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A POSTERIORI ESTIMATOR OF NONCONFORMING FINITE ELEMENT METHOD FOR FOURTH ORDER ELLIPTIC PERTURBATION PROBLEMS*

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Abstract

In this paper, we consider the nonconforming finite element approximations of fourth order elliptic perturbation problems in two dimensions. We present an *a posteriori* error estimator under certain conditions, and give an h-version adaptive algorithm based on the error estimation. The local behavior of the estimator is analyzed as well. This estimator works for several nonconforming methods, such as the modified Morley method and the modified Zienkiewicz method, and under some assumptions, it is an optimal one. Numerical examples are reported, with a linear stationary Cahn-Hilliard-type equation as a model problem.

Mathematics subject classification: 65N30.
Key words: Fourth order elliptic perturbation problems, Nonconforming finite element method, A posteriori error estimator, Adaptive algorithm, Local behavior.

1. Introduction

The parabolic perturbation problems, such as the Cahn-Hilliard-type equations, are frequently encountered in applications, see, e.g., [12, 14, 20]. Their stationary formations, namely the corresponding elliptic perturbation problems, are important for both theoretical analysis and computation. The numerical solution to such problems has been an interesting and practical topic in computational mathematics. Various finite element methods, both standard and nonstandard, have been developed for this problem, and their convergences were proven; see, e.g., [16, 17, 25, 30].

The adaptive finite element methods, in particular the h-version methods, are very useful for efficient numerical solutions. As to these methods, the key features are *a posteriori* error estimation and the strategy of mesh refinement. The *a posteriori* error estimation can be treated as an indicator of the distribution of the error on certain mesh. According to the *a posteriori* error estimation, the numerical solution can be carried out in the local, parallel or adaptive ways, see, e.g., [32]. In all these methods, an *a posteriori* error estimator is utilized as an indicator of the quality of the mesh.

It is pointed by Bank [9] that the notion of using *a posteriori* error estimates to measure and control the error in practical finite element calculations was first suggested by Babuska and Rheinboldt [5]. The approach in [5] provided the earliest general way for *a posteriori* error estimation with firm theoretical foundations. So far, *a posteriori* error estimation for conforming finite element methods, especially on second order problems, has been the subject of extensive investigation, see, e.g., [2, 5, 6, 9–12, 23, 33], the reviews [3, 8, 19, 21, 22] and the

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monographs [4, 7, 27]. However, the treatment of nonconforming methods has been subjected to sporadic attention. Dari et al. [18] considered the error as a combination of a conforming part and a nonconforming part, where the nonconforming part is estimated via the difference between the nonconforming solution and its smooth approximation. The idea has been carried out on the second order problems, with the help of the orthogonal decomposition (*Helmholtz* decomposition) of L^2. According to this, many ways were put forward to extend the method for conforming methods to nonconforming ones; see, e.g., [1] and the references therein. Castensen et al. [15] followed the idea and developed a technique to present a framework of *a posteriori* estimation for a class of nonconforming methods on parallelogram meshes. The framework has been shown to be effective for problems of second order. Even though the *a posteriori* estimation for fourth order problems can date back to [27], however, partially because that few nonconforming finite element spaces contain a subspace consisting of C^1 continuous functions, there are few works dealing with the nonconforming methods directly. A general framework of *a posteriori* error estimation is presented in [26] for the nonconforming methods, and it is shown that the methodology of decomposing the errors can be used for problems with arbitrarily high order.

The error estimators obtained in such ways give upper bounds of the global error, and can be computed in *a posteriori* way. However, in local sense, they may provide upper bounds of error as well as mesh indicators. Xu and Zhou [32] showed a local upper bound for *conforming* methods applied to second order problems. Wang and Zhang [31] proved that a local *a posteriori* error estimator can be a local upper bound of the error up to higher order terms for the *nonconforming* finite element methods to two dimensional biharmonic equations.

