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ON NON-ISOTROPIC JACOBI PSEUDOSPECTRAL METHOD*

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Abstract

In this paper, a non-isotropic Jacobi pseudospectral method is proposed and its applications are considered. Some results on the multi-dimensional Jacobi-Gauss type interpolation and the related Bernstein-Jackson type inequalities are established, which play an important role in pseudospectral method. The pseudospectral method is applied to a two-dimensional singular problem and a problem on axisymmetric domain. The convergence of proposed schemes is established. Numerical results demonstrate the efficiency of the proposed method.

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Key words: Jacobi pseudospectral method in multiple dimensions, Jacobi-Gauss type interpolation, Bernstein-Jackson type inequalities, Singular problem, Problem on axisymmetric domain.

1. Introduction

The main advantage of spectral method is its high accuracy, see [4–9]. However, this merit may be seriously affected by singularities of genuine solutions, which could be caused by several factors, such as degenerating coefficients of leading terms in differential equations. Moreover, the coefficients of derivatives of different orders involved in underlying problems may degenerate in completely different way. For solving such problems, Guo [11, 12] developed the Jacobi approximation in certain non-uniformly weighted Sobolev space, and proposed the corresponding Jacobi spectral method with its applications to one-dimensional singular differential equations. We also refer to the work on the Jacobi approximation in [1, 7, 16, 20]. The Jacobi spectral method is also very useful for many kinds of other related problems, e.g., differential equations on unbounded domains and axisymmetric domains, [3, 10, 13, 14, 25]. On the other hand, some results on the Jacobi approximation have been successfully applied to the analysis of various rational spectral methods,see e.g., [15, 17, 18, 22, 23, 26].

In practice, it is more important and interesting to solve multi-dimensional singular problems and related problems numerically. Guo and Wang [21] provided the Jacobi spectral method in two-dimensions. It is well known that the pseudospectral method is more preferable in actual computations, since it only needs to evaluate unknown functions at interpolation nodes. This feature simplifies calculations and saves a lot of work. Furthermore, it is much easier to deal with nonlinear terms. Guo and Wang [19] investigated the Jacobi pseudospectral method for one-dimensional singular problems. However, no existing works have been found for considering the Jacobi pseudospectral method in multiple dimensions.

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This paper is devoted to the Jacobi pseudospectral method in multiple dimensions and its applications. In the next section, we recall some basic results on the one-dimensional Jacobi approximation. In Section 3, we establish the main results on the Jacobi-Gauss type interpolation in multi-dimensional space, which play important role in designing and analyzing various Jacobi pseudospectral schemes for singular problems and other related problems. We also derive a series of sharp results on the Legendre-Gauss type interpolation and the related Bernstein-Jackson type inequalities, which are very useful for pseudospectral methods of partial differential equations with non-constant coefficients. It is noted that Canuto, Hussaini et.al [6], and Bernardi and Maday [4] first studied the Legendre-Gauss type interpolation in the Sobolev spaces, and Quarteroni [24] first considered the Bernstein-Jackson type inequalities in L_p-space for the Legendre orthogonal approximation. We improve and generalize some of those results in this paper. As examples of applications, we consider a two-dimensional singular problem in Section 4, and a problem defined on an axisymmetric domain in Section 5. The convergence of proposed schemes is proved. Numerical results confirm the theoretical predictions. The final section provides some concluding remarks.

2. Preliminaries

We first recall some basic results on the one-dimensional Jacobi approximation. Let $\Lambda = (-1,1)$, and $\chi(x)$ be a certain weight function. Denote by $\mathbb{N}$ the set of all non-negative integers. For any $r \in \mathbb{N}$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ in the usual way, with the inner product $(u,v)_{r,\chi,\Lambda}$, semi-norm $|v|_{r,\chi,\Lambda}$ and norm $\|v\|_{r,\chi,\Lambda}$, respectively. In particular,

$$L^2_\chi(\Lambda) = H^0_\chi(\Lambda), \quad (u,v)_{\chi,\Lambda} = (u,v)_{0,\chi,\Lambda},$$

and $\|v\|_{\chi,\Lambda} = \|v\|_{0,\chi,\Lambda}$. For any $r > 0$, the space $H^r_\chi(\Lambda)$ and its norms are defined by space interpolation as in [2]. The space $H^r_{0,\chi}(\Lambda)$ stands for the closure in $H^r_\chi(\Lambda)$ of the set $\mathcal{D}(\Lambda)$ consisting of all infinitely differentiable functions with compact support in Λ. Besides,

$${}_0H^r_\chi(\Lambda) = \{v \mid v \in H^r_\chi(\Lambda), v(-1) = 0\}.$$

Whenever $\chi(x) \equiv 1$, we omit the subscript χ in the notations.

Let α, $\beta > -1$. The Jacobi polynomials $J^{(\alpha,\beta)}_l(x)$, $l = 0,1,2,\ldots$, are the eigenfunctions of Sturm-Liouville problem

$$\partial_x((1-x)^{\alpha+1}(1+x)^{\beta+1}\partial_x v(x)) + \lambda(1-x)^\alpha(1+x)^\beta v(x) = 0, \qquad x \in \Lambda,$$

with the corresponding eigenvalues

$$\lambda^{(\alpha,\beta)}_l = l(l+\alpha+\beta+1) \quad l = 0,1,2,\ldots$$

Let $\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$. We have that

$$\int_\Lambda J^{(\alpha,\beta)}_l(x) J^{(\alpha,\beta)}_{l'}(x)\chi^{(\alpha,\beta)}(x)\,\mathrm{d}x = \gamma^{(\alpha,\beta)}_l \delta_{l,l'},$$

where $\delta_{l,l'}$ is the Kronecker symbol, and

$$\gamma^{(\alpha,\beta)}_l = \frac{2^{\alpha+\beta+1}\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{(2l+\alpha+\beta+1)\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}.$$

Thus, for any $v \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$, we have

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l^{(\alpha,\beta)} J_l^{(\alpha,\beta)}(x),$$

where

$$\hat{v}_l^{(\alpha,\beta)} = \frac{1}{\gamma_l^{(\alpha,\beta)}} \int_\Lambda v(x) J_l^{(\alpha,\beta)}(x) \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x.$$

Now, let $N \in \mathbb{N}$, and denote by $\mathcal{P}_N(\Lambda)$ the set of all algebraic polynomials of degree at most N. Moreover,

$${}_0\mathcal{P}_N(\Lambda) = \{v \mid v \in \mathcal{P}_N(\Lambda), v(-1) = 0\},$$

and

$$\mathcal{P}_N^0(\Lambda) = \{v \mid v \in \mathcal{P}_N(\Lambda), v(\pm 1) = 0\}.$$

Throughout this paper, we denote by c a generic positive constant independent of any function and N.

The orthogonal projection $P_{N,\alpha,\beta,\Lambda}$: $L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}_N(\Lambda)$ is defined by

$$(P_{N,\alpha,\beta,\Lambda} v - v)_{\chi^{(\alpha,\beta)},\Lambda} = 0, \qquad \forall \phi \in \mathcal{P}_N(\Lambda). \tag{2.1}$$

For description of approximation results, we define the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$, $r \in \mathbb{N}$, with the semi-norm

$$|v|_{r,\chi^{(\alpha,\beta)},A,\Lambda} = \|\partial_x^r v\|_{\chi^{(\alpha+r,\beta+r)},\Lambda}$$

and the norm

$$\|v\|_{r,\chi^{(\alpha,\beta)},A,\Lambda} = \Big(\sum_{k=0}^{r} |v|^2_{k,\chi^{(\alpha,\beta)},A,\Lambda}\Big)^{\frac{1}{2}}.$$

For any $r > 0$, the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ is defined by space interpolation.

According to Theorem 2.1 of [20], for any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$, $r \in \mathbb{N}$ and $0 \leqslant \mu \leqslant r$,

$$\|P_{N,\alpha,\beta,\Lambda} v - v\|_{\mu,\chi^{(\alpha,\beta)},A,\Lambda} \leqslant c N^{\mu - r} |v|_{r,\chi^{(\alpha,\beta)},A,\Lambda}. \tag{2.2}$$

We now turn to the Jacobi-Gauss type interpolation. Let $\zeta^{(\alpha,\beta)}_{G,N,j}$, $\zeta^{(\alpha,\beta)}_{R,N,j}$ and $\zeta^{(\alpha,\beta)}_{L,N,j}$, $0 \leqslant j \leqslant N$, be the zeros of polynomials $J^{(\alpha,\beta)}_{N+1}(x)$, $(1+x)J^{(\alpha,\beta+1)}_N(x)$ and $(1-x^2)\partial_x J^{(\alpha,\beta)}_N(x)$, respectively. We denote by $\omega^{(\alpha,\beta)}_{Z,N,j}$, $0 \leqslant j \leqslant N$, $Z = G, R, L$, the corresponding Christoffel numbers such that

$$\int_\Lambda \phi(x) \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x = \sum_{j=0}^{N} \phi(\zeta^{(\alpha,\beta)}_{Z,N,j}) \omega^{(\alpha,\beta)}_{Z,N,j}, \qquad \forall \phi \in \mathcal{P}_{2N+\lambda_Z}(\Lambda), \tag{2.3}$$

where $\lambda_Z = 1$ for $Z = G$, $\lambda_Z = 0$ for $Z = R$, and $\lambda_Z = -1$ for L, respectively.

We introduce the following discrete inner product and norm,

$$(u,v)_{\chi^{(\alpha,\beta)},Z,N,\Lambda} = \sum_{j=0}^{N} u(\zeta^{(\alpha,\beta)}_{Z,N,j}) v(\zeta^{(\alpha,\beta)}_{Z,N,j}) \omega^{(\alpha,\beta)}_{Z,N,j}, \qquad \|v\|_{\chi^{(\alpha,\beta)},Z,N,\Lambda} = (v,v)^{\frac{1}{2}}_{\chi^{(\alpha,\beta)},Z,N,\Lambda}.$$

By the exactness of (2.3), we have that

$$(\phi,\psi)_{\chi^{(\alpha,\beta)},Z,N,\Lambda} = (\phi,\psi)_{\chi^{(\alpha,\beta)},\Lambda}, \quad \forall \phi \cdot \psi \in \mathcal{P}_{2N+\lambda_Z}(\Lambda), \qquad Z = G,\, R,\, L. \tag{2.4}$$

Let $\Lambda^{(\alpha,\beta)}_{Z,N} = \{x\,|\,x = \zeta^{(\alpha,\beta)}_{Z,N,j},\ 0 \leqslant j \leqslant N\}$. The Jacobi-Gauss-type interpolation $\mathcal{I}_{Z,N,\alpha,\beta,\Lambda}v \in \mathcal{P}_N(\Lambda)$ is determined uniquely by

$$\mathcal{I}_{Z,N,\alpha,\beta,\Lambda}v(x) = v(x), \qquad x \in \Lambda^{(\alpha,\beta)}_{Z,N}.$$

Here we assum that $v \in C(\Lambda)$ for $Z = G$, $v \in C(\Lambda \cup \{x = -1\})$ for $Z = R$, and $v \in C(\bar{\Lambda})$ for $Z = L$. They are named as the Jacobi-Gauss interpolation for $Z = G$, the Jacobi-Gauss-Radau interpolation for $Z = R$, and the Jacobi-Gauss-Lobatto interpolation for $Z = L$, respectively.

According to Theorems 4.1, 4.5 and 4.9 of [20], we have the following results.

(i) For any $v \in C(\Lambda) \cap H^1_{\chi^{(\alpha+k,\beta+l)},A}(\Lambda)$, $k, l \in \mathbb{N}$ and $0 \leqslant k + l \leqslant 1$,

$$\|\mathcal{I}_{G,N,\alpha,\beta,\Lambda}v\|_{\chi^{(\alpha+k,\beta+l)},\Lambda} \leqslant c\|v\|_{\chi^{(\alpha+k,\beta+l)},\Lambda} + cN^{-1}\|\partial_x v\|_{\chi^{(\alpha+k+1,\beta+l+1)},\Lambda}. \tag{2.5}$$

(ii) For any $v \in C(\Lambda \cup \{x = -1\}) \cap H^1_{\chi^{(\alpha+k,\beta-l)},A}(\Lambda)$ with $v(-1) = 0$, $k, l \in \mathbb{N}$, $0 \leqslant k \leqslant l \leqslant 1$ and $l < \beta + 1$,

$$\|\mathcal{I}_{R,N,\alpha,\beta,\Lambda}v\|_{\chi^{(\alpha+k,\beta-l)},\Lambda} \leqslant c\|v\|_{\chi^{(\alpha+k,\beta-l)},\Lambda} + cN^{-1}\|\partial_x v\|_{\chi^{(\alpha+k+1,\beta-l+1)},\Lambda}. \tag{2.6}$$

(iii) For any $v \in C(\bar{\Lambda}) \cap H^1_{\chi^{(\alpha-k,\beta-l)},A}(\Lambda)$ with $v(\pm 1) = 0$, $k, l \in \mathbb{N}$, $0 \leqslant k, l \leqslant 1, k < \alpha + 1$ and $l < \beta + 1$,

$$\|\mathcal{I}_{L,N,\alpha,\beta,\Lambda}v\|_{\chi^{(\alpha-k,\beta-l)},\Lambda} \leqslant c\|v\|_{\chi^{(\alpha-k,\beta-l)},\Lambda} + cN^{-1}\|\partial_x v\|_{\chi^{(\alpha-k+1,\beta-l+1)},\Lambda}. \tag{2.7}$$

Furthermore, as the special cases of Theorems 4.2, 4.6 and 4.10 of [20], we have the following results.

