad + (\alpha u_h, \pi_h^c u - u) - (yp - y(u_h)p(u_h), u - u_h) \\
&= (\alpha u_h - y_h p_h, \pi_h^c u - u) + (y_h p_h - y(u_h)p(u_h), u_h - u) \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}
\tag{4.23}
$$

where

$$
\begin{aligned}
I_1 &= (yp - y(u_h)p(u_h), \pi_h^c u - u), \\
I_2 &= (y(u_h)p(u_h) - y_h p_h, \pi_h^c u - u), \\
I_3 &= (y_h p_h - y(u_h)p(u_h), u_h - u), \\
I_4 &= (yp, \pi_h^c u - u) = (yp - \pi_h^c(yp), \pi_h^c u - u).
\end{aligned}
\tag{4.24}
$$

Now let us to estimate term by term. It is easy to show that

$$
I_1 \leq C \|u - \pi_h^c u\|_{L^2(\Omega)}^2 + \frac{c_0}{4} \|u - u_h\|_{L^2(\Omega)}^2,
\tag{4.25}
$$

$$
I_2 \leq C[\ \|u - \pi_h^c u\|_{L^2(\Omega)}^2 + \|p_h - p(u_h)\|_{L^2(\Omega)}^2 + \|y_h - y(u_h)\|_{L^2(\Omega)}^2\ ],
\tag{4.26}
$$

$$
I_3 \leq \frac{c_0}{4} \|u - u_h\|_{L^2(\Omega)}^2 + C[\|p_h - p(u_h)\|_{L^2(\Omega)}^2 + \|y_h - y(u_h)\|_{L^2(\Omega)}^2],
\tag{4.27}
$$

$$
I_4 \leq \|u - \pi_h^c u\|_{L^2(\Omega)}^2 + \frac{c_0}{4} \|yp - \pi_h^c(yp)\|_{L^2(\Omega)}^2.
\tag{4.28}
$$

Thus, by using the estimates above and Lemma 4.1, we have

$$
\begin{aligned}
\|u - u_h\|_{L^2(\Omega)} &\leq C[\ \|p_h - p(u_h)\|_{L^2(\Omega)} + \|y_h - y(u_h)\|_{L^2(\Omega)} \\
&\quad + \|u - \pi_h^c u\|_{L^2(\Omega)} + \|yp - \pi_h^c(yp)\|_{L^2(\Omega)}\ ] \leq C(h^2 + h_U).
\end{aligned}
\tag{4.29}
$$

Then, it follows from the Lemma 4.2 that

$$
\begin{aligned}
&\|y - y_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \\
&\leq \|y(u) - y_h(u)\|_{L^2(\Omega)} + \|p(u) - p_h(u)\|_{L^2(\Omega)} \\
&\quad + \|y_h(u) - y_h(u_h)\|_{L^2(\Omega)} + \|p_h(u) - p_h(u_h)\|_{L^2(\Omega)} \\
&\leq Ch^2 + \|u - u_h\|_{H^{-1}(\Omega)} \leq C(h^2 + h_U).
\end{aligned}
$$

The proof of Theorem 4.3 is completed. $\qquad\square$

5. Superconvergence Analysis

In this section, we will provide some super-close results. Let us start from the superconvergence analysis for the control u. Let

$$
\Omega_+ = \{\mathbf{x} : \ u(\mathbf{x}) > 0\}, \quad \Omega_0 = \{\mathbf{x} : \ u(\mathbf{x}) = 0\}.
$$

We decompose Ω^h into three parts:

$$
\begin{aligned}
\Omega_+^h &= \{\tau_U : \ \tau_U \subset \Omega_+, \}, \quad \Omega_0^h = \{\tau_U : \ \tau_U \subset \Omega_0\}, \\
\Omega_b^h &= \Omega^h \setminus G^h, \quad G^h = \Omega_+^h \cup \Omega_0^h.
\end{aligned}
$$