In this paper, we study the *a posteriori* error estimation for nonconforming finite element methods for the elliptic perturbation problems. A two dimensional linear stationary Cahn-Hilliard-type equation is used as a model problem. The rest of the paper is organized as follows. In Section 2, some preliminary materials are provided. In Section 3, global *a posteriori* error estimator is obtained for general nonconforming finite element discretization methods on shape-regular grids for the model problem. The deduction uses the same idea of the framework [26]. The efficiency of the estimator is devised and analyzed in Section 4. Based upon certain convergence assumptions, the estimator is optimal in the sense that the *a posteriori* error estimator has the same convergence order as that of the *a priori* error estimator. An h-version adaptive method is discussed and some numerical experiments are reported in Section 5. In the final section, further discussions are presented and the local behavior of the estimator is analyzed.

2. Preliminaries

In this section, we describe the model problem and the corresponding nonconforming finite element methods.

Let Ω be a bounded domain in R^2, with the boundary $\partial\Omega$ and ν the unit outer normal vector to $\partial\Omega$. For nonnegative integer s, we shall use the standard notation $H^s(\Omega)$ for Sobolev space, $\|\cdot\|_{s,\Omega}$ the associated norm and $|\cdot|_{s,\Omega}$ the associated seminorm. We shall omit s and not distinguish the norm and the seminorm when $s=0$. In addition we denote

$$H_0^1(\Omega)=\{v\in H^1(\Omega):v|_{\partial\Omega}=0\},\quad H_0^2(\Omega)=\left\{v\in H^2(\Omega)\cap H_0^1(\Omega):\frac{\partial v}{\partial\nu}\big|_{\partial\Omega}=0\right\}.$$

2.1. A model problem

In this paper, we consider the nonconforming finite element methods for the elliptic perturbation problems (EPPs) of the form

$$\begin{cases} \varepsilon^2\Delta^2 u - \Delta u = f, \ \text{in } \Omega, \\ u = 0, \ \dfrac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Δ is the standard Laplacian operator, $0 < \varepsilon < 1$ is a small parameter, and $f \in L^2(\Omega)$. The equation can be viewed as a linear stationary Cahn-Hilliard-type equation.

Let $i, j \in \{1, 2\}$ and set $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij} = \partial_i\partial_j$. For a domain $B \subset R^2$, define the bilinear forms

$$a_B(u, v) = \int_B \Big\{\Delta u\Delta v + (1-\sigma)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v)\Big\}$$

for $u, v \in H^2(B)$, where $\sigma \in [0, \frac{1}{2}]$, and

$$b_B(u, v) = \int_B \sum_{i=1}^{2} \partial_i u\, \partial_i v$$

for $u, v \in H^1(B)$. The weak problem (WP) is to find $u \in H_0^2(\Omega)$ such that

$$\varepsilon^2 a_\Omega(u, v) + b_\Omega(u, v) = (f, v), \quad \forall\, v \in H_0^2(\Omega), \tag{2.2}$$

where by $(\cdot, \cdot)$ we denote the inner product of $L^2(\Omega)$. Without loss of generality, throughout this paper, we set $\sigma = 0$.

Some mathematical properties of (EPP) have been studied, which are useful in this work. Let u^0 be the solution of following problem (DP):

$$\begin{cases} -\Delta u = f, \quad \text{in } \Omega, \\ u = 0, \qquad\ \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

The following lemma can be found in [25]:

Lemma 2.1. *Let u and u^0 be the solutions of (WP) and (DP) respectively. If Ω is convex, then there exists a constant C independent of ε such that, for all $f \in L^2(\Omega)$,*

$$|u|_{2,\Omega} + \varepsilon|u|_{3,\Omega} \leqslant C\varepsilon^{-1/2}\|f\|_\Omega, \tag{2.4}$$

$$\|u^0\|_{2,\Omega} \leqslant C\|f\|_\Omega, \tag{2.5}$$

$$|u - u^0|_{1,\Omega} \leqslant C\varepsilon^{1/2}\|f\|_\Omega. \tag{2.6}$$

2.2. Finite element methods

By a triple (T, P_T, Φ_T) we denote a finite element with T the geometric shape, P_T the shape function space and Φ_T the set of nodal parameters, and let Φ_T be P_T-unisolvent. Let $\{\mathcal{T}_h(\Omega)\}$ be a family of triangulations corresponding to element (T, P_T, Φ_T) and the mesh size h, with $h \geqslant h_T$, the mesh-size of element T, on $\forall\, T \in \mathcal{T}(\Omega)$. In this paper, we always assume that the geometric shape is triangle or rectangle, the triangulation is shape-regular.