(i) For any $v \in C(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ and integer $r \geq 1$,

$$\|\mathcal{I}_{G,N,\alpha,\beta,\Lambda}v - v\|_{\chi^{(\alpha,\beta)},\Lambda} \leqslant cN^{-r}\|\partial_x^r v\|_{\chi^{(\alpha+r,\beta+r)},\Lambda}. \tag{2.8}$$

(ii) For any $v \in C(\Lambda \cup \{x = -1\}) \cap H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ and integer $r \geq 1$,

$$\|\mathcal{I}_{R,N,\alpha,\beta,\Lambda}v - v\|_{\chi^{(\alpha,\beta)},\Lambda} \leqslant cN^{-r}\|\partial_x^r v\|_{\chi^{(\alpha+r,\beta+r)},\Lambda}. \tag{2.9}$$

(iii) Let $-1 < \alpha, \beta \leq 0$ or $0 < \alpha, \beta \leq 1$. If $v \in C(\bar{\Lambda}), \partial_x^r v \in L^2_{\chi^{(\alpha+r-1,\beta+r-1)}}(\Lambda)$ and integer $r \geq 1$, then

$$\|\mathcal{I}_{L,N,\alpha,\beta,\Lambda}v - v\|_{\chi^{(\alpha,\beta)},\Lambda} \leqslant cN^{-r}\|\partial_x^r v\|_{\chi^{(\alpha+r-1,\beta+r-1)},\Lambda}. \tag{2.10}$$

The result (2.8) improves one of the results in (13.12) of [4], where

$$\|\partial_x^\mu(\mathcal{I}_{G,N,0,0,\Lambda}v - v)\|_\Lambda \leqslant cN^{\mu-r}, \|v\|_{H^r(\Lambda)},\ \mu = 0, 1.$$

But it does not imply the result (13.13) of [4], where

$$\|\mathcal{I}_{G,N,0,0,\Lambda}v - v\|_\Lambda \leqslant cN^{-r}\ln N\|v\|_{D^{\frac{r}{2}}(\Lambda)}.$$

In fact, both the semi-norm $\|\partial_x^r v\|_{\chi^{(r,r)},\Lambda}$ and the norm $\|v\|_{D^{\frac{r}{2}}(\Lambda)}$ are bounded above by the norm $\|v\|_{H^r(\Lambda)}$. However, so far, it has not been clear if

$$\|\partial_x^r v\|_{\chi^{(r,r)},\Lambda} \leq c\|v\|_{D^{\frac{r}{2}}(\Lambda)}.$$

In pseudospectral method, we need more precise results on the Legendre-Gauss-Lobatto interpolation which was considered by Bernardi and Maday [4]. We now derive some sharper results on such interpolation, and establish the related Bernstein-Jackson type inequalities.

Let

$$J_l^{(-1,-1)}(x) = (1-x^2)J_l^{(1,1)}(x), \quad l = 0, 1, \dots.$$

They form a complete $L^2_{\chi^{(-1,-1)}}(\Lambda)$-orthogonal system, see [16]. Further, we set

$$Q_N(\Lambda) = \mathrm{span}\{J_l^{(-1,-1)}(x),\, 0 \leqslant l \leqslant N-2\}.$$

The orthogonal projection $P_{N,-1,-1,\Lambda}v : L^2_{\chi^{(-1,-1)}}(\Lambda) \to Q_N(\Lambda)$ is defined by

$$(P_{N,-1,-1,\Lambda}v - v, \phi)_{\chi^{(-1,-1)},\Lambda} = 0, \qquad \forall \phi \in Q_N(\Lambda).$$

According to (1.8) of [16], for integer $r \geq 1$,

$$\|\partial_x^\mu (P_{N,-1,-1,\Lambda}v - v)\|_{\chi^{(\mu-1,\mu-1)},\Lambda} \leqslant cN^{\mu-r}\|\partial_x^r v\|_{\chi^{(r-1,r-1)},\Lambda}, \quad \mu = 0, 1. \tag{2.11}$$

We now estimate $\|\partial_x(\mathcal{I}_{L,N,0,0,\Lambda}v - v)\|_\Lambda$. In fact, due to (4.27) of [19], for any $v \in H_0^1(\Lambda) \cap L^2_{\chi^{(-1,-1)}}(\Lambda)$,

$$\|\mathcal{I}_{L,N,0,0,\Lambda}v\|_{\chi^{(-1,-1)},\Lambda} \leqslant c(\|v\|_{\chi^{(-1,-1)},\Lambda} + N^{-1}\|\partial_x v\|_\Lambda). \tag{2.12}$$

On the other hand, by virtue of (3.4) of [19], for any $\phi \in \mathcal{P}_N^0(\Lambda)$,

$$\|\partial_x \phi\|_\Lambda \leqslant cN\|\phi\|_{\chi^{(-1,-1)},\Lambda}. \tag{2.13}$$

Obviously,

$$\begin{aligned}&\|\partial_x(\mathcal{I}_{L,N,0,0,\Lambda}v - v)\|_\Lambda \\ &\leqslant \|\partial_x(P_{N,-1,-1,\Lambda}v - v)\|_\Lambda + \|\partial_x(\mathcal{I}_{L,N,0,0,\Lambda}v - P_{N,-1,-1,\Lambda}v)\|_\Lambda.\end{aligned} \tag{2.14}$$

Moreover, using (2.13), (2.12) and (2.11) successively, we deduce that if $v \in H_0^1(\Omega), \partial_x^r v \in L^2_{\chi^{(r-1,r-1)}}(\Lambda)$ and integer $r \geq 1$, then

$$\begin{aligned}&\|\partial_x(\mathcal{I}_{L,N,0,0,\Lambda}v - P_{N,-1,-1}v)\|_\Lambda = \|\partial_x \mathcal{I}_{L,N,0,0,\Lambda}(v - P_{N,-1,-1,\Lambda}v)\|_\Lambda \\ &\leqslant cN\|\mathcal{I}_{L,N,0,0,\Lambda}(v - P_{N,-1,-1,\Lambda}v)\|_{\chi^{(-1,-1)},\Lambda} \\ &\leqslant cN\|v - P_{N,-1,-1,\Lambda}v\|_{\chi^{(-1,-1)},\Lambda} + c\|\partial_x(v - P_{N,-1,-,1,\Lambda}v)\|_\Lambda \\ &\leqslant cN^{1-r}\|\partial_x^r v\|_{\chi^{(r-1,r-1)},\Lambda}.\end{aligned}$$

Substituting the above estimate and (2.11) with $\mu = 1$ into (2.14), we conclude that if $v \in H_0^1(\Omega), \partial_x^r v \in L^2_{\chi^{(r-1,r-1)}}(\Lambda)$ and integer $r \geq 1$, then

$$\|\partial_x(\mathcal{I}_{L,N,0,0,\Lambda}v - v)\|_\Lambda \leqslant cN^{1-r}\|\partial_x^r v\|_{\chi^{(r-1,r-1)},\Lambda}. \tag{2.15}$$

We next turn to estimate $\|\partial_x(\mathcal{I}_{L,N,0,0,\Lambda}v - v)\|_\Lambda$, without the condition $v(\pm 1) = 0$. We set

$$\tilde{v}(x) = v(x) - \frac{1}{2}(1+x)v(1) - \frac{1}{2}(1-x)v(-1),$$

and define the interpolation $\tilde{\mathcal{I}}_{L,N,0,0,\Lambda}v(x)$ as

$$\tilde{\mathcal{I}}_{L,N,0,0,\Lambda}v(x) = \mathcal{I}_{L,N,0,0,\Lambda}\tilde{v}(x) + \frac{1}{2}(x+1)v(1) + \frac{1}{2}(1-x)v(-1).$$

It can be checked that

$$\tilde{\mathcal{I}}_{L,N,0,0,\Lambda}v(x) = \mathcal{I}_{L,N,0,0,\Lambda}v(x) \quad \text{on } \Lambda_{L,N}^{(0,0)}$$

This implies $\tilde{\mathcal{I}}_{L,N,0,0,\Lambda}v = \mathcal{I}_{L,N,0,0,\Lambda}v$. It is easy to show that $\tilde{v}(\pm 1) = 0$ and

$$\tilde{\mathcal{I}}_{L,N,0,0,\Lambda}v(x) - v(x) = \mathcal{I}_{L,N,0,0,\Lambda}\tilde{v}(x) - \tilde{v}(x).$$

Accordingly, due to (2.15), we have that if $v \in H^1(\Omega), \partial_x^r v \in L^2_{\chi^{(r-1,r-1)}}(\Lambda)$ and integer $r \geq 2$, then

$$\begin{aligned}
&\|\partial_x(\mathcal{I}_{L,N,0,0,\Lambda}v - v)\|_\Lambda \\
&= \|\partial_x(\tilde{\mathcal{I}}_{L,N,0,0,\Lambda}v - v)\|_\Lambda = \|\partial_x(\mathcal{I}_{L,N,0,0,\Lambda}\tilde{v} - \tilde{v})\|_\Lambda \\
&\leqslant cN^{1-r}\|\partial_x^r \tilde{v}\|_{\chi^{(r-1,r-1)},\Lambda} = cN^{1-r}\|\partial_x^r v\|_{\chi^{(r-1,r-1)},\Lambda}.
\end{aligned} \tag{2.16}$$

If $r = 1$, then by the Cauchy inequality, we assert that

$$\|\partial_x \tilde{v}\|_\Lambda \leqslant \|\partial_x v\|_\Lambda + \frac{\sqrt{2}}{2}|v(1) - v(-1)| \leqslant 2\|\partial_x v\|_\Lambda.$$

This implies the validity of (2.16) for any integer $r \geq 1$.

The above result improves the corresponding result of [4], where

$$\|\partial_x^\mu(\mathcal{I}_{L,N,0,0,\Lambda}v - v)\|_\Lambda \leqslant cN^{\mu-r}\|v\|_{H^r(\Lambda)}, \quad \mu = 0, 1.$$

The improved result will be used in Section 4 of this paper.

We now derive the Bernstein-Jackson type inequalities. By (2.10) with $\alpha = \beta = 0$, (2.16) and the imbedding inequality, we verify that if $v \in C(\bar{\Lambda}), \partial_x^r v \in L^2_{\chi^{(r-1,r-1)}}(\Lambda)$ and integer $r \geq 1$, then

$$\begin{aligned}
&\|\mathcal{I}_{L,N,0,0,\Lambda}v - v\|_{C(\Lambda)} \\
&\leq c\|\mathcal{I}_{L,N,0,0,\Lambda}v - v\|_\Lambda^{\frac{1}{2}}|\mathcal{I}_{L,N,0,0,\Lambda}v - v|_{H^1(\Lambda)}^{\frac{1}{2}} \leq cN^{\frac{1}{2}-r}\|\partial_x^r v\|_{\chi^{(r-1,r-1)},\Lambda}.
\end{aligned} \tag{2.17}$$

3. Jacobi-Gauss Type Interpolation In Two Dimensions

In this section, we investigate the Jacobi-Gauss type interpolation in two dimensions, which serves as the mathematical foundation for the Jacobi pseudospectral method.

3.1. Orthogonal Jacobi approximation in two dimensions

We first recall some basic results on the Jacobi orthogonal projections in two dimensions. Let

$$\Lambda_q = \{x_q \,|\, -1 < x_q < 1\}, \quad \Omega = \Lambda_1 \times \Lambda_2, \quad x = (x_1, x_2),$$

and $\chi(x)$ be a certain weight function. We define the weighted Sobolev spaces $H^r_\chi(\Omega)$ and $H^r_{0,\chi}(\Omega)$ as usual, with the norm $||v||_{r,\chi}$. Besides,

$${}_0H^r_\chi(\Omega) = \{v \mid v \in H^r_\chi(\Omega),\ v(-1, x_2) = v(x_1, -1) = 0\}.$$

In particular, $L^2(\Omega) = H^0_\chi(\Omega)$, with the inner product $(u, v)_\chi$ and the norm $||v||_\chi$. Whenever $\chi(x) \equiv 1$, we omit the subscript χ in the notations.

For $\alpha_q, \tilde{\alpha}_q, \beta_q, \tilde{\beta}_q, \gamma_q, \delta_q > -1$, $q = 1, 2$, we set that

$$\alpha = (\alpha_1, \tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_2), \quad \beta = (\beta_1, \tilde{\beta}_1, \beta_2, \tilde{\beta}_2), \quad \gamma = (\gamma_1, \gamma_2), \quad \delta = (\delta_1, \delta_2).$$

The two-dimensional Jacobi weight functions are given by

$$\chi_1^{(\alpha,\beta)}(x) = \chi^{(\alpha_1,\beta_1)}(x_1)\chi^{(\tilde{\alpha}_2,\tilde{\beta}_2)}(x_2), \qquad \chi_2^{(\alpha,\beta)}(x) = \chi^{(\tilde{\alpha}_1,\tilde{\beta}_1)}(x_1)\chi^{(\alpha_2,\beta_2)}(x_2),$$
$$\chi^{(\gamma,\delta)}(x) = \chi^{(\gamma_1,\delta_1)}(x_1)\chi^{(\gamma_2,\delta_2)}(x_2).$$

Let $N=(N_1,N_2), N_1,N_2 \in \mathbb{N}$. $\mathcal{P}_N$ stands for the set of all polynomials of degree at most N_q with respect to the variables $x_q, q=1,2$.