5.1. Super-close result for u

In this section, we assume that $\partial\Omega_+$ is piecewise smooth and has a finite length such that

$$meas(\Omega_b^h) \leq Ch_U. \tag{5.1}$$

We also assume that $p, y \in H^2(\Omega) \bigcap W^{1,\infty}(\Omega)$, and then $u \in W^{1,\infty}(\Omega)$. Introduce a function:

$$\tilde{u} = \begin{cases} \pi_h^c u, & in \ \tau_U \subset G^h; \\ 0, & in \ \tau_U \subset \Omega_b^h \ and \ u_h = 0; \\ \alpha^{-1}\pi_h^c(yp), & in \ \tau_U \subset \Omega_b^h \ and \ u_h > 0; \end{cases} \tag{5.2}$$

and we have the following result.

Theorem 5.1. *Assume the condition (5.1) holds. Let u be the solution of (2.7) and u_h be the solution of (3.3) strongly converging to u. Then,*

$$\|u_h - \tilde{u}\|_{L^2(\Omega)} \leq Ch_U^2. \tag{5.3}$$

To prove Theorem 5.1, we need to prove two lemmas.

Lemma 5.1. *Let u and u_h be the solutions of (2.7) and (3.3), respectively. Then,*

$$(\alpha\tilde{u} - yp, \tilde{u} - u_h) \leq 0 \tag{5.4}$$

and

$$(\alpha u_h - y_h p_h, u_h - \tilde{u}) \leq 0. \tag{5.5}$$

Proof. It is clear that

$$u = \begin{cases} \alpha^{-1}yp & yp \geq 0, \\ 0 & yp < 0; \end{cases}$$

so that $\pi_h^c(u) = \alpha^{-1}\pi_h^c(yp)$ in Ω_+^h and $\pi_h^c(u) = 0$ in Ω_0^h. We have

$$\alpha\tilde{u}(\tilde{u} - u_h) = \begin{cases} \pi_h^c(yp)(\tilde{u} - u_h) & in \ \Omega_+^h; \\ 0 \leq \pi_h^c(yp)(\tilde{u} - u_h) & in \ \Omega_0^h; \ (yp \leq 0, \ \tilde{u} = 0) \\ \pi_h^c(yp)(\tilde{u} - u_h) & in \ \Omega_b^h \ and \ u_h > 0; \\ 0 = \pi_h^c(yp)(\tilde{u} - u_h) & in \ \Omega_b^h \ and \ u_h = 0; \end{cases}$$

or equivalently,

$$(\alpha\tilde{u}, \tilde{u} - u_h) \leq (\pi_h^c(yp), \tilde{u} - u_h).$$

Noting $(\alpha\tilde{u} - yp, \tilde{u} - u_h) = (\alpha\tilde{u} - \pi_h^c(yp), \tilde{u} - u_h)$, we thus have derived (5.4).

Similarly, noting that

$$u_h = \begin{cases} \alpha^{-1}\pi_h^c(y_h p_h) & \pi_h^c(y_h p_h) \geq 0; \\ 0 & \pi_h^c(y_h p_h) < 0; \end{cases}$$

and $\tilde{u} \geq 0$ if $u_h = 0$, we have

$$\alpha u_h(u_h - \tilde{u}) = \begin{cases} \pi_h^c(y_h p_h)(u_h - \tilde{u}) & \pi_h^c(y_h p_h) \geq 0; \\ 0 \leq \pi_h^c(yp)(u_h - \tilde{u}) & \pi_h^c(y_h p_h) < 0; \ (u_h = 0) \end{cases}$$

or equivalently,

$$(\alpha u_h, u_h - \tilde{u}) \leq (\pi_h^c(y_h p_h), u_h - \tilde{u}).$$

This leads to (5.5).   □

Lemma 5.2. *Assume the condition* (5.1) *holds. Then,*

$$\|y_h(\tilde{u}) - y_h(u)\|_{H^1(\Omega)} + \|p_h(\tilde{u}) - p_h(u)\|_{H^1(\Omega)} \leq Ch_U^2. \tag{5.6}$$