Given a mesh parameter h and a triangulation $\mathcal{T}_h(\Omega)$, denote by $V^h(\Omega)$ and $V_0^h(\Omega)$ the finite element spaces with respect to $H^2(\Omega)$ and $H_0^2(\Omega)$ respectively. We do not need that

$V^h(\Omega) \subset H^2(\Omega)$ or $V_0^h(\Omega) \subset H_0^2(\Omega)$. For a domain $G \subset \Omega$, we denote by $\mathcal{T}_h(G)$ and $V^h(G)$ the restrictions of $\mathcal{T}_h(\Omega)$ and $V^h(\Omega)$ to G, respectively. Accordingly we denote

$$V_0^h(G) = \big\{v \in V_0^h(\Omega) : \ \mathrm{supp}(v) \subset \overline{G}\big\}.$$

For ease of exposition, we assume that any $G \subset \Omega$ mentioned below would align with $\mathcal{T}_h(\Omega)$ when necessary, which is reasonable. We introduce the piecewise space

$$H^s\big(\mathcal{T}_h(G)\big) = \{v \in L^2(G) : \ v\big|_T \in H^s(T), \forall\, T \in \mathcal{T}_h(G)\}$$

associated with the triangulation for nonnegative integer s, and use $\|\cdot\|_{s,h,G}$ and $|\cdot|_{s,h,G}$ to denote the norms and semi-norms on $H^s\big(\mathcal{T}_h(G)\big)$. In particular, for simplicity and convenience, we use the notations $H^s(\mathcal{T}_h)$, $\|\cdot\|_{s,h}$ and $|\cdot|_{s,h}$ when $G = \Omega$, and $H^s(T)$, $\|\cdot\|_{s,T}$ and $|\cdot|_{s,T}$ when $G = T$, $T \in \mathcal{T}_h(\Omega)$.

Define

$$\begin{aligned} a_h(v,w) &= \sum_{T\in\mathcal{T}_h(\Omega)} a_T(v,w), \quad \forall v,w \in H^2(\mathcal{T}_h), \\ b_h(v,w) &= \sum_{T\in\mathcal{T}_h(\Omega)} b_T(v,w), \quad \forall v,w \in H^1(\mathcal{T}_h). \end{aligned}$$

Let $L_h : V^h(\Omega) \to H^2(\mathcal{T}_h)$ be a linear interpolation operator. We consider the following discrete weak problem (DWP): Find $u^h \in V_0^h(\Omega)$, such that

$$\varepsilon^2 a_h(u^h, v^h) + b_h(L_h u^h, L_h v^h) = (f, L_h v^h), \qquad \forall\, v^h \in V_0^h(\Omega). \tag{2.7}$$

Note that the standard finite element method is a special case while L_h is chosen to be the identity operator.

2.3. Basic assumptions

Let $p \in H^2(\mathcal{T}_h)$ be C^1-continuous in each element T. Then p is said to be weakly continuous on $\overline{\Omega}$ if for any mutual side F of two neighbor elements, there exists a point on F, at which p is continuous. If p is weakly continuous, and for each mutual side F of two neighbor elements, there exists a point on F at which $\partial_l p$, $l = 1, 2$, are continuous, then p is said to be weakly C^1 continuous on $\overline{\Omega}$. Besides, if for any side $F \subset \partial\Omega$, there exists a point on F such that p vanishes there, then p is said to satisfy the weakly homogeneous boundary condition. Further, if p satisfies the weakly homogeneous boundary condition, and for any side $F \subset \partial\Omega$, there exists a point on F such that $\partial_l p$, $l = 1, 2$, vanish there, then p is said to satisfy the weakly C^1 homogeneous boundary condition.