The $L^2_{\chi^{(\gamma,\delta)}}(\Omega)$-orthogonal projection $P_{N,\gamma,\delta}: L^2_{\chi^{(\gamma,\delta)}}(\Omega) \to \mathcal{P}_N$ is defined by

$$(P_{N,\gamma,\delta}v - v, \phi)_{\chi^{(\gamma,\delta)}} = 0, \qquad \forall \phi \in \mathcal{P}_N.$$

We also introduce the non-isotropic space $H^{r,s}_{\chi^{(\gamma,\delta)},A}(\Omega)$, $r,s \in \mathbb{N}$, as

$$H^{r,s}_{\chi^{(\gamma,\delta)},A}(\Omega) = \big\{v \in L^2_{\chi^{(\gamma,\delta)}}(\Omega) \,|\, \partial^r_{x_1}v \in L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2; L^2_{\chi^{(\gamma_1+r,\delta_1+r)}}(\Lambda_1)$$
$$\text{and } \partial^s_{x_2}v \in L^2_{\chi^{(\gamma_2+s,\delta_2+s)}}(\Lambda_2; L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1))\big\}.$$

Its semi-norm $|v|_{r,s;\chi^{(\gamma,\delta)},A}$ is given by

$$\left(\|\partial^r_{x_1}v\|^2_{L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+r,\delta_1+r)}}(\Lambda_1))} + \|\partial^s_{x_2}v\|^2_{L^2_{\chi^{(\gamma_2+s,\delta_2+s)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1))}\right)^{\frac{1}{2}}.$$

The corresponding norm is

$$\|v\|_{r,s;\chi^{(\gamma,\delta)},A} = \left(\|v\|^2_{\chi^{(\gamma,\delta)}} + |v|^2_{r,s;\chi^{(\gamma,\delta)},A}\right)^{\frac{1}{2}}.$$

We have from Theorem 3.1 of [21] that for any $v \in H^{r,s}_{\chi^{(\gamma,\delta)},A}(\Omega)$ and integers $r,s \geqslant 0$,

$$\|P_{N,\gamma,\delta}v - v\|_{\chi^{(\gamma,\delta)}} \leqslant c(N_1^{-r} + N_2^{-s})|v|_{r,s;\chi^{(\gamma,\delta)},A}. \tag{3.1}$$

In the forthcoming discussions, we need another orthogonal projection. To do this, let $H^0_{\alpha,\beta,\gamma,\delta}(\Omega) = L^2_{\chi^{(\gamma,\delta)}}(\Omega)$, and

$$H^1_{\alpha,\beta,\gamma,\delta}(\Omega) = \{v \mid \|v\|_{1,\alpha,\beta,\gamma,\delta} < \infty\},$$

equipped with the semi-norm and norm as

$$|v|_{1,\alpha,\beta,\gamma,\delta} = \left(\|\partial_{x_1}v\|^2_{\chi_1^{(\alpha,\beta)}} + \|\partial_{x_2}v\|^2_{\chi_2^{(\alpha,\beta)}}\right)^{\frac{1}{2}}, \quad \|v\|_{1,\alpha,\beta,\gamma,\delta} = \left(|v|^2_{1,\alpha,\beta,\gamma,\delta} + \|v\|^2_{\chi^{(\gamma,\delta)}}\right)^{\frac{1}{2}}.$$

For $0<\mu<1$, the space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Omega)$ is defined by the space interpolation, with the norm $\|v\|_{\mu,\alpha,\beta,\gamma,\delta}$.We define the bilinear form as

$$a_{\alpha,\beta,\gamma,\delta}(u,v) = (\partial_{x_1}u, \partial_{x_1}v)_{\chi_1^{(\alpha,\beta)}} + (\partial_{x_2}u, \partial_{x_2}v)_{\chi_2^{(\alpha,\beta)}} + (u,v)_{\chi^{(\gamma,\delta)}}, \quad \forall u,v \in H^1_{\alpha,\beta,\gamma,\delta}(\Omega).$$

The orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta}: H^1_{\alpha,\beta,\gamma,\delta}(\Omega) \to \mathcal{P}_N$ is defined by

$$a_{\alpha,\beta,\gamma,\delta}(P^1_{N,\alpha,\beta,\gamma,\delta}v - v, \phi) = 0, \qquad \forall \phi \in \mathcal{P}_N.$$

To derive the approximation results, we introduce the non-isotropic space $Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}(\Omega)$, with integers $r,s \geqslant 1$ and $\sigma,\lambda = 1,2$. For $r=\sigma=s=\lambda=1$,

$$Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}(\Omega) = H^1_{\alpha,\beta,\gamma,\delta}(\Omega),$$

For $r+\sigma \geqslant 3$ and $s+\lambda \geqslant 3$,

$$Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}(\Omega) = \{v \mid \|v\|_{Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}} < \infty\},$$

with the following semi-norm and norm,

$$\begin{aligned}
|v|_{Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}} = \Bigg(& ||\partial^r_{x_1} v||_{L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2; L^2_{\chi^{(\alpha_1+r-1,\beta_1+r-1)}}(\Lambda_1))} \\
& + ||\partial^s_{x_2} v||_{L^2_{\chi^{(\alpha_2+s-1,\beta_2+s-1)}}(\Lambda_2; L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1))} \\
& + ||\partial^{r+\sigma-2}_{x_1}\partial_{x_2} v||_{L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2; L^2_{\chi^{(\alpha_1+r+\sigma-3,\beta_1+r+\sigma-3)}}(\Lambda_1))} \\
& + ||\partial_{x_1}\partial^{s+\lambda-2}_{x_2} v||_{L^2_{\chi^{(\alpha_2+s+\lambda-3,\beta_2+s+\lambda-3)}}(\Lambda_2; L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1))}\Bigg)^{\frac{1}{2}}, \\
\|v\|_{Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}} = \Bigg(& |v|^2_{Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}} + ||v||^2_{1,\alpha,\beta,\gamma,\delta}\Bigg)^{\frac{1}{2}}.
\end{aligned}$$

For any $r,s>1$, the space $Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}(\Omega)$ is defined by space interpolation.

We know from Theorem 3.2 of [21] that if $\gamma_q \leqslant \tilde{\alpha}_q$, $\delta_q \leqslant \tilde{\beta}_q$, $q=1,2$, and

$$\alpha_1 \leqslant \gamma_1+\sigma,\ \beta_1 \leqslant \delta_1+\sigma,\ \alpha_2 \leqslant \gamma_2+\lambda,\ \beta_2 \leqslant \delta_2+\lambda,\ \sigma,\lambda = 1 \text{ or } 2, \tag{3.2}$$

then for any $v \in Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}(\Omega)$ and $r,s \geqslant 1$,

$$\|P^1_{N,\alpha,\beta,\gamma,\delta} v - v\|_{1,\alpha,\beta,\gamma,\delta} \leqslant c(N_1^{1-r} + N_2^{1-s})\|v\|_{Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}}. \tag{3.3}$$

Taking the boundary conditions into consideration, we have to deal with functions vanishing on the whole boundary or some parts of the boundary. So we need to consider other orthogonal projections. Denote by V_1, V_2, V_3 and V_4 the corners $(-1,-1), (1,-1), (1,1)$ and $(-1,1)$ of the square Ω, respectively. Γ_j $(j=1,2,3,4)$ stand for the edges with the endpoints V_{j-1} and V_j $(V_0 = V_4)$, respectively. Let $\Gamma \subseteq \partial\Omega$,

$$\begin{aligned}
&H^{1,\Gamma}_{0,\alpha,\beta,\gamma,\delta}(\Omega) = \{v \mid v \in H^1_{\alpha,\beta,\gamma,\delta}(\Omega),\ v(x)=0 \text{ on } \Gamma\}, \\
&\mathcal{P}^{\Gamma,0}_N = H^{1,\Gamma}_{0,\alpha,\beta,\gamma,\delta}(\Omega) \cap \mathcal{P}_N.
\end{aligned}$$

The orthogonal projection $P^{1,\Gamma}_{\alpha,\beta,\gamma,\delta} : H^{1,\Gamma}_{0,\alpha,\beta,\gamma,\delta}(\Omega) \to \mathcal{P}^{\Gamma,0}_N$ is defined by

$$a_{\alpha,\beta,\gamma,\delta}(P^{1,\Gamma}_{\alpha,\beta,\gamma,\delta} v - v, \phi) = 0, \qquad \forall \phi \in \mathcal{P}^{\Gamma,0}_N.$$

In particular, for $\Gamma = \partial\Omega$, we denote $H^{1,\Gamma}_{0,\alpha,\beta,\gamma,\delta}(\Omega)$ and $\mathcal{P}^{\Gamma,0}_N$ by $H^1_{0,\alpha,\beta,\gamma,\delta}(\Omega)$ and $\mathcal{P}^0_N$, respectively.

We can estimate the difference between v and $P^{1,\Gamma}_{\alpha,\beta,\gamma,\delta} v$. For instance, let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_4$. Then we have from Theorem 3.4 of [21] that if $-1 < \alpha_2, \beta_2 < 1$, $\gamma_q \leqslant \tilde{\alpha}_q$, $\delta_q \leqslant \tilde{\beta}_q$, $q=1,2$, and one of the following conditions holds:

(i) $0 < \alpha_1 < 1,\ \alpha_1 \leq \gamma_1 + 1,\ \beta_1 < 1,\ \beta_1 \leq \delta_1 + 2,$

(ii) $\alpha_1 \leq \gamma_1 + 2,\ \beta_1 \leq 0,\ \delta_1 \geq 0,$

then for any $v \in H^{1,\Gamma}_{0,\alpha,\beta,\gamma,\delta}(\Omega) \cap Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}(\Omega)$ and $r,s \geq 1$,

$$||P^{1,\Gamma}_{N,\alpha,\beta,\gamma,\delta} v - v||_{1,\alpha,\beta,\gamma,\delta} \leq c(N_1^{1-r} + N_2^{1-s})||v||_{Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}}, \qquad \sigma=\lambda=2. \tag{3.4}$$

Moreover, if (i) holds and $\beta_1 \leq \delta_1+1$, then (3.4) is also valid with $\sigma = 1$; while if $-1 \leq \alpha_2, \beta_2 \leq 0$ or $0 < \alpha_2, \beta_2 < 1$, then (3.4) also holds with $\lambda = 1$.

In this paper, we shall use another specific orthogonal projection. For this purpose, let

$$\bar{a}_{\alpha,\beta}(u,v) = (\partial_{x_1}u, \partial_{x_1}v)_{\chi_1^{(\alpha,\beta)}} + (\partial_{x_2}u, \partial_{x_2}v)_{\chi_2^{(\alpha,\beta)}}.$$

The orthogonal projection $\bar{P}^{1,\Gamma}_{N,\alpha,\beta} : H^{1,\Gamma}_{0,\alpha,\beta,\gamma,\delta}(\Omega) \to \mathcal{P}^{\Gamma,0}_N$ is defined by

$$\bar{a}_{\alpha,\beta}(\bar{P}^{1,\Gamma}_{N,\alpha,\beta}v - v, \phi) = 0, \qquad \forall \phi \in \mathcal{P}^{\Gamma,0}_N.$$

Let

$$|v|_{1,\alpha,\beta} = (\|\partial_{x_1}v\|^2_{\chi_1^{(\alpha,\beta)}} + \|\partial_{x_2}v\|^2_{\chi_2^{(\alpha,\beta)}})^{\frac{1}{2}},$$

If the conditions for which (3.4) holds are fulfilled, then by using the projection theorem and (3.4), we deduce that

$$\begin{aligned}
&|\bar{P}^{1,\Gamma}_{N,\alpha,\beta}v - v|_{1,\alpha,\beta} \leq |P^{1,\Gamma}_{N,\alpha,\beta,\gamma,\delta}v - v|_{1,\alpha,\beta} \\
&\leq ||P^{1,\Gamma}_{N,\alpha,\beta,\gamma,\delta}v - v||_{1,\alpha,\beta,\gamma,\delta} \\
&\leq c(N_1^{1-r} + N_2^{1-s})||v||_{Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}},
\end{aligned} \tag{3.5}$$

where the values of σ and λ are the exactly same as in (3.4).

3.2. Jacobi-Gauss type interpolation

We are now in position to study the Jacobi-Gauss type interpolation. Let $Z = (Z_1, Z_2)$, $Z_q = G,\ R,\ L,\ q = 1, 2$. We define the discrete inner product and norm in two dimensions by

$$\begin{aligned}
&(u,v)_{\chi^{(\gamma,\delta)},Z,N} = \sum_{j_1=0}^{N_1}\sum_{j_2=0}^{N_2} u\left(\zeta^{(\gamma_1,\delta_1)}_{Z_1,N_1,j_1}, \zeta^{(\gamma_2,\delta_2)}_{Z_2,N_2,j_2}\right) v\left(\zeta^{(\gamma_1,\delta_1)}_{Z_1,N_1,j_1}, \zeta^{(\gamma_2,\delta_2)}_{Z_2,N_2,j_2}\right) \omega^{(\gamma_1,\delta_1)}_{Z_1,N_1,j_1}\omega^{(\gamma_2,\delta_2)}_{Z_2,N_2,j_2}, \\
&\|v\|_{\chi^{(\gamma,\delta)},Z,N} = (v,v)^{\frac{1}{2}}_{\chi^{(\gamma,\delta)},Z,N}.
\end{aligned}$$

By (2.4), we have

$$(\phi_1\phi_2, \psi_1\psi_2)_{\chi^{(\gamma,\delta)},Z,N} = (\phi_1\phi_2, \psi_1\psi_2)_{\chi^{(\gamma,\delta)}}, \quad \forall \phi_q.\psi_q \in \mathcal{P}_{2N_q+\lambda_{Z_q}}, \tag{3.6}$$

with $Z_q = G, R, L,\ \ q = 1, 2$. Next, let

$$\Omega^{(\gamma,\delta)}_{Z,N} = \left\{(x_1,x_2)\,|\,x_1 = \zeta^{(\gamma_1,\delta_1)}_{Z_1,N_1,j_1},\ x_2 = \zeta^{(\gamma_2,\,\delta_2)}_{Z_2,N_2,j_2},\ 0 \leqslant j_q \leqslant N_q,\ q = 1, 2\right\}.$$

The Jacobi-Gauss-type interpolation $\mathcal{I}_{Z,N,\gamma,\delta}v(x) \in \mathcal{P}_N$, is determined uniquely by

$$\mathcal{I}_{Z,\,N,\,\gamma,\,\delta}v(x) = v(x), \qquad x \in \Omega^{(\gamma,\delta)}_{Z,\,N}.$$

Here, we assume that $v \in C(\Omega)$ for $Z = (G,G)$, $v \in C(\Omega \cup \{(x_1,x_2) \mid x_1 = -1 \text{ or } x_2 = -1\})$ for $Z = (R,R)$, and $v \in C(\bar{\Omega})$ for $Z = (L,L)$, etc.. For simplicity, we use the notations $\mathcal{I}_{G,N,\gamma,\delta}v$ for $Z = (G,G)$, $\mathcal{I}_{R,N,\gamma,\delta}v$ for $Z = (R,R)$, and $\mathcal{I}_{L,N,\gamma,\delta}v$ for $Z = (L,L)$, respectively. They are the standard two-dimensional Jacobi-Gauss interpolation, Jacobi-Gauss-Radau interpolation and Jacobi-Gauss-Lobatto interpolation, respectively. In the Jacobi pseudospectral method coupled with domain decomposition, we also use frequently the mixed interpolation $\mathcal{I}_{Z,N,\gamma,\delta}v$ with $Z=(R,L)$, denoted by $\mathcal{I}_{RL,N,\gamma,\delta}v$.