Proof. It follows from the definition of $y_h(u)$ and $y_h(\tilde{u})$ that for each $w_h \in V_h$,

$$a(y_h(u), w_h) + (uy_h(u), w_h) = (f, w_h)$$

and

$$a(y_h(\tilde{u}), w_h) + (\tilde{u}y_h(\tilde{u}), w_h) = (f, w_h).$$

Thus

$$
\begin{aligned}
&a(y_h(\tilde{u}) - y_h(u), w_h) + (u(y_h(\tilde{u}) - y_h(u)), w_h) \\
&= (u - \tilde{u}, (y_h(\tilde{u}) - y(\tilde{u}))w_h) + (u - \tilde{u}, y(\tilde{u})w_h).
\end{aligned}
\tag{5.7}
$$

Taking $w_h = y_h(\tilde{u}) - y_h(u)$ in (5.7), we have

$$
\begin{aligned}
\|y_h(\tilde{u}) - y_h(u)\|_{H^1(\Omega)}^2 &\leq C|(u - \tilde{u}, y(\tilde{u})(y_h(\tilde{u}) - y_h(u)))| \\
&\leq C[\ |(u - \tilde{u}, y(\tilde{u})(y_h(\tilde{u}) - y_h(u)) - \pi_h^c(y(\tilde{u})(y_h(\tilde{u}) - y_h(u))))_{G^h}| \\
&\quad + |(u - \tilde{u}, y(\tilde{u})(y_h(\tilde{u}) - y_h(u)))_{\Omega_b^h}|\ ] \\
&\leq C[h_U^2\|u\|_{H^1(\Omega)}\|y_h(\tilde{u}) - y_h(u)\|_{H^1(\Omega)} + |(u - \tilde{u}, y(\tilde{u})(y_h(\tilde{u}) - y_h(u)))_{\Omega_b^h}|\ ].
\end{aligned}
\tag{5.8}
$$

By using the trace theory of Sobolev space $\|v\|_{L^1(\partial\Omega_+)} \leq C\|v\|_{H^1(\Omega)}$, and noting that

$$
\begin{aligned}
&|(u - \tilde{u}, y(\tilde{u})(y_h(\tilde{u}) - y_h(u)))_{\Omega_b^h}| \\
&\leq |(u - \tilde{u}, y(\tilde{u})(y_h(\tilde{u}) - y_h(u)) - \mathcal{Q}(y(\tilde{u})(y_h(\tilde{u}) - y_h(u))))_{\Omega_b^h}| \\
&\quad + |(u - \tilde{u}, \mathcal{Q}(y(\tilde{u})(y_h(\tilde{u}) - y_h(u))))_{\Omega_b^h}| \\
&\leq Ch_U^2\|u\|_{W^{1,\infty}(\Omega)}[\ \|y(\tilde{u})(y_h(\tilde{u}) - y_h(u))\|_{H^1(\Omega_b^h)} \\
&\quad + \|\mathcal{Q}(y(\tilde{u})(y_h(\tilde{u}) - y_h(u)))\|_{L^1(\partial\Omega_+^h)}] \\
&\leq Ch_U^2\|u\|_{W^{1,\infty}(\Omega)}\|y_h(\tilde{u}) - y_h(u)\|_{H^1(\Omega)},
\end{aligned}
\tag{5.9}
$$

where $\mathcal{Q}$ is an orthogonal projection from $H^1(\Omega_b^h)$ onto $L^1(\partial\Omega_+)$, we have

$$\|y_h(\tilde{u}) - y_h(u)\|_{H^1(\Omega)} \leq Ch_U^2. \tag{5.10}$$

From the definition of $p_h(u)$ and $p_h(\tilde{u})$, we have that for each $w_h \in V_h$,

$$a(q_h, p_h(u)) + (up_h(u), q_h) = (y_h(u) - y_d, q_h)$$

and

$$a(q_h, p_h(\tilde{u})) + (\tilde{u}p_h(\tilde{u}), q_h) = (y_h(\tilde{u}) - y_d, q_h).$$