For any nonnegative integer k and $T \in \mathcal{T}_h(\Omega)$, let $P_k(T)$ denote the set of all polynomials with degree not greater than k, and $Q_k(T)$ the set of all polynomials with degree of each variable not greater than k. Throughout this paper, Q_T is the polynomial space of the Bell element when T is a triangle, and is the bi-cubic polynomial space when T is a rectangle, $V_C^h(\Omega)$ and $V_{C0}^h(\Omega)$ are the corresponding conforming finite element spaces for biharmonic problems and

$$P_{ch}(\Omega) = \big\{p \in L^2(\Omega) : \ p|_T \in Q_T, \forall\, T \in \mathcal{T}(\Omega)\big\}.$$

Further we introduce the following notations. For a set $B \subset \Omega$,

$$S_h(B) = \big\{T \in \mathcal{T}_h(\Omega) : \overline{T} \cap \overline{B} \neq \varnothing\big\}.$$

For $D \subset \overline{G} \subset \overline{\Omega}$, we use the notation $D \subset\subset G$ to mean that $\mathrm{dist}(\partial D \setminus \partial\Omega, \partial G \setminus \partial\Omega) > 0$, see also [32]. Throughout this paper, we use the letter C (with or without subscripts) to denote a generic positive constant which may stand for different values at its different occurrences. For convenience, similar to [32], the symbols $\lesssim$, $\gtrsim$ and $\eqsim$ are used in this paper. Specially,

$$x_1 \lesssim y_1, \quad x_2 \gtrsim y_2, \quad x_3 \eqsim y_3,$$

mean that

$$x_1 \leqslant C_1 y_1, \quad x_2 \geqslant c_2 y_2, \quad c_3 x_3 \leqslant y_3 \leqslant C_3 x_3$$

for some constants C_1, c_2, c_3 and C_3, which are independent of the varying parameters.

Certain basic assumptions on the finite element methods, listed below, are needed.

A1. (*Approximation.*) There exists $r_1 \geq 2$ such that given an $s \in [0, r_1]$, for $\forall\, v \in H^{s+1}(\Omega)$, there exists a $v^h \in V^h(\Omega)$, such that

$$\sum_{j=0}^{t+1} h_T^{2j} |v - v^h|_{j,T}^2 \lesssim \sum_{T' \in S_h(T)} h_{T'}^{2(t+1)} |v|_{t+1,T'}^2, \quad \forall\, T \in \mathcal{T}_h(\Omega), \quad 0 \leqslant t \leqslant s. \tag{2.8}$$

In addition, if $v \in H^{s+1}(\Omega) \cap H_0^2(\Omega)$, we can choose $v^h \in V_0^h(\Omega)$ which satisfies (2.8). For the interpolation operator, it is assumed that

$$\sum_{j=0}^{2} h_T^{2j} |v^h - L_h v^h|_{j,T}^2 \lesssim \sum_{T' \in S_h(T)} h_{T'}^4 |v^h|_{2,T'}^2, \quad \forall\, v^h \in V^h(\Omega). \tag{2.9}$$

A2. (*Consistency.*) There exists $r_2 \geq 1$ such that given an $s \in [1, r_2]$, for $G \subset \Omega$ and $i, j \in \{1, 2\}$, the following are true:

$$\Big| \sum_{T \in \mathcal{T}_h(G)} \int_T (\partial_{ij} w \varphi + \partial_i w \partial_j \varphi) \Big| \lesssim \sum_{T \in \mathcal{T}_h(G)} h_T^s |w|_{2,T} |\varphi|_{s,T},$$
$$\forall\, w \in H_0^2(G) + V_0^h(G),\ \forall\, \varphi \in H^s(G); \tag{2.10}$$
$$\Big| \sum_{T \in \mathcal{T}_h(G)} \int_T (\partial_i w \varphi + w \partial_i \varphi) \Big| \lesssim \sum_{T \in \mathcal{T}_h(G)} h_T^s |w|_{1,T} |\varphi|_{s,T},$$
$$\forall\, w \in H_0^1(G) + L_h V_0^h(G),\ \forall\, \varphi \in H^s(G). \tag{2.11}$$

A3. (*Weak Continuity.*) Let $v^h \in V^h(\Omega)$. Then it possesses the following properties: 1) v^h is weakly C^1 continuous; 2) if $v^h \in V_0^h(\Omega)$, it also satisfies the weak C^1 homogeneous boundary condition; 3) $L_h v^h$ is weakly continuous; further when $v^h \in V_0^h(\Omega)$, $L_h v^h$ satisfies the weakly homogeneous boundary condition.