We now consider the stability of Jacobi-Gauss-type interpolation. Let $k_q, l_q \in \mathbb{N}, k = (k_1,k_2), l=(l_1,l_2)$ and

$$\chi^{(\gamma+k,\delta+l)}(x)=\chi^{(\gamma_1+k_1,\delta_1+l_1)}(x_1)\chi^{(\gamma_2+k_2,\delta_2+l_2)}(x_2).$$

We define $\chi^{(\gamma+k,\delta-l)}\ (x)$ and $\chi^{(\gamma-k,\delta-l)}(x)$ similarly. For simplicity of statements, we introduce the non-isotropic spaces

$$\begin{aligned}M_{G,\gamma,\delta,k,l}(\Omega)=&\Big\{v\,|\,v\in L^2_{\chi^{(\gamma+k,\delta+l)}}(\Omega),\ \partial_{x_1}v\in L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1)),\\&\quad\partial_{x_2}v\in L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1)),\ \partial_{x_1}\partial_{x_2}v\in L^2_{\chi^{(\gamma+k+1,\delta+l+1)}}(\Omega)\Big\},\\M_{R,\gamma,\delta,k,l}(\Omega)=&\Big\{v\,|\,v\in L^2_{\chi^{(\gamma+k,\delta-l)}}(\Omega),\ \partial_{x_1}v\in L^2_{\chi^{(\gamma_2+k_2,\delta_2-l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1-l_1+1)}}(\Lambda_1)),\\&\quad\partial_{x_2}v\in L^2_{\chi^{(\gamma_2+k_2+1,\delta_2-l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1-l_1)}}(\Lambda_1)),\ \partial_{x_1}\partial_{x_2}v\in L^2_{\chi^{(\gamma+k+1,\delta-l+1)}}(\Omega)\Big\},\\M_{L,\gamma,\delta,k,l}(\Omega)=&\Big\{v\,|\,v\in L^2_{\chi^{(\gamma-k,\delta-l)}}(\Omega),\ \partial_{x_1}v\in L^2_{\chi^{(\gamma_2-k_2,\delta_2-l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1-k_1+1,\delta_1-l_1+1)}}(\Lambda_1)),\\&\quad\partial_{x_2}v\in L^2_{\chi^{(\gamma_2-k_2+1,\delta_2-l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1-k_1,\delta_1-l_1)}}(\Lambda_1)),\ \partial_{x_1}\partial_{x_2}v\in L^2_{\chi^{(\gamma-k+1,\delta-l+1)}}(\Omega)\Big\},\end{aligned}$$

and

$$\begin{aligned}M_{RL,\gamma,\delta,k,l}(\Omega)=\ &\Big\{v\,|\,v\in L^2_{\chi^{(\gamma_2-k_2,\delta_2-l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1-l_1+1)}}(\Lambda_1)),\\&\partial_{x_1}v\in L^2_{\chi^{(\gamma_2-k_2,\delta_2-l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1-l_1+1)}}(\Lambda_1)),\\&\partial_{x_2}v\in L^2_{\chi^{(\gamma_2-k_2+1,\delta_2-l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1-l_1)}}(\Lambda_1)),\\&\partial_{x_1}\partial_{x_2}v\in L^2_{\chi^{(\gamma_2-k_2+1,\delta_2-l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1-l_1+1)}}(\Lambda_1))\Big\}.\end{aligned}$$

Like the spaces ${}_0H^r(\Omega)$ and $H^r_{0,\chi}(\Omega)$, we can define the spaces ${}_0M_{R,\gamma,\delta,k,l}(\Omega)$ and $M^0_{L,\gamma,\delta,k,l}(\Omega)$. In addition,

$$M^*_{RL,\gamma,\delta,k,l}(\Omega)=\{v\ |\ v(-1,x_2)=v(x_1,-1)=v(x_1,1)=0\}.$$

The stability of Jacobi-Gauss interpolation is stated below.

Theorem 3.1. *For any $v\in C(\Omega)\cap M_{G,\gamma,\delta,k,l}(\Omega)$, $k_q,l_q\in\mathbb{N}$ and $0\leqslant k_q+l_q\leqslant 1$, $q=1,2$,*

$$\begin{aligned}&\|\mathcal{I}_{G,N,\gamma,\delta}v\|_{\chi^{(\gamma+k,\delta+l)}}\\&\leqslant c\|v\|_{\chi^{(\gamma+k,\delta+l)}}+cN_1^{-1}\|\partial_{x_1}v\|_{L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1))}\\&\quad+\ cN_2^{-1}\|\partial_{x_2}v\|_{L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1))}\\&\quad+\ cN_1^{-1}N_2^{-1}\|\partial_{x_1}\partial_{x_2}v\|_{\chi^{(\gamma+k+1,\delta+l+1)}}.\end{aligned}$$

Proof. Due to $\mathcal{I}_{G,N,\gamma,\delta}v=\mathcal{I}_{G,N_2,\gamma_2,\delta_2,\Lambda_2}(\mathcal{I}_{G,N_1,\gamma_1,\delta_1,\Lambda_1}v)$, we use (2.5) to verify that

$$\begin{aligned}
&\|\mathcal{I}_{G,N,\gamma,\delta}v\|_{\chi^{(\gamma+k,\delta+l)}}\leq c\|\mathcal{I}_{G,\,N_1,\,\gamma_1,\,\delta_1,\Lambda_1}v\|_{L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1))}\\
&\quad+cN_2^{-1}\|\partial_{x_2}\mathcal{I}_{G,\,N_1,\gamma_1,\,\delta_1,\Lambda_1}v\|_{L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1))}\\
&\leqslant c\|v\|_{L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1))}\\
&\quad+cN_1^{-1}\|\partial_{x_1}v\|_{L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1))}\\
&\quad+cN_2^{-1}\|\partial_{x_2}v\|_{L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1))}\\
&\quad+cN_1^{-1}N_2^{-1}\|\partial_{x_1}\partial_{x_2}v\|_{L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1))}.
\end{aligned}$$

This leads to the desired result. ■

In the same manner as for the proof of the last theorem, we use (2.6) and (2.7) to derive the following results.

Theorem 3.2. *For any* $v\in C(\Omega\cup\{(x_1,x_2)\mid x_1=-1 \text{ or } x_2=-1\})\cap{}_0M_{R,\gamma,\delta,k,l}(\Omega)$, $k_q,l_q\in\mathbb{N}$, $0\leqslant k_q\leqslant l_q\leqslant 1$ *and* $l_q<\delta_q+1$, $q=1,2$,*we have*

$$\begin{aligned}
&\|\mathcal{I}_{R,N,\gamma,\delta}v\|_{\chi^{(\gamma+k,\delta-l)}}\\
&\leqslant c\|v\|_{\chi^{(\gamma+k,\delta-l)}}+cN_1^{-1}\|\partial_{x_1}v\|_{L^2_{\chi^{(\gamma_2+k_2,\delta_2-l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1-l_1+1)}}(\Lambda_1))}\\
&\quad+cN_2^{-1}\|\partial_{x_2}v\|_{L^2_{\chi^{(\gamma_2+k_2+1,\delta_2-l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1-l_1)}}(\Lambda_1))}\\
&\quad+cN_1^{-1}N_2^{-1}\|\partial_{x_1}\partial_{x_2}v\|_{\chi^{(\gamma+k+1,\delta-l+1)}}.
\end{aligned}$$

Theorem 3.3. *For any* $v\in C(\bar{\Omega})\cap M^0_{L,\gamma,\delta,k,l}(\Omega)$, $k_q,l_q\in\mathbb{N},0\leqslant k_q,l_q\leqslant 1$, $k_q<\gamma_q+1$ *and* $l_q<\delta_q+1$, $q=1,2$,*we have*

$$\begin{aligned}
&\|\mathcal{I}_{L,N,\gamma,\delta}v\|_{\chi^{(\gamma-k,\delta-l)}}\\
&\leqslant c\|v\|_{\chi^{(\gamma-k,\delta-l)}}+cN_1^{-1}\|\partial_{x_1}v\|_{L^2_{\chi^{(\gamma_2-k_2,\delta_2-l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1-k_1+1,\delta_1-l_1+1)}}(\Lambda_1))}\\
&\quad+cN_2^{-1}\|\partial_{x_2}v\|_{L^2_{\chi^{(\gamma_2-k_2+1,\delta_2-l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1-k_1,\delta_1-l_1)}}(\Lambda_1))}\\
&\quad+cN_1^{-1}N_2^{-1}\|\partial_{x_1}\partial_{x_2}v\|_{\chi^{(\gamma-k+1,\delta-l+1)}}.
\end{aligned}$$

Theorem 3.4. *For any* $v\in C(\Omega\cup\{(x_1,x_2)\mid x_1=-1 \text{ or } x_2=\pm1\})\cap M^*_{RL,\gamma,\delta,k,l}(\Omega)$, $k_1,l_1,k_2,l_2\in\mathbb{N}$, $0\leqslant k_1\leqslant l_1\leqslant 1,l_1<\delta_1+1$, $0\leqslant k_2,l_2\leqslant 1,k_2<\gamma_2+1$ *and* $l_2<\delta_2+1$,*,we have*

$$\begin{aligned}
&\|\mathcal{I}_{RL,N,\gamma,\delta}v\|_{L^2_{\chi^{(\gamma_2-k_2,\delta_2-l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1-l_1)}}(\Lambda_1))}\\
&\leqslant c\|v\|_{L^2_{\chi^{(\gamma_2-k_2,\delta_2-l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1-l_1+1)}}(\Lambda_1))}\\
&\quad+cN_1^{-1}\|\partial_{x_1}v\|_{L^2_{\chi^{(\gamma_2-k_2,\delta_2-l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1-l_1+1)}}(\Lambda_1))}\\
&\quad+cN_2^{-1}\|\partial_{x_2}v\|_{L^2_{\chi^{(\gamma_2-k_2+1,\delta_2-l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1-l_1)}}(\Lambda_1))}\\
&\quad+cN_1^{-1}N_2^{-1}\|\partial_{x_1}\partial_{x_2}v\|_{L^2_{\chi^{(\gamma_2-k_2+1,\delta_2-l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1-l_1+1)}}(\Lambda_1))}.
\end{aligned}$$

3.3. Error estimates of Jacobi-Gauss type interpolation

In this subsection, we present the main results on the Jacobi-Gauss type interpolation. For integers $r, s \geq 1$, we introduce the non-isotropic spaces

$$\begin{aligned} B^{r,s}_{\gamma,\delta}(\Omega) = \Big\{ v \mid v \in L^2_{\chi^{(\gamma,\delta)}}(\Omega), \partial^r_{x_1} v \in L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2; L^2_{\chi^{(\gamma_1+r,\delta_1+r)}}(\Lambda_1)), \\ \partial^s_{x_2} v \in L^2_{\chi^{(\gamma_2+s,\delta_2+s)}}(\Lambda_2; L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)), \\ \partial_{x_1}\partial^{s-1}_{x_2} v \in L^2_{\chi^{(\gamma_2+s-1,\delta_2+s-1)}}(\Lambda_2; L^2_{\chi^{(\gamma_1+1,\delta_1+1)}}(\Lambda_1)) \Big\}, \end{aligned}$$

$$\begin{aligned} C^{r,s}_{\gamma,\delta}(\Omega) = \Big\{ v \mid v \in L^2_{\chi^{(\gamma,\delta)}}(\Omega), \partial^r_{x_1} v \in L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2; L^2_{\chi^{(\gamma_1+r-1,\delta_1+r-1)}}(\Lambda_1)), \\ \partial^s_{x_2} v \in L^2_{\chi^{(\gamma_2+s-1,\delta_2+s-1)}}(\Lambda_2; L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)), \\ \partial_{x_1}\partial^{s-1}_{x_2} v \in L^2_{\chi^{(\gamma_2+s-2,\delta_2+s-2)}}(\Lambda_2; L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)) \Big\} \end{aligned}$$

and

$$\begin{aligned} D^{r,s}_{\gamma,\delta}(\Omega) = \Big\{ v \mid v \in L^2_{\chi^{(\gamma,\delta)}}(\Omega), \partial^r_{x_1} v \in L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2; L^2_{\chi^{(\gamma_1+r,\delta_1+r)}}(\Lambda_1)), \\ \partial^s_{x_2} v \in L^2_{\chi^{(\gamma_2+s-1,\delta_2+s-1)}}(\Lambda_2; L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)), \\ \partial_{x_1}\partial^{s-1}_{x_2} v \in L^2_{\chi^{(\gamma_2+s-2,\delta_2+s-2)}}(\Lambda_2; L^2_{\chi^{(\gamma_1+1,\delta_1+1)}}(\Lambda_1)) \Big\}. \end{aligned}$$

Their semi-norms and norms are defined in the usual way. For instance,

$$\begin{aligned} |v|_{B^{r,s}_{\gamma,\delta}} = \Big(||\partial^r_{x_1} v||^2_{L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2; L^2_{\chi^{(\gamma_1+r,\delta_1+r)}}(\Lambda_1))} + ||\partial^s_{x_2} v||^2_{L^2_{\chi^{(\gamma_2+s,\delta_2+s)}}(\Lambda_2; L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1))} \\ + ||\partial_{x_1}\partial^{s-1}_{x_2} v||^2_{L^2_{\chi^{(\gamma_2+s-1,\delta_2+s-1)}}(\Lambda_2; L^2_{\chi^{(\gamma_1+1,\delta_1+1)}}(\Lambda_1))} \Big)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} |v|_{D^{r,s}_{\gamma,\delta}} = \Big(||\partial^r_{x_1} v||^2_{L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2; L^2_{\chi^{(\gamma_1+r,\delta_1+r)}}(\Lambda_1))} + ||\partial^s_{x_2} v||^2_{L^2_{\chi^{(\gamma_2+s-1,\delta_2+s-1)}}(\Lambda_2; L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1))} \\ + ||\partial_{x_1}\partial^{s-1}_{x_2} v||^2_{L^2_{\chi^{(\gamma_2+s-2,\delta_2+s-2)}}(\Lambda_2; L^2_{\chi^{(\gamma_1+1,\delta_1+1)}}(\Lambda_1))} \Big)^{\frac{1}{2}}. \end{aligned}$$

For any real $r, s > 0$, the spaces $B^{r,s}_{\gamma,\delta}(\Omega)$, $C^{r,s}_{\gamma,\delta}(\Omega)$ and $D^{r,s}_{\gamma,\delta}(\Omega)$ are defined by space interpolation.