So that for each $q_h \in V_h$,

$$
a(q_h, p_h(\tilde{u}) - p_h(u)) + (u(p_h(\tilde{u}) - p_h(u)), q_h)
$$
$$
=(u - \tilde{u}, (p_h(\tilde{u}) - p(\tilde{u}))q_h) + (u - \tilde{u}, p(\tilde{u})q_h) + (y_h(\tilde{u}) - y_h(u), q_h). \tag{5.11}
$$

Taking $q_h = p_h(\tilde{u}) - p_h(u)$ in (5.11), we have

$$
\|p_h(\tilde{u}) - p_h(u)\|_{H^1(\Omega)}^2
$$
$$
\leq C[\,|(u - \tilde{u}, p(\tilde{u})(p_h(\tilde{u}) - p_h(u)))| + |(y_h(\tilde{u}) - y_h(u), p_h(\tilde{u}) - p_h(u))|\,]
$$
$$
\leq C[\,|(u - \tilde{u}, p(\tilde{u})(p_h(\tilde{u}) - p_h(u)) - \pi_h^c(p(\tilde{u})(p_h(\tilde{u}) - p_h(u))))_{G^h}|
$$
$$
+ |(u - \tilde{u}, p(\tilde{u})(p_h(\tilde{u}) - p_h(u)))_{\Omega_b^h}| + \|y_h(\tilde{u}) - y_h(u)\|_{L^2(\Omega)}^2\,]
$$
$$
\leq C[h_U^2 \|u\|_{H^1(\Omega)} \|p_h(\tilde{u}) - p_h(u)\|_{H^1(\Omega)} + \|y_h(\tilde{u}) - y_h(u)\|_{L^2(\Omega)}^2
$$
$$
+ |(u - \tilde{u}, p(\tilde{u})(p_h(\tilde{u}) - p_h(u)))_{\Omega_b^h}|\,]. \tag{5.12}
$$

By using (5.10) and bounding the last term on the right-hand side of (5.12), similarly to (5.9), we also can show that

$$
\|p_h(\tilde{u}) - p_h(u)\|_{H^1(\Omega)} \leq Ch_U^2. \tag{5.13}
$$

Thus (5.6) is derived. This ends the proof of Lemma 5.2.   □

Now we are in the position of proving Theorem 5.1.

Proof of Theorem 5.1. Note that u_h and $\tilde{u}$ strongly converge to u in $L^2(\Omega)$. It follows from Lemma 4.3 and Lemma 5.1 that

$$
c_0 \|u_h - \tilde{u}\|_{L^2(\Omega)}^2 \leq (J_h'(u_h) - J_h'(\tilde{u}), u_h - \tilde{u})
$$
$$
=\alpha(u_h - \tilde{u}, u_h - \tilde{u}) - (y_h p_h - y_h(\tilde{u})p_h(\tilde{u}), u_h - \tilde{u})
$$
$$
\leq (y_h p_h, u_h - \tilde{u}) - (yp, u_h - \tilde{u}) - (y_h p_h - y_h(\tilde{u})p_h(\tilde{u}), u_h - \tilde{u})
$$
$$
=(y_h(\tilde{u})p_h(\tilde{u}) - yp, u_h - \tilde{u}) = R_1 + R_2, \tag{5.14}
$$

where

$$
R_1 = (y_h(u)p_h(u) - yp, u_h - \tilde{u}),
$$
$$
R_2 = (y_h(\tilde{u})p_h(\tilde{u}) - y_h(u)p_h(u), u_h - \tilde{u}).
$$