A4. $V^h(\Omega) \subset P_{ch}(\Omega)$; $L_h V^h(\Omega) \subset P_{ch}(\Omega)$.

The above assumptions are natural in some sense, and they are satisfied by some known finite element methods, such as the modified Morley method and the modified Zienkiewicz method.

2.4. Global *a posteriori* estimation: A framework

The following lemma would be used as a framework. The same idea can be found in [26].

Lemma 2.2. *Assume that X and X^h, with the mesh parameter h, are subspaces of a linear space. For each h and parameter τ, assume that there exist a bilinear form $d_{\tau,h}(\cdot,\cdot)$ defined on $(X+X^h)\times(X+X^h)$, and $\|\cdot\|_{\tau,h}$ be a norm on $X+X^h$. Suppose that $d_{\tau,h}(\cdot,\cdot)$ possesses continuity in the sense that*

$$d_{\tau,h}(w,v)\lesssim \|w\|_{\tau,h}\|v\|_{\tau,h},\qquad \forall\, w,v\in X+X^h, \tag{2.12}$$

and is coercive on $X\times X$ in the sense that

$$d_{\tau,h}(w,w)\gtrsim \|w\|^2_{\tau,h},\qquad \forall\, w\in X. \tag{2.13}$$

Then it holds for $v\in X$ and $v^h\in X^h$ that

$$\|v-v^h\|_{\tau,h}\eqsim \sup_{0\neq w\in X}\frac{\left|d_{\tau,h}(v,w)-d_{\tau,h}(v^h,w)\right|}{\|w\|_{\tau,h}}+\inf_{w\in X}\|v^h-w\|_{\tau,h}. \tag{2.14}$$

Proof. We first prove that

$$\|v-v^h\|_{\tau,h}\lesssim \sup_{0\neq w\in X}\frac{\left|d_{\tau,h}(v,w)-d_{\tau,h}(v^h,w)\right|}{\|w\|_{\tau,h}}+\inf_{w\in X}\|v^h-w\|_{\tau,h}. \tag{2.15}$$

For any h and τ, let $v\in X$. Then by the coercivity condition (2.13), we derive for $v^h\in X^h$ and arbitrary $w'\in X$ that

$$\begin{aligned}\|v-w'\|^2_{\tau,h}&\lesssim d_{\tau,h}(v-w',v-w')\\&=d_{\tau,h}(v-v^h,v-w')-d_{\tau,h}(w'-v^h,v-w').\end{aligned}$$

Hence by the continuity condition (2.12), we have that

$$\begin{aligned}\|v-w'\|_{\tau,h}&\lesssim \frac{\left|d_{\tau,h}(v,v-w')-d_{\tau,h}(v^h,v-w')\right|}{\|v-w'\|_{\tau,h}}+\|w'-v^h\|_{\tau,h}\\&\lesssim \sup_{0\neq w\in V}\frac{\left|d_{\tau,h}(v,w)-d_{\tau,h}(v^h,w)\right|}{\|w\|_{\tau,h}}+\|w'-v^h\|_{\tau,h}.\end{aligned}$$

Noting that

$$\|v-v^h\|_{\tau,h}\leqslant \|v-w'\|_{\tau,h}+\|w'-v^h\|_{\tau,h},$$

by the arbitrariness of $w'\in X$, we obtain (2.15). Since

$$\left|d_{\tau,h}(v,w)-d_{\tau,h}(v^h,w)\right|\lesssim \|v-v^h\|_{\tau,h}\|w\|_{\tau,h},\quad \forall w\in X,$$

we can prove the remaining immediately. $\square$

Remark 2.1. Let X be the Sobolev space corresponding to some elliptic equation and X^h the nonconforming finite element space. Then this lemma can be regarded as an analogue of the second Strang lemma, with X and X^h exchanging their positions.