Theorem 3.5. *For any $v \in C(\Omega) \cap B^{r,s}_{\gamma,\delta}(\Omega)$ and integers $r, s \geqslant 1$,*

$$\|\mathcal{I}_{G,N,\gamma,\delta} v - v\|_{\chi^{(\gamma,\delta)}} \leqslant c(N_1^{-r} + N_2^{-s} + N_1^{-1} N_2^{1-s}) |v|_{B^{r,s}_{\gamma,\delta}}.$$

Proof. We have that

$$||\mathcal{I}_{G,N,\gamma,\delta} v - v||_{\chi^{(\gamma,\delta)}} \leq W_1 + W_2,$$

where

$$W_1 = \|\mathcal{I}_{G_1,N_1,\gamma_1,\delta_1,\Lambda_1} v - v\|_{\chi^{(\gamma,\delta)}}, \quad W_2 = \|\mathcal{I}_{G_1,N_1,\gamma_1,\delta_1,\Lambda_1}(\mathcal{I}_{G_2,N_2,\gamma_2,\delta_2,\Lambda_2} v - v)\|_{\chi^{(\gamma,\delta)}}.$$

By virtue of (2.8),

$$W_1 \le cN_1^{-r}||\partial_{x_1}^r v||_{L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+r,\delta_1+r)}}(\Lambda_1))}.$$

Using (2.5) and (2.8) yields that

$$\begin{aligned}W_2 &\le c||\mathcal{I}_{G_2,N_2,\gamma_2,\delta_2,\Lambda_2}v - v||_{L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1))} \\ &\quad + cN_1^{-1}||\mathcal{I}_{G_2,N_2,\gamma_2,\delta_2,\Lambda_2}\partial_{x_1}v - \partial_{x_1}v||_{L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+1,\delta_1+1)}}(\Lambda_1))} \\ &\le cN_2^{-s}||\partial_{x_2}^s v||_{L^2_{\chi^{(\gamma_2+s,\delta_2+s)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1))} \\ &\quad + cN_1^{-1}N_2^{1-s}||\partial_{x_1}\partial_{x_2}^{s-1}v||_{L^2_{\chi^{(\gamma_2+s-1,\delta_2+s-1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+1,\delta_1+1)}}(\Lambda_1))}.\end{aligned}$$

Then the desired result follows from the combination of previous two estimates. ■

By the same argument as in the proof of Theorem 3.5, we can use (2.6), (2.7), (2.9) and (2.10) to derive the following results.

Theorem 3.6. *For any* $v \in C(\Omega \cup \{(x_1,x_2) \mid x_1 = -1 \text{ or } x_2 = -1\}) \cap B^{r,s}_{\gamma,\delta}(\Omega)$ *and* $r,s \geqslant 1$,*we have*

$$\|\mathcal{I}_{R,N,\gamma,\delta}v - v\|_{\chi^{(\gamma,\delta)}} \leqslant c(N_1^{-r} + N_2^{-s} + N_1^{-1}N_2^{1-s})|v|_{B^{r,s}_{\gamma,\delta}}.$$

Theorem 3.7. *Let* $-1 < \gamma_q, \delta_q \le 0$ *or* $0 < \gamma_q, \delta_q \le 1, q = 1,2$. *Then for any* $v \in C(\bar{\Omega}) \cap C^{r,s}_{\gamma,\delta}(\Omega)$ *and* $r,s \geqslant 1$,*we have*

$$\|\mathcal{I}_{L,N,\gamma,\delta}v - v\|_{\chi^{(\gamma,\delta)}} \leqslant c(N_1^{-r} + N_2^{-s} + N_1^{-1}N_2^{1-s})|v|_{C^{r,s}_{\gamma,\delta}}.$$

Theorem 3.8. *Let* $-1 < \gamma_2, \delta_2 \le 0$ *or* $0 < \gamma_2, \delta_2 \le 1$. *Then for any* $v \in C(\Omega \cup \{(x_1,x_2) \mid x_1 = -1 \text{ or } x_2 = \pm 1\}) \cap D^{r,s}_{\gamma,\delta}(\Omega)$ *and* $r,s \geqslant 1$,*we have*

$$\|\mathcal{I}_{RL,N,\gamma,\delta}v - v\|_{\chi^{(\gamma,\delta)}} \leqslant c(N_1^{-r} + N_2^{-s} + N_1^{-1}N_2^{1-s})|v|_{D^{r,s}_{\gamma,\delta}}.$$

Remark 3.1. We may derive other estimates. For instance,

$$\begin{aligned}&\|\mathcal{I}_{G,N,\gamma,\delta}v - v\|_{\chi^{(\gamma,\delta)}} \\ &\leqslant c(N_1^{-r} + N_2^{-s} + N_1^{1-r}N_2^{-1})\Big(||\partial_{x_1}^r v||^2_{L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+r,\delta_1+r)}}(\Lambda_1))} \\ &\quad + ||\partial_{x_2}^s v||^2_{L^2_{\chi^{(\gamma_2+s,\delta_2+s)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1))} \\ &\quad + ||\partial_{x_1}^{r-1}\partial_{x_2}v||^2_{L^2_{\chi^{(\gamma_2+1,\delta_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+r-1,\delta_1+r-1)}}(\Lambda_1))}\Big)^{\frac{1}{2}}.\end{aligned}$$

3.4. Some results on Legendre-Gauss-Lobatto interpolation

We now focus on the two-dimensional Legendre-Gauss-Lobatto interpolation and the related Bernstein-Jackson type inequalities, which will be used in the sequel.

For $v \in C(\bar{\Omega})$, we define the Legendre-Gauss-Lobatto interpolation as

$$\mathcal{I}_{L,N,0,0}v = \mathcal{I}_{L,N_1,0,0,\Lambda_1}(\mathcal{I}_{L,N_2,0,0,\Lambda_2}v).$$

Clearly,

$$||\mathcal{I}_{L,N,0,0}v - v|| \le W_1 + W_2,$$

where

$$W_1 = ||\mathcal{I}_{L,N_1,0,0,\Lambda_1} v - v||, \qquad W_2 = ||\mathcal{I}_{L,N_1,0,0,\Lambda_1}(\mathcal{I}_{L,N_2,0,0,\Lambda_2} v - v)||.$$

Due to (2.10) with $\alpha = \beta = 0$,

$$W_1 \le cN_1^{-r} \|\partial_{x_1}^r v\|_{L^2(\Lambda_2; L^2_{\chi^{(r-1,r-1)}}(\Lambda_1))}.$$

Also, we have from (2.10) with $r = 1$ that

$$||\mathcal{I}_{L,N_1,0,0,\Lambda_1} v||_{\Lambda_1} \le ||v||_{\Lambda_1} + cN_1^{-1} ||\partial_{x_1} v||_{\Lambda_1}.$$

This with (2.16) leads to that

$$\begin{aligned} W_2 &\le ||\mathcal{I}_{L,N_2,0,0,\Lambda_2} v - v|| + cN_1^{-1} ||\partial_{x_1}(\mathcal{I}_{I,N_2,0,0,\Lambda_2} v - v)|| \\ &\le cN_2^{-s} ||\partial_{x_2}^s v||_{L^2_{\chi^{(s-1,s-1)}}(\Lambda_2; L^2(\Lambda_1))} + cN_1^{-1} N_2^{1-s} ||\partial_{x_1} \partial_{x_2}^{s-1} v||_{L^2_{\chi^{(s-2,s-2)}}(\Lambda_2; L^2(\Lambda_1))}. \end{aligned}$$

Therefore

$$||\mathcal{I}_{L,N,0,0} v - v|| \le c(N_1^{-r} + N_2^{-s} + cN_1^{-1} N_2^{1-s}) |v|_{E^{r,s}} \tag{3.7}$$

where

$$\begin{aligned} |v|_{E^{r,s}} = \Big(& \|\partial_{x_1}^r v\|^2_{L^2(\Lambda_2; L^2_{\chi^{(r-1,r-1)}}(\Lambda_1))} + ||\partial_{x_2}^s v||^2_{L^2_{\chi^{(s-1,s-1)}}(\Lambda_2; L^2(\Lambda_1))} \\ & + ||\partial_{x_1} \partial_{x_2}^{s-1} v||^2_{L^2_{\chi^{(s-2,s-2)}}(\Lambda_2; L^2(\Lambda_1))} + ||\partial_{x_1}^{r-1} \partial_{x_2} v||^2_{L^2(\Lambda_2; L^2_{\chi^{(r-2,r-2)}}(\Lambda_1))} \Big)^{\frac{1}{2}}. \end{aligned}$$

Next, we estimate $||\partial_{x_1}(\mathcal{I}_{L,N,0,0} v - v)||$. We have that

$$||\partial_{x_1}(\mathcal{I}_{L,N,0,0} v - v)|| \le W_3 + W_4,$$

where

$$W_3 = ||\partial_{x_1}(\mathcal{I}_{L,N_1,0,0,\Lambda_1} v - v)||, \qquad W_4 = ||\partial_{x_1} \mathcal{I}_{L,N_1,0,0,\Lambda_1}(\mathcal{I}_{L,N_2,0,0,\Lambda_2} v - v)||.$$

With the aid of (2.16), we obtain that

$$W_3 \le cN_1^{1-r} \|\partial_{x_1}^r v\|_{L^2(\Lambda_2; L^2_{\chi^{(r-1,r-1)}}(\Lambda_1))}.$$

Using (2.10) and (2.16) with $r = 1$, we derive that

$$\begin{aligned} W_4 &\le c||\partial_{x_1}(\mathcal{I}_{L,N_2,0,0,\Lambda_2} v - v)|| \\ &\le cN_2^{1-s} ||\partial_{x_1} \partial_{x_2}^{s-1} v||_{L^2_{\chi^{(s-2,s-2)}}(\Lambda_2; L^2(\Lambda_1))}. \end{aligned}$$

We can estimate $||\partial_{x_2}(\mathcal{I}_{L,N,0,0} v - v)||$ similarly. Finally, the previous statements yeild

$$|\mathcal{I}_{L,N,0,0} v - v|_{H^1(\Omega)} \le c(N_1^{1-r} + N_2^{1-s}) |v|_{E^{r,s}}. \tag{3.8}$$

In the end of this section, we consider the Bernstein-Jackson type inequalities. Obviously,

$$||\mathcal{I}_{L,N,0,0} v - v||_{C(\Omega)} \le W_5 + W_6,$$

where

$$W_5 = ||\mathcal{I}_{L,N_1,0,0,\Lambda_1} v - v||_{C(\Omega)}, \qquad W_6 = ||\mathcal{I}_{L,N_1,0,0,\Lambda_1}(\mathcal{I}_{L,N_2,0,0,\Lambda_2} v - v)||_{C(\Omega)}.$$

Further, by (2.17) and the imbedding inequality, we have that for $0 < \eta < \frac{1}{2}$,

$$\begin{aligned} W_5 &\le cN_1^{\frac{1}{2}-r}\|\partial_{x_1}^r v\|_{C(\Lambda_2;L^2_{\chi^{(r-1,r-1)}}(\Lambda_1))} \\ &\le cN_1^{\frac{1}{2}-r}\|\partial_{x_1}^r v\|_{H^{\frac{1}{2}+\eta}(\Lambda_2;L^2_{\chi^{(r-1,r-1)}}(\Lambda_1))}. \end{aligned}$$