Then it gives

$$
|R_1| \leq |((y_h(u) - y)p, u_h - \tilde{u})_U| + |(y_h(u)(p_h(u) - p), u_h - \tilde{u})|
$$
$$
\leq \frac{c_0}{4} \|u_h - \tilde{u}\|_{L^2(\Omega)}^2 + C(\|y_h(u) - y\|_{L^2(\Omega)}^2 + \|p_h(u) - p\|_{L^2(\Omega)}^2)
$$
$$
\leq \frac{c_0}{4} \|u_h - \tilde{u}\|_{L^2(\Omega)}^2 + Ch^4(\|y\|_{H^2(\Omega)}^2 + \|p\|_{H^2(\Omega)}^2) \tag{5.15}
$$

and

$$
|R_2| \leq |(y_h(\tilde{u})(p_h(\tilde{u}) - p_h(u)), u_h - \tilde{u})| + |((y_h(\tilde{u}) - y_h(u))p_h(u), u_h - \tilde{u})|
$$
$$
\leq \frac{c_0}{4} \|u_h - \tilde{u}\|_{L^2(\Omega)}^2 + C(\|y_h(\tilde{u}) - y_h(u)\|_{H^1(\Omega)}^2 + \|p_h(\tilde{u}) - p_h(u)\|_{H^1(\Omega)}^2). \tag{5.16}
$$

Thus, we have

$$
\|u_h - \tilde{u}\|_{L^2(\Omega)} \leq C(\|y_h(\tilde{u}) - y_h(u)\|_{H^1(\Omega)} + \|p_h(\tilde{u}) - p_h(u)\|_{H^1(\Omega)}). \tag{5.17}
$$

Applying Lemma 5.2 to (5.17) leads to (5.3). This ends the proof of Theorem 5.1.   □

5.2. Super-close results for y and p

Theorem 5.2. *Assume that all the conditions in Theorem 5.1 are valid. Let y_h, p_h be the solutions of (3.3). Then,*

$$\|y_h - y_h(u)\|_{H^1(\Omega)} + \|p_h - p_h(u)\|_{H^1(\Omega)} \le C(h^2 + h_U^2). \tag{5.18}$$

Proof. Noting that

$$a_0\|y_h - y_h(u)\|_{H^1(\Omega)}^2$$
$$\le a(y_h - y_h(u), y_h - y_h(u)) + (u(y_h - y_h(u)), y_h - y_h(u)) = I_1 + I_2 + I_3, \tag{5.19}$$

where

$$\begin{aligned}
I_1 &= ((u_h - \tilde{u})y_h, y_h - y_h(u)), \\
I_2 &= -(\tilde{u} - u, y_h(y_h - y_h(u)))_{\Omega_b^h}, \\
I_3 &= -(\tilde{u} - u, y_h(y_h - y_h(u)) - \pi_h^c(y_h(y_h - y_h(u))))_{G^h}.
\end{aligned} \tag{5.20}$$

Now we have

$$\begin{aligned}
|I_2| &\le |(u - \tilde{u}, y_h(y_h - y_h(u)) - \mathcal{Q}(y_h(y_h - y_h(u))))_{\Omega_b^h}| \\
&\quad + |(u - \tilde{u}, \mathcal{Q}(y_h(y_h - y_h(u))))_{\Omega_b^h}| \\
&=: I_{21} + I_{22},
\end{aligned} \tag{5.21}$$

where I_{21} and I_{22} satisfy

$$\begin{aligned}
I_{21} &\le Ch_U^2\|u\|_{W^{1,\infty}(\Omega_U^b)}\|y_h - y_h(u)\|_{H^1(\Omega_b^h)}, \\
I_{22} &\le Ch_U^2\|u\|_{W^{1,\infty}(\Omega)}\|\mathcal{Q}(y_h(y_h - y_h(u)))\|_{L^1(\partial\Omega_+)}.
\end{aligned} \tag{5.22}$$

Consequently,

$$I_2 \le Ch_U^2\|u\|_{W^{1,\infty}(\Omega)}\|y_h - y_h(u)\|_{H^1(\Omega)}. \tag{5.23}$$

Using standard techniques, it is easy to estimate $I_1 + I_3$ where we omit the details. It gives

$$\|y_h - y_h(u)\|_{H^1(\Omega)} \le C(h_U^2 + h^2).$$

Noting that

$$\begin{aligned}
a_0\|p_h - p_h(u)\|_{H^1(\Omega)}^2 &\le a(p_h - p_h(u), p_h - p_h(u)) + (u(p_h - p_h(u)), p_h - p_h(u)) \\
&= (y_h - y_h(u), p_h - p_h(u)) - ((u_h - u)p_h, p_h - p_h(u)), \tag{5.24}
\end{aligned}$$

and similarly to estimate $\|y_h - y_h(u)\|_{H^1(\Omega)}$, we also have

$$\|p_h - p_h(u)\|_{H^1(\Omega)} \le C(h_U^2 + h^2).$$

The proof of Theorem 5.2 is completed. $\qquad\square$

In the next section we will discuss a superconvergence result for u and some applications.