Next, using (2.17) with $r = 1$ and $r = s-1$ successively, along with the imbedding inequality, gives that

$$\begin{aligned} W_6 &\le ||\mathcal{I}_{L,N_2,0,0,\Lambda_2}v - v||_{C(\Omega)} + cN_1^{-\frac{1}{2}}||\partial_{x_1}(\mathcal{I}_{L,N_2,0,0,\Lambda_2}v - v)||_{C(\Lambda_2;L^2(\Lambda_1))} \\ &\le cN_2^{\frac{1}{2}-s}\|\partial_{x_2}^s v\|_{L^2_{\chi^{(s-1,s-1)}}(\Lambda_2;C(\Lambda_1))} + cN_1^{-\frac{1}{2}}N_2^{\frac{3}{2}-s}||\partial_{x_1}\partial_{x_2}^{s-1}v||_{L^2_{\chi^{(s-2,s-2)}}(\Lambda_2;L^2(\Lambda_1))} \\ &\le cN_2^{\frac{1}{2}-s}\|\partial_{x_2}^s v\|_{L^2_{\chi^{(s-1,s-1)}}(\Lambda_2;H^{\frac{1}{2}+\eta}(\Lambda_1))} + cN_1^{-\frac{1}{2}}N_2^{\frac{3}{2}-s}||\partial_{x_1}\partial_{x_2}^{s-1}v||_{L^2_{\chi^{(s-2,s-2)}}(\Lambda_2;L^2(\Lambda_1))}. \end{aligned}$$

Consequently, for $r, s \ge 1$ and $0 < \eta < \frac{1}{2}$,

$$||\mathcal{I}_{L,N,0,0,0}v - v||_{C(\Omega)} \le c\left(N_1^{\frac{1}{2}-r} + N_2^{\frac{1}{2}-s} + N_1^{-\frac{1}{2}}N_2^{\frac{3}{2}-s}\right)|v|_{F_\eta^{r,s}} \tag{3.9}$$

where

$$\begin{aligned} |v|_{F_\eta^{r,s}} =&\Bigg(\|\partial_{x_1}^r v\|^2_{H^{\frac{1}{2}+\eta}(\Lambda_2;L^2_{\chi^{(r-1,r-1)}}(\Lambda_1))} + \|\partial_{x_2}^s v\|^2_{L^2_{\chi^{(s-1,s-1)}}(\Lambda_2;H^{\frac{1}{2}+\eta}(\Lambda_1))} \\ &+ ||\partial_{x_1}\partial_{x_2}^{s-1}v||_{L^2_{\chi^{(s-2,s-2)}}(\Lambda_2;L^2(\Lambda_1))}\Bigg)^{\frac{1}{2}}. \end{aligned}$$

Remark 3.2. In the forthcoming discussions, we denote by $F_\eta^{r,s}(\Omega)$ the space with the norm

$$||v||_{F_\eta^{r,s}} = (||v||^2_{C(\Omega)} + |v|^2_{F_\eta^{r,s}})^{\frac{1}{2}}.$$

Remark 3.3. The estimate (3.9) is still valid, if $|v|_{F_\eta^{r,s}}$ is replaced by

$$\begin{aligned} |v|_{F_*^{r,s}} =&\Bigg(\|\partial_{x_1}^r v\|^2_{C(\Lambda_2;L^2_{\chi^{(r-1,r-1)}}(\Lambda_1))} + \|\partial_{x_2}^s v\|^2_{L^2_{\chi^{(s-1,s-1)}}(\Lambda_2;C(\Lambda_1))} \\ &+ ||\partial_{x_1}\partial_{x_2}^{s-1}v||_{L^2_{\chi^{(s-2,s-2)}}(\Lambda_2;L^2(\Lambda_1))}\Bigg)^{\frac{1}{2}}. \end{aligned}$$

4. Jacobi Pseudospectral Method for Singular Problems

We consider the model problem

$$-\partial_{x_1}(a_1(x)\partial_{x_1}U(x)) - \partial_{x_2}(a_2(x)\partial_{x_2}U(x)) + a_0(x)U(x) = f(x), \qquad x \in \Omega. \tag{4.1}$$

Without lose of generality, we suppose that

$$a_0(x) = \tilde{a}_0(x)\chi^{(\gamma,\delta)}(x), \quad a_1(x) = \tilde{a}_1(x)\chi_1^{(\alpha,\beta)}(x), \quad a_2(x) = \tilde{a}_2(x)\chi_2^{(\alpha,\beta)}(x), \tag{4.2}$$

where

$$\begin{aligned} &\tilde{a}_q(x) \in F_\eta^{s_q,s_q'}(\Omega), \quad s_q, s_q' > 1, 0 < \eta < \frac{1}{2}; \\ &\tilde{a}_q(x) \geqslant \tilde{a}_{\min}^{(q)} > 0, \quad \forall x \in \bar{\Omega}, \quad q = 0,1,2. \end{aligned} \tag{4.3}$$

We look for the solution of (4.1) such that

$$\lim_{x\to\Gamma_1\cup\Gamma_3} a_1(x)U(x)\partial_{x_1}U(x) = \lim_{x\to\Gamma_2\cup\Gamma_4} a_2(x)U(x)\partial_{x_2}U(x) = 0.$$

To do this, let

$$\begin{aligned} A_{\alpha,\beta,\gamma,\delta}(u,v) =& (\tilde{a}_1\partial_{x_1}u,\partial_{x_1}v)_{\chi_1^{(\alpha,\beta)}} + (\tilde{a}_2\partial_{x_2}u,\partial_{x_2}v)_{\chi_2^{(\alpha,\beta)}} \\ &+ (\tilde{a}_0 u, v)_{\chi^{(\gamma,\delta)}}, \quad \forall u,v\in H^1_{\alpha,\beta,\gamma,\delta}(\Omega). \end{aligned}$$

A weak formulation of (4.1) is to find $U\in H^1_{\alpha,\beta,\gamma,\delta}(\Omega)$ such that

$$A_{\alpha,\beta,\gamma,\delta}(U,v) = (f,v)_{L^2(\Omega)}, \quad \forall v\in H^1_{\alpha,\beta,\gamma,\delta}(\Omega). \tag{4.4}$$

By Remark 3.2, we verify that for any $u,v\in H^1_{\alpha,\beta,\gamma,\delta}(\Omega)$,

$$\begin{aligned} |A_{\alpha,\beta,\gamma,\delta}(u,v)| &\leqslant (\|\tilde{a}_1\|_{L^\infty(\Omega)} + \|\tilde{a}_2\|_{L^\infty(\Omega)} + \|\tilde{a}_0\|_{L^\infty(\Omega)})\|u\|_{1,\alpha,\beta,\gamma,\delta}\|v\|_{1,\alpha,\beta,\gamma,\delta} \\ &\leqslant c(\|\tilde{a}_1\|_{F^{1,1}_\eta} + \|\tilde{a}_2\|_{F^{1,1}_\eta} + \|\tilde{a}_0\|_{F^{1,1}_\eta})\|u\|_{1,\alpha,\beta,\gamma,\delta}\|v\|_{1,\alpha,\beta,\gamma,\delta}, \qquad (4.5) \\ |A_{\alpha,\beta,\gamma,\delta}(u,u)| &\geqslant \min(\tilde{a}^{(0)}_{\min},\tilde{a}^{(1)}_{\min},\tilde{a}^{(2)}_{\min})\|u\|^2_{1,\alpha,\beta,\gamma,\delta}. \qquad (4.6) \end{aligned}$$

Therefor, if $f\in L^2_{\chi^{(-\gamma,-\delta)}}(\Omega)$, then by (4.3), (4.5), (4.6) and the Lax-Milgram Lemma, the problem (4.4) has a unique solution such that $\|U\|_{1,\alpha,\beta,\gamma,\delta}\leqslant c\|f\|_{\chi^{(-\gamma,-\delta)}}$.

We denote the discrete inner product and norm with $Z=(G,G)$, by $(u,v)_{\chi,G,N}$ and $||v||_{\chi,G,N}$, respectively. Let $N=(N_1,N_2)$ be any pair of positive even numbers, $\hat{a}_q(x) = I_{L,N/2,0,0}\tilde{a}_q(x)$, $q=0,1,2$, and $\hat{f}(x) = \chi^{(-\gamma,-\delta)}f(x)$.

Assume $f\in C(\Omega)$. Let

$$\hat{A}_{\alpha,\beta,\gamma,\delta,N}(u,v) = (\hat{a}_1\partial_{x_1}u,\partial_{x_1}v)_{\chi_1^{(\alpha,\beta)},G,N} + (\hat{a}_2\partial_{x_2}u,\partial_{x_2}v)_{\chi_2^{(\alpha,\beta)},G,N} + (\hat{a}_0 u,v)_{\chi^{(\gamma,\delta)},G,N}.$$

The Jacobi paeudospectral scheme for solving (4.4) is to find $u_N\in\mathcal{P}_N$ such that

$$\hat{A}_{\alpha,\beta,\gamma,\delta,N}(u_N,\phi) = (\hat{f},\phi)_{\chi^{(\gamma,\delta)},G,N}, \qquad \forall\phi\in\mathcal{P}_N. \tag{4.7}$$

We know from (3.6) that for any $\phi,\psi\in\mathcal{P}_N$,

$$\begin{aligned} |\hat{A}_{\alpha,\beta,\gamma,\delta,N}(\phi,\psi)| &\leqslant \|\hat{a}_1\|_{C(\Omega)}\|\partial_{x_1}\phi\|_{\chi_1^{(\alpha,\beta)},G,N}\|\partial_{x_1}\psi\|_{\chi_1^{(\alpha,\beta)},G,N} \\ &\quad + \|\hat{a}_2\|_{C(\Omega)}\|\partial_{x_2}\phi\|_{\chi_2^{(\alpha,\beta)},G,N}\|\partial_{x_2}\psi\|_{\chi_2^{(\alpha,\beta)},G,N} + \|\hat{a}_0\|_{C(\Omega)}\|\phi\|_{\chi^{(\gamma,\delta)},G,N}\|\psi\|_{\chi^{(\gamma,\delta)},G,N} \\ &\leq \big(\|\hat{a}_1\|_{C(\Omega)} + \|\hat{a}_2\|_{C(\Omega)} + \|\hat{a}_0\|_{C(\Omega)}\big)\|\phi\|_{1,\alpha,\beta,\gamma,\delta}\|\psi\|_{1,\alpha,\beta,\gamma,\delta}. \end{aligned}$$

Furthermore, let $N_1=\mathcal{O}(N_2)$. Thanks to (3.9) and (4.3), there exists a constant $c^*>0$, depending only on $\|\tilde{a}_q\|_{F^{1,1}_\eta}, q=0,1,2$, such that

$$\sum_{q=0}^{2}\|\hat{a}_q\|_{C(\Omega)}\leqslant c^*. \tag{4.8}$$

Consequently, for any $\phi,\psi\in\mathcal{P}_N$,

$$|\hat{A}_{\alpha,\beta,\gamma,\delta,N}(\phi,\psi)|\leqslant c^*\|\phi\|_{1,\alpha,\beta,\gamma,\delta}\|\psi\|_{1,\alpha,\beta,\gamma,\delta}. \tag{4.9}$$

On the other hand, according to (3.6), (3.9) and (4.3), we find that for any $\phi \in \mathcal{P}_N$ and suitably large mode N, the coefficients $\hat{a}_q(x), q = 0,1,2$, are uniformly bounded below by a constant $c_* > 0$, namely,

$$\hat{A}_{\alpha,\beta,\gamma,\delta,N}(\phi,\phi) \geqslant c_* \|\phi\|^2_{1,\alpha,\beta,\gamma,\delta}. \tag{4.10}$$

Hence, by the Lax-Milgram lemma, (4.7) has a unique solution such that

$$\|u_N\|_{1,\alpha,\beta,\gamma,\delta} \leqslant c\|\mathcal{I}_{G,N,\gamma,\delta}\tilde{f}\|_{\chi^{(\gamma,\delta)}}.$$

We now deal with the convergence of numerical solution u_N.

Theorem 4.1. *Let* (4.2) *and* (4.3) *hold, and* $\gamma_q \leqslant \tilde{\alpha}_q$, $\delta_q \leqslant \tilde{\beta}_q$, $q = 1,2$. *If* $U \in Y^{r_1,r_2,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}(\Omega)$, $\hat{f} \in B^{r_1',r_2'}_{\gamma,\delta}(\Omega)$ *and* $r_1, r_2, r_1', r_2' \geqslant 1$, *then*

$$\begin{aligned}\|u_N - U\|_{1,\alpha,\beta,\gamma,\delta} \leqslant & cc^*\big(N_1^{1-r_1} + N_2^{1-r_2}\big)\|U\|_{Y^{r_1,r_2,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}} + cA^s_N(U,\tilde{a})\\ & + c\big(N_1^{-r_1'} + N_2^{-r_2'} + N_1^{-1}N_2^{1-r_2'}\big)\|\hat{f}\|_{B^{r_1',r_2'}_{\gamma,\delta}},\end{aligned}$$

where

$$\begin{aligned}A^s_N(U,\tilde{a}) = & \left[\Big(\frac{N_1}{2}\Big)^{\frac12 - s_1} + \Big(\frac{N_2}{2}\Big)^{\frac12 - s_1'} + \Big(\frac{N_1}{2}\Big)^{-\frac12}\Big(\frac{N_2}{2}\Big)^{\frac32 - s_1'}\right] |\tilde{a}_1|_{F^{s_1,s_1'}_\eta} ||\partial_{x_1} v||_{\chi_1^{(\alpha,\beta)}}\\ & + \left[\Big(\frac{N_1}{2}\Big)^{\frac12 - s_2} + \Big(\frac{N_2}{2}\Big)^{\frac12 - s_2'} + \Big(\frac{N_1}{2}\Big)^{-\frac12}\Big(\frac{N_2}{2}\Big)^{\frac32 - s_2'}\right] |\tilde{a}_2|_{F^{s_2,s_2'}_\eta} ||\partial_{x_2} v||_{\chi_2^{(\alpha,\beta)}}\\ & + \left[\Big(\frac{N_1}{2}\Big)^{\frac12 - s_0} + \Big(\frac{N_2}{2}\Big)^{\frac12 - s_0'} + \Big(\frac{N_1}{2}\Big)^{-\frac12}\Big(\frac{N_2}{2}\Big)^{\frac32 - s_0'}\right] |\tilde{a}_0|_{F^{s_0,s_0'}_\eta} ||v||_{\chi_1^{(\gamma,\delta)}}.\end{aligned}$$

Proof. Let $U_N = P^1_{N,\alpha,\beta,\gamma,\delta}U$. By (4.4), (4.7) and the ellipticity (4.10),

$$\begin{aligned} c_*\|u_N - U_N\|^2_{1,\alpha,\beta,\gamma,\delta} & \leqslant \hat{A}_{\alpha,\beta,\gamma,\delta,N}(u_N - U_N, u_N - U_N)\\ & = (\hat{f}, u_N - U_N)_{\chi^{(\gamma,\delta)},G,N} - \hat{A}_{\alpha,\beta,\gamma,\delta,N}(U_N, u_N - U_N)\\ & = A_{\alpha,\beta,\gamma,\delta}(U, u_N - U_N) - \hat{A}_{\alpha,\beta,\gamma,\delta,N}(U_N, u_N - U_N)\\ & \quad + (\hat{f}, u_N - U_N)_{\chi^{(\gamma,\delta)},G,N} - (\hat{f}, u_N - U_N)_{\chi^{(\gamma,\delta)}}.\end{aligned}$$

Thus

$$\begin{aligned}\|u_N - U_N\|^2_{1,\alpha,\beta,\gamma,\delta} \leqslant & c\Big(|A_{\alpha,\beta,\gamma,\delta}(U, u_N - U_N) - \hat{A}_{\alpha,\beta,\gamma,\delta,N}(U_N, u_N - U_N)|\\ & + (|\hat{f}, u_N - U_N)_{\chi^{(\gamma,\delta)},G,N} - (\hat{f}, u_N - U_N)_{\chi^{(\gamma,\delta)}}|\Big).\end{aligned} \tag{4.11}$$

We now estimate the first term at the right side of (4.11). For simplicity, let

$$\hat{A}_{\alpha,\beta,\gamma,\delta}(u,\phi) = (\hat{a}_1\partial_{x_1}u, \partial_{x_1}\phi)_{\chi_1^{(\alpha,\beta)}} + (\hat{a}_2\partial_{x_2}u, \partial_{x_2}\phi)_{\chi_2^{(\alpha,\beta)}} + (\hat{a}_0 u, \phi)_{\chi^{(\gamma,\delta)}}.$$

Then

$$A_{\alpha,\beta,\gamma,\delta}(U,\phi) - \hat{A}_{\alpha,\beta,\gamma,\delta,N}(U_N,\phi) = G_1(\phi) + G_2(\phi), \tag{4.12}$$

where

$$G_1(\phi) = A_{\alpha,\beta,\gamma,\delta}(U,\phi) - \hat{A}_{\alpha,\beta,\gamma,\delta}(U,\phi), \quad G_2(\phi) = \hat{A}_{\alpha,\beta,\gamma,\delta}(U,\phi) - \hat{A}_{\alpha,\beta,\gamma,\delta,N}(U_N,\phi).$$

By using (3.9), we find that

$$\begin{aligned}|G_1(\phi)| &\leqslant \Big(\|\hat{a}_1-\tilde{a}_1\|_{L^\infty(\Omega)}||\partial_{x_1}U||_{\chi_1^{(\alpha,\beta)}}+\|\hat{a}_2-\tilde{a}_2\|_{L^\infty(\Omega)}||\partial_{x_2}U||_{\chi_2^{(\alpha,\beta)}}\\ &\quad+\|\hat{a}_0-\tilde{a}_0\|_{L^\infty(\Omega)}||U||_{\chi^{(\gamma,\delta)}}\Big)\|\phi\|_{1,\alpha,\beta,\gamma,\delta}\\ &\leq cA_N^s(U,\tilde{a})||\phi||_{1,\alpha,\beta,\gamma,\delta}.\end{aligned}\tag{4.13}$$

On the other hand, by using (3.6) and the definitions of $\hat{A}_{\alpha,\beta,\gamma,\delta,N}(u,\phi)$ and $\hat{a}_q(x)$ gives

$$\hat{A}_{\alpha,\beta,\gamma,\delta,N}(P^1_{N/2,\alpha,\beta,\gamma,\delta}U,\phi)=\hat{A}_{\alpha,\beta,\gamma,\delta}(P^1_{N/2,\alpha,\beta,\gamma,\delta}U,\phi),\quad\forall\phi\in\mathcal{P}_N$$

whence

$$|G_2(\phi)|\leqslant|\hat{A}_{\alpha,\beta,\gamma,\delta}(P^1_{N/2,\alpha,\beta,\gamma,\delta}U-U,\phi)|+|\hat{A}_{\alpha,\beta,\gamma,\delta,N}(P^1_{N/2,\alpha,\beta,\gamma,\delta}U-U_N,\phi)|.$$

With the aid of (4.9), we obtain that

$$|\hat{A}_{\alpha,\beta,\gamma,\delta}(P^1_{N/2,\alpha,\beta,\gamma,\delta}U-U,\phi)|\leq c^*\|P^1_{N/2,\alpha,\beta,\gamma,\delta}U-U\|_{1,\alpha,\beta,\gamma,\delta}\|\phi\|_{1,\alpha,\beta,\gamma,\delta}.\tag{4.14}$$

Moreover,

$$\begin{aligned}&|\hat{A}_{\alpha,\beta,\gamma,\delta,N}(P^1_{N/2,\alpha,\beta,\gamma,\delta}U-U_N,\phi)|\\ &\leqslant c^*\|P^1_{N/2,\alpha,\beta,\gamma,\delta}U-U_N\|_{1,\alpha,\beta,\gamma,\delta}\|\phi\|_{1,\alpha,\beta,\gamma,\delta}\\ &\leqslant c^*\left(\|P^1_{N/2,\alpha,\beta,\gamma,\delta}U-U\|_{1,\alpha,\beta,\gamma,\delta}+\|U_N-U\|_{1,\alpha,\beta,\gamma,\delta}\right)\|\phi\|_{1,\alpha,\beta,\gamma,\delta}.\end{aligned}$$

Along with the above and (4.14), we use (3.3) to reach that

$$\begin{aligned}|G_2(\phi)|&\leqslant c^*\left(\|P^1_{N/2,\alpha,\beta,\gamma,\delta}U-U\|_{1,\alpha,\beta,\gamma,\delta}+\|U_N-U\|_{1,\alpha,\beta,\gamma,\delta}\right)\|\phi\|_{1,\alpha,\beta,\gamma,\delta}\\ &\leqslant cc^*(N_1^{1-r_1}+N_2^{1-r_2})\|U\|_{Y^{r_1,r_2,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}}\|\phi\|_{1,\alpha,\beta,\gamma,\delta}.\end{aligned}\tag{4.15}$$

Next, due to (3.6),

$$(\hat{f},\phi)_{\chi^{(\gamma,\delta)},G,N}=(\mathcal{I}_{G,N,\gamma,\delta}\hat{f},\phi)_{\chi^{(\gamma,\delta)},G,N}=(\mathcal{I}_{G,N,\gamma,\delta}\hat{f},\phi)_{\chi^{(\gamma,\delta)}}.$$

Thereby, we use Theorem 3.5 to obtain that

$$\begin{aligned}&|(f,\phi)-(\hat{f},\phi)_{\chi^{(\gamma,\delta)},G,N}|\leqslant\|\mathcal{I}_{G,N,\gamma,\delta}\hat{f}-\hat{f}\|_{\chi^{(\gamma,\delta)}}\|\phi\|_{\chi^{(\gamma,\delta)}}\\ &\leqslant c\left(N_1^{-r_1'}+N_2^{-r_2'}+N_1^{-1}N_2^{1-r_2'}\right)\|\hat{f}\|_{B^{r_1',r_2'}_{\gamma,\delta}}\|\phi\|_{1,\alpha,\beta,\gamma,\delta}.\end{aligned}\tag{4.16}$$

A combination of (4.11)-(4.13), (4.15) and (4.16) with $\phi=u_N-U_N$ leads to that

$$\begin{aligned}\|u_N-U_N\|_{1,\alpha,\beta,\gamma,\delta}&\leqslant cc^*\big(N_1^{1-r_1}+N_2^{1-r_2}\big)\|U\|_{Y^{r_1,r_2,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}}+cA_N^s(U,\tilde{a})\\ &\quad+c\big(N_1^{-r_1'}+N_2^{-r_2'}+N_1^{-1}N_2^{1-r_2'}\big)\|\hat{f}\|_{B^{r_1',r_2'}_{\gamma,\delta}}.\end{aligned}$$

Finally, the desired result follows from the above estimate and (3.3). ■

Remark 4.1. Let $\mathcal{P}_{N_1,N_2} = \mathcal{P}_N$ with $N = (N_1, N_2)$. If, instead of (4.2), $\tilde{a}_1 \in \mathcal{P}_{k_1+2,k_2}$, $\tilde{a}_2 \in \mathcal{P}_{k_1,k_2+2}$, $\tilde{a}_0 \in \mathcal{P}_{k_1,k_2}$, $0 \leqslant k_1 \leqslant N_1$ and $0 \leqslant k_2 \leqslant N_2$, then we have that

$$\begin{aligned}&\|u_N - U\|_{1,\alpha,\beta,\gamma,\delta}\\ &\leqslant cc^*(N_1^{1-r_1} + N_2^{1-r_2})\|U\|_{Y^{r_1,r_2,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}} + c(N_1^{-r_1'} + N_2^{-r_2'} + N_1^{-1}N_2^{1-r_2'})\|\hat{f}\|_{B^{r_1',r_2'}_{\gamma,\delta}}.\end{aligned}$$

Remark 4.2. If f is not continuous, we may take

$$\hat{a}_q(x) = P_{N/2,0,0}\tilde{a}_q(x)$$

or $\hat{a}_q(x) = P^1_{N/2,0,0,0,0}\ \tilde{a}_q(x)$ in (4.7), instead of $\hat{a}_q(x) = I_{L,N/2,0,0}\tilde{a}_q(x)$, $q = 0, 1, 2$. In these cases, we have the error estimates similar to those stated in Theorem 4.1.

Remark 4.3. With the aid of (4.3)-(4.5) and Theorems 3.5-3.8, we could design and analyze the Jacobi pseudospectral schemes for singular problems coupled with certain boundary conditions and other related problems.

We close this section by presenting some numerical results.

Example 4.1. We consider problem (4.1) with

$$a_1(x) = a_2(x) = (1 - x_1^2)(1 - x_2^2)$$

and $a_0(x) = 1$. Take the test solution

$$U(x) = \arcsin(x_1x_2)e^{x_1x_2}. \tag{4.17}$$

Clearly, $|\partial_{x_q}U| \to \infty$, $q = 1, 2$, as x tends to V_j, $1 \leq j \leq 4$. We use (4.7) to solve (4.4) numerically.

Let ζ_{Z_q,N_q,j_q} and Ω_{Z_q,N_q,j_q} be nodes and weights of the one-dimensional Legendre-Gauss quadrature. We measure the numerical errors by

$$E(u_N) = \left(\sum_{j_1=0}^{N_1}\sum_{j_2=0}^{N_2}(U(\zeta_{Z_1,N_1,j_1},\zeta_{Z_2,N_1,j_1}) - u_N(\zeta_{Z_1,N_1,j_1},\zeta_{Z_2,N_1,j_1}))^2\Omega_{Z_1,N_1,j_1}\Omega_{Z_2,N_2,j_2}\right)^{\frac{1}{2}}. \tag{4.18}$$

We present the errors $E(u_N)$ in Table 4.1. Obviously the scheme (4.7) provides accurate numerical solutions even for small mode N, and for the solution with singular derivatives. Moreover, the numerical solution converges to the exact solution rapidly, as N increases. This confirms the theoretical analysis in the previous parts.

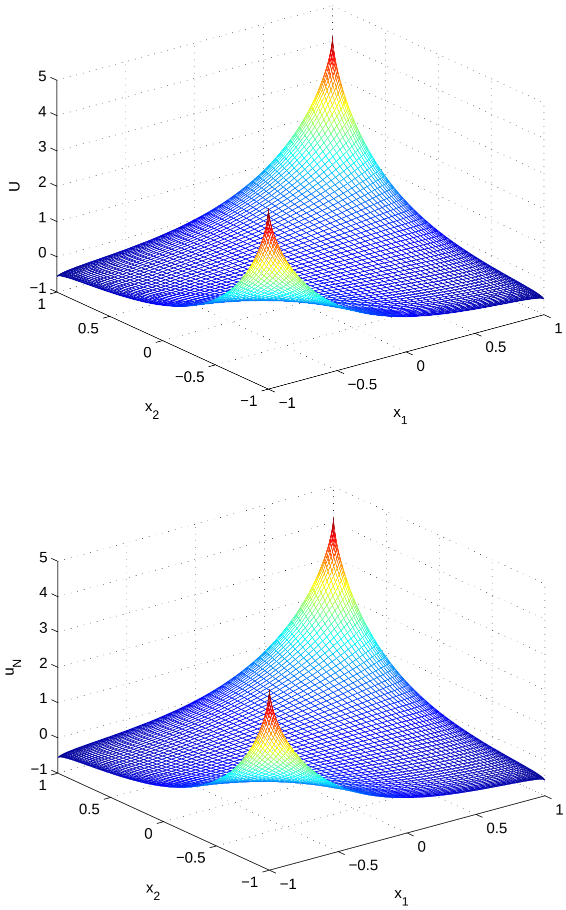
Table 4.1: The errors $E(u_N)$.