6. Superconvergence Result for u and Some Applications

As a consequence of super-close results in the section 5, we can derive the following optimal a priori error estimates in L^2-norm.

6.1. L^2-norm and L^∞-norm error estimates

Theorem 6.1. *Assume that all the conditions in Theorem* 5.1 *and* 5.2 *are valid. Then there holds the following a priori error estimate:*

$$\|y - y_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \le C(h_U^2 + h^2). \tag{6.1}$$

Proof. It follows from results in Theorem 5.2 that

$$\|y - y_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)}$$
$$\le \|y - y_h(u)\|_{L^2(\Omega)} + \|p - p_h(u)\|_{L^2(\Omega)} + \|y_h(u) - y_h\|_{L^2(\Omega)} + \|p_h(u) - p_h\|_{L^2(\Omega)}$$
$$\le C(h_U^2 + h^2).$$

This completes the proof of Theorem 6.1.   □

Furthermore, we can have the following almost optimal a priori error estimates in L^∞-norm.

Theorem 6.2. *Assume that all the conditions in Theorems* 5.1 *and* 5.2 *are valid. If* $y, p \in W^{2,\infty}(\Omega)$*, then there hold the following a priori error estimates:*

$$\|y - y_h\|_{L^\infty(\Omega)} + \|p - p_h\|_{L^\infty(\Omega)} \le C(h_U^2 + h^2)|\ln h|^{\frac{1}{2}} \tag{6.2}$$

and

$$\|u - u_h\|_{L^\infty(\Omega)} \le C(h + h_U). \tag{6.3}$$

Proof. In the 2-d case, it follows from the known result

$$\|w_h\|_{L^\infty(\Omega)} \le C|\ln h|^{\frac{1}{2}}\|\nabla w_h\|_{L^2(\Omega)} \quad \forall\, w_h \in V_h,$$

and results in Theorem 5.2 that we have

$$\|y - y_h\|_{L^\infty(\Omega)} + \|p - p_h\|_{L^\infty(\Omega)}$$
$$\le \|y - y_h(u)\|_{L^\infty(\Omega)} + \|p - p_h(u)\|_{L^\infty(\Omega)}$$
$$+ C|\ln h|^{\frac{1}{2}}[\ \|\nabla(y_h(u) - y_h)\|_{L^2(\Omega)} + \|\nabla(p_h(u) - p_h)\|_{L^2(\Omega)}\ ]$$
$$\le C|\ln h|^{\frac{1}{2}}(h^2 + h_U^2).$$

This is (6.2). On the other hand, by using the inverse property of finite element spaces,

$$\|v_h\|_{L^\infty(\Omega)} \le Ch_U^{-1}\|v_h\|_{L^2(\Omega)} \quad \forall\, v_h \in U^h$$

and the results in Theorem 5.1, we have

$$\|u - u_h\|_{L^\infty(\Omega)} \le \|u - \tilde{u}\|_{L^\infty(\Omega)} + \|\tilde{u} - u_h\|_{L^\infty(\Omega)}$$
$$\le \|u - \tilde{u}\|_{L^\infty(\Omega)} + Ch_U^{-1}\|\tilde{u} - u_h\|_{L^2(\Omega)}$$
$$\le C(h + h_U).$$

This is (6.3). The proof of Theorem 6.2 is then completed.   □

6.2. Superconvergence result for u and a posteriori error estimators

As a consequence of the optimal L^2-norm estimation for y_h, p_h, we have the following superconvergence result for u.