$N=8$	$N=16$	$N=24$	$N=32$	$N=40$	$N=48$	$N=56$	$N=64$
3.06e-3	4.97e-4	1.62e-4	7.15e-5	3.77e-5	2.23e-5	1.42e-5	9.63e-6

In Fig. 4.1, we plot the exact solution of (4.4) and the numerical solution of (4.7) with $N = 64$. Clearly, the numerical solution matches the exact solution very well.

Example 4.2. We next use (4.7) to solve (4.4) with the exact solution (4.17), $a_0 = 1$, and

$$\begin{aligned}a_1(x) &= (1 - x_1^2)(1 - x_2^2)(1 + \frac{1}{10}\sin(\pi x_1)\cos(\pi x_2)),\\ a_2(x) &= (1 - x_1^2)(1 - x_2^2)(1 + \frac{1}{10}\cos(\pi x_1)\sin(\pi x_2)).\end{aligned}$$



(a)

(b)

Fig. 4.1. (a) Exact solution of (4.4) and (b) numerical solution using scheme (4.7).

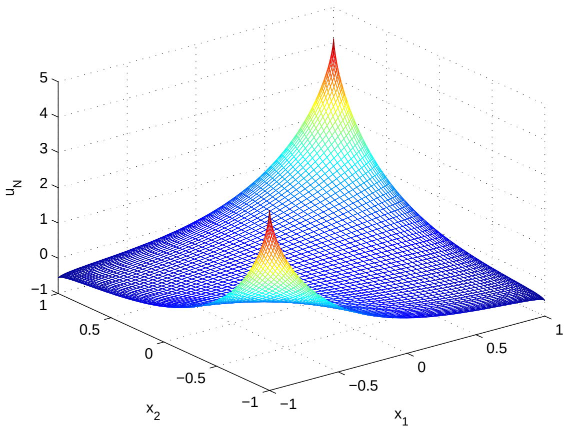


Fig. 4.2. Numerical solution of scheme (4.7).

These functions oscillate as x_1 and x_2 vary. In Fig. 4.2, we plot the numerical solution of (4.7) with $N = 64$. Clearly, the numerical solution also fits well with the exact solution as shown in Fig. 4.1.

5. Jacobi Pseudospectral Method for Axisymmetric Domains

The non-isotropic Jacobi pseudospectral method can be also applied to some problems on axisymmetric domains. As an example, we consider the following cylindrically symmetrical problem

$$\begin{cases} -\partial_\rho^2 V(\rho,z) - \frac{1}{\rho}\partial_\rho V(\rho,z) - \partial_z^2 V(\rho,z) = F(\rho,z), \qquad 0 \le \rho < 2, -1 < z < 1, \\ V(2,z) = V(\rho,-1) = V(\rho,1) = 0. \end{cases} \tag{5.1}$$

We make the variable transformation

$$x_1 = 1-\rho, \quad x_2 = z, \quad U(x) = V(1-x_1, z), \quad f(x) = F(1-x_1, z).$$

Then (5.1) is reformulated to the following problem,

$$\begin{cases} -\partial_{x_1}((1-x_1)\partial_{x_1} U(x)) - (1-x_1)\partial_{x_2}^2 U(x) = (1-x)f(x), \quad -1 < x_1 \leqslant 1, -1 < x_2 < 1, \\ U(-1,x_2) = U(x_1,-1) = U(x_1,1) = 0. \end{cases} \tag{5.2}$$

We take

$$\alpha_1 = \tilde{\alpha}_1 = \gamma_1 = 1, \quad \beta_1 = \tilde{\alpha}_2 = \tilde{\beta}_2 = \alpha_2 = \beta_2 = \tilde{\beta}_1 = \delta_1 = \gamma_2 = \delta_2 = 0. \tag{5.3}$$

Then, a weak formulation of (5.2) is to find $U \in H^{1,\Gamma}_{0,\alpha,\beta,\gamma,\delta}(\Omega)$ such that

$$\bar{a}_{\alpha,\beta}(U,v) = (f,v)_{\chi^{(\gamma,\delta)}}, \quad \forall v \in H^{1,\Gamma}_{0,\alpha,\beta,\gamma,\delta}(\Omega). \tag{5.4}$$

Obviously, $\bar{a}_{\alpha,\beta}(u,v)$ is elliptic. If $f \in L^2_{\chi^{(\gamma,\delta)}}(\Omega)$, then by the Lax-Milgram Lemma, the problem (5.4) has a unique solution such that $|U|_{1,\alpha,\beta} \leqslant c\|f\|_{\chi^{(\gamma,\delta)}}$.

We denote the discrete inner product and norm with $Z = (R, L)$, corresponding to the weight $\chi(x)$, by $(u,v)_{\chi,RL,N}$ and $||v||_{\chi,RL,N}$, respectively. Let $f \in C(\Omega \cup \{(x_1,x_2) \mid x_1 = -1 \text{ or } x_2 = \pm 1\})$ and

$$\bar{a}_{\alpha,\beta,N}(u,v) = (\partial_{x_1}u, \partial_{x_1}v)_{\chi_1^{(\alpha,\beta)},RL,N} + (\partial_{x_2}u, \partial_{x_2}v)_{\chi_2^{(\alpha,\beta)},RL,N}.$$

The Jacobi pseudospectral scheme for solving (5.4) is to find $u_N \in \mathcal{P}_N^{\Gamma,0}$ such that

$$\bar{a}_{\alpha,\beta,N}(u_N,\phi) = (f,\phi)_{\chi^{(\gamma,\delta)},RL,N}, \qquad \forall \phi \in \mathcal{P}_N^{\Gamma,0}. \tag{5.5}$$

It is easy to verify that (5.5) has a unique solution. Moreover, by the boundary conditions and the Poincaré inequality, we use (3.6) to derive that

$$||u_N||_{1,\alpha,\beta,\gamma,\delta} \leqslant c|u_N|_{1,\alpha,\beta} \leqslant c\|\mathcal{I}_{RL,N,\gamma,\delta}f\|_{\chi^{(\gamma,\delta)}}.$$

We now deal with the convergence of numerical solution u_N.

Theorem 5.1. *If* $U \in Y^{r_1,r_2,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}(\Omega)$, $f \in D^{r_1',r_2'}_{\gamma,\delta}(\Omega)$ *and* $r_1, r_2, r_1', r_2' \geqslant 1$, *then*

$$\begin{aligned}&||u_N - U||_{1,\alpha,\beta,\gamma,\delta}\\ &\leqslant c\bigg((N_1^{1-r_1} + N_2^{1-r_2})||U||_{Y^{r,s,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}} + (N_1^{-r_1'} + N_2^{-r_2'} + N_1^{-1}N_2^{1-r_2'})|f|_{D^{r_1',r_2'}_{\gamma,\delta}}\bigg),\end{aligned}$$

where $\sigma, \lambda = 1, 2$.

Proof. Let $\bar{P}^{1,\Gamma}_{N,\alpha,\beta}$ be the same as in (3.5) and $U_N = \bar{P}^{1,\Gamma}_{N,\alpha,\beta}U$. By the ellipticity of the bilinear form $\bar{a}_{\alpha,\beta}(u, v)$, we use (5.3)-(5.5) to deduce that for certain a constant $c_0 > 0$,

$$\begin{aligned}c_0|u_N - U_N|^2_{1,\alpha,\beta} &\leqslant \bar{a}_{\alpha,\beta,N}(u_N - U_N, u_N - U_N)\\ &= (f, u_N - U_N)_{\chi^{(\gamma,\delta)},RL,N} - \bar{a}_{\alpha,\beta,N}(U_N, u_N - U_N)\\ &= \bar{a}_{\alpha,\beta}(U, u_N - U_N) - \bar{a}_{\alpha,\beta,N}(U_N, u_N - U_N)\\ &\quad + (f, u_N - U_N)_{\chi^{(\gamma,\delta)},RL,N} - (f, u_N - U_N)_{\chi^{(\gamma,\delta)}}.\end{aligned} \tag{5.6}$$

Next, thanks to (5.3), we have that $\alpha_1 \leqslant \gamma_1 + 2$, $\beta_1 \leqslant 0$ and $\delta_1 = 0$. Therefore, by (3.5) and an argument similar to the derivation of (4.15), we obtain that for r_1, $r_2 \geqslant 1$

$$\begin{aligned}&|\bar{a}_{\alpha,\beta}(U, u_N - U_N) - \bar{a}_{\alpha,\beta,N}(U_N, u_N - U_N)|\\ &\leqslant c(N_1^{1-r_1} + N_2^{1-r_2})||U||_{Y^{r_1,r_2,\sigma,\lambda}_{\alpha,\beta,\gamma,\delta}}|u_N - U_N|_{1,\alpha,\beta}.\end{aligned} \tag{5.7}$$

Furthermore, with the aid of Theorem 3.8, we know that for r_1', $r_2' \geqslant 1$,

$$\begin{aligned}&|(f, u_N - U_N)_{\chi^{(\gamma,\delta)},RL,N} - (f, u_N - U_N)_{\chi^{(\gamma,\delta)}}|\\ &\leqslant c(N_1^{-r_1'} + N_2^{-r_2'} + N_1^{-1}N_2^{1-r_2'})|f|_{D^{r_1',r_2'}_{\gamma,\delta}}\|u_N - U_N\|_{\chi^{(\gamma,\delta)}}.\end{aligned} \tag{5.8}$$

Moreover, by the boundary conditions and the Poincaré inequality,

$$\|u_N - U_N\|_{\chi^{(\gamma,\delta)}} \leqslant c|u_N - U_N|_{1,\alpha,\beta}. \tag{5.9}$$

Finally the desired result comes from a combination of (3.5) and (5.6)-(5.9). ■

Example 5.1. We now use (5.5) to solve (5.4) with the exact solution

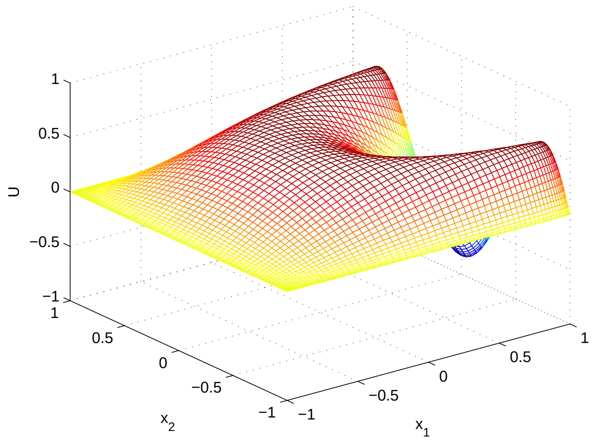
$$U(x) = (1 + x_1)(1 - x_2^2)\cos((1 + x_1)(1 - x_2^2)). \tag{5.10}$$

The error of numerical solution is measured by $E(u_N)$ as in (4.18). We present the errors $E(u_N)$ in Table 5.1. It shows the convergence and the high accuracy of scheme (5.5). In Fig. 5.1, we plot the exact solution of (5.4) and the numerical solution of (5.5) with $N = 64$, respectively. Clearly, the numerical solution fits the exact solution well.

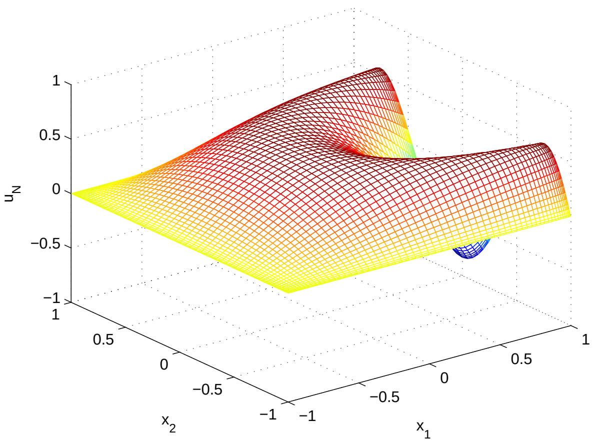
Table 5.1: The errors $E(u_N)$.

$N = 8$	$N = 16$	$N = 24$	$N = 32$	$N = 40$	$N = 48$	$N = 56$	$N = 64$
3.36e-3	5.03e-4	1.74e-4	7.40e-5	3.97e-5	2.41e-5	1.51e-5	9.85e-6

Remark 5.1. We may combine the Fourier approximation and the method described in this section to solve differential equations on three-dimensional axisymmetric domains. For the Laplace equation on a cylinder as discussed in [3], we can design the corresponding pseudospectral scheme and derive sharp error estimate for the numerical solution. The main reason is that in our case, there exist Jacobi weights in the norms of exact solution, appearing in the error estimates.



(a)



(b)

Fig. 5.1. (a) Exact solution of (5.4) and (b) numerical solution using scheme (5.5).

6. Concluding Remarks

In this paper, we proposed the non-isotropic Jacobi pseudospectral method with its applications. As examples, we considered a singular problem in two dimensions and a problem on an axisymmetric domain. The numerical results demonstrated the spectral accuracy of proposed schemes, and agreed well with the theoretical analysis. Indeed, the multi-dimensional Jacobi pseudospectral method coupled with variable transformations is also applicable to certain problems on unbounded domains.

We established some basic results on the Jacobi-Gauss type interpolation, with which the convergence of proposed schemes followed. These results play important role in designing and analyzing Jacobi pseudospectral methods for various practical problems. In fact, they also serve as important tools in the analysis of multiple-dimensional rational pseudospectral method induced by the Jacobi polynomials.

We also derived a series of sharp results on the Legendre-Gauss type interpolation and the related Bernstein-Jackson type inequalities, which improve and generalize the existing results. They are very useful for pseudospectral method of partial differential equations with non-constant coefficients, as well as numerical solutions of initial value problems of nonlinear ordinary differential equations. We shall report the related results in the future.

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