Theorem 6.3. *Assume that all the conditions in Theorems* 5.1 *and* 5.2 *hold. Let*

$$\widehat{u}_h = \max(0, \frac{1}{\alpha} y_h p_h).$$

There holds a superconvergence error estimate:

$$\|u - \widehat{u}_h\|_{L^2(\Omega)} \leq C(h_U^2 + h^2). \tag{6.4}$$

Proof. Since max is Lipschitz continuous, hence

$$|u - \widehat{u}_h| = |\max\{0, \frac{1}{\alpha} yp\} - \max\{0, \frac{1}{\alpha} y_h p_h\}|$$
$$\leq \frac{1}{\alpha} |yp - y_h p_h| \leq C[\, |p - p_h| + |y - y_h| \,].$$

By using Theorem 6.1, we derive (6.4).   □

As another application of the results in Theorems 5.1 and 5.2, we derive super-convergence for the states y, p and the optimal control u. To this end, let us adopt the same recovery operator $G_h v = (R_h v_x, R_h v_y)$ as in [26], where R_h is the recovery operator defined for the recovery of u in [26], $v_x = \partial v/\partial x$ and $v_y = \partial v/\partial y$. It should be noted that G_h is same as the Z-Z gradient recovery (see, e.g., [28, 29]) in our piecewise linear case. Then we have:

Theorem 6.4. *Suppose that all the conditions in Theorems* 5.1 *and* 5.2 *are valid. Moreover, we assume that* $y, p \in H^3(\Omega)$. *Then,*

$$\|G_h y_h - \nabla y\|_{L^2(\Omega)} + \|G_h p_h - \nabla p\|_{L^2(\Omega)} \leq C(h^2 + h_U^2). \tag{6.5}$$

Proof. First note that

$$\|G_h y_h - \nabla y\|_{L^2(\Omega)} \leq \|G_h y_h - G_h y_h(u)\|_{L^2(\Omega)} + \|G_h y_h(u) - \nabla y\|_{L^2(\Omega)}. \tag{6.6}$$

It follows from Theorem 5.2 that

$$\|G_h y_h - G_h y_h(u)\|_{L^2(\Omega)} \leq C\|\nabla(y_h - y_h(u))\|_{L^2(\Omega)} \leq C(h^2 + h_U^2). \tag{6.7}$$

It has been proved in [27] (Remark 3.2 and Theorem 3.2) that $G_h v_I = \nabla v$ on τ if v is a quadratic function on the neighborhood of τ_U $(\cup_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} \{\tau'\})$. Then, it follows from the standard interpolation error estimate technique (see, e.g., [7]) that

$$\|G_h y_h(u) - \nabla y\|_{L^2(\Omega)} \leq \|G_h y_h(u) - G_h y_I\|_{L^2(\Omega)} + \|G_h y_I - \nabla y\|_{L^2(\Omega)}$$
$$\leq C h^2 (|y|_{3,\Omega} + \|y\|_{2,\Omega}). \tag{6.8}$$

Here y_I is the linear interpolation of y. Therefore, it follows from (6.6)-(6.8) that

$$\|G_h y_h - \nabla y\|_{L^2(\Omega)} \leq C(h^2 + h_U^2). \tag{6.9}$$

Similarly, it can be proved that

$$\|G_h p_h - \nabla p\|_{L^2(\Omega)} \leq C(h^2 + h_U^2). \tag{6.10}$$

Therefore, (6.5) follows from (6.9) and (6.10).   □

Using the above two results, we can easily construct a posteriori error estimators for the control problem as follows:

$$\eta_{\tau_U} = \|u_h - \widehat{u}_h\|_{L^2(\tau_U)}, \quad \xi_\tau = \|\nabla y_h - G_h y_h\|_{L^2(\tau)}, \quad \zeta_\tau = \|\nabla p_h - G_h p_h\|_{L^2(\tau)}. \qquad (6.11)$$

They can be used as error indicators of finite element approximation with adaptive meshes. It is clear that such estimators are asymptotically exact on uniform meshes, see [26] for more details.

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