3. Construction of an Estimator: Reliability

In this section, we follow Lemma 2.2 to estimate the two terms on the righthand of (2.14) and to establish an estimator for the model problem. Denote by $\partial\mathcal{T}_h$ the set consisting of all sides of all elements in $\mathcal{T}_h(\Omega)$. Define

$$\partial\mathcal{T}_h^b=\{F\in\partial\mathcal{T}_h:\ F\subset\partial\Omega\},\quad \partial\mathcal{T}_h^i=\partial\mathcal{T}_h\setminus\partial\mathcal{T}_h^b.$$

For $T\in\mathcal{T}_h(\Omega)$ and a side $F\subset\partial T$, we denote by $[\cdot]^T_{J,F}$ the jump of a function through F from the interior of T to the outer when $F\in\partial\mathcal{T}_h^i$, and itself when $F\in\partial\mathcal{T}_h^b$. Let $|T|$ and $|F|$ denote the measures of T and F respectively. For w on which the definition makes sense, define

$$\begin{aligned}J_2^T(w)=&\sum_{F\subset\partial T,F\in\partial\mathcal{T}_h^i}|F|\Big(\Big\|\Big[\frac{\partial^2 w}{\partial {\nu^T}^2}\Big]^T_{J,F}\Big\|_F^2+\Big\|\Big[\frac{\partial^2 w}{\partial\nu^T\partial s^T}\Big]^T_{J,F}\Big\|_F^2+\Big\|\Big[\frac{\partial^2 w}{\partial {s^T}^2}\Big]^T_{J,F}\Big\|_F^2\Big)\\&+\sum_{F\subset\partial T,F\in\partial\mathcal{T}_h^b}|F|\Big(\Big\|\Big[\frac{\partial^2 w}{\partial\nu^T\partial s^T}\Big]^T_{J,F}\Big\|_F^2+\Big\|\Big[\frac{\partial^2 w}{\partial {s^T}^2}\Big]^T_{J,F}\Big\|_F^2\Big),\end{aligned}\tag{3.1}$$

$$J_1^T(w)=\sum_{F\subset\partial T}|F|\Big(\Big\|\Big[\frac{\partial w}{\partial\nu^T}\Big]^T_{J,F}\Big\|_F^2+\Big\|\Big[\frac{\partial w}{\partial s^T}\Big]^T_{J,F}\Big\|_F^2\Big),\tag{3.2}$$

$$J_0^T(w)=\sum_{F\subset\partial T}|F|\,\big\|[w]^T_{J,F}\big\|_F^2,\tag{3.3}$$

where $\nu^T=(\nu_1^T,\nu_2^T)^\top$ is the unit outer normal of ∂T, and $s^T=(s_1^T,s_2^T)^\top=(-\nu_2^T,\nu_1^T)^\top$ is the unit tangent vector of ∂T. For convenience, in the remaining, we make use of the following notations. Let $\alpha=(\alpha_1,\alpha_2)$ be a multiple index, $\alpha_1,\alpha_2\geqslant 0$, $|\alpha|=\alpha_1+\alpha_2$. By $\partial^\alpha_{\nu s}$ we denote $\partial^{|\alpha|}/\partial\nu^{\alpha_1}\partial s^{\alpha_2}$, where ν and s are corresponding normal and tangent unit vectors, if there is no extra announcement.

Furthermore, we define

$$\begin{aligned}R^T(u^h)=&\varepsilon^2h_T^2|u^h|^2_{3,T}+\varepsilon^2J_2^T(u^h)+h_T^2|L_hu^h|^2_{2,T}+J_1^T(L_hu^h)\\&+\min(h_T^2,\varepsilon^2)|u^h|^2_{2,T}+\min(h_T^2,\frac{h_T^4}{\varepsilon^2})\|f-\varepsilon^2\Delta^2u^h+\Delta L_hu^h\|^2_T.\end{aligned}\tag{3.4}$$

Then we are to show that

$$\varepsilon^2|u-u^h|^2_{2,h}+|u-L_hu^h|^2_{1,h}\lesssim\sum_{T\in\mathcal{T}_h}R^T(u^h).$$

Remark 3.1. If nonhomogeneous boundary conditions are used, the estimator will work once we take into consideration the difference between the boundary data and the traces of u^h and L_hv^h. In fact, (3.1)-(3.4) are special cases corresponding to the homogeneous boundary conditions.

Lemma 3.1. *For any $v^h\in P_{ch}(\Omega)$, we have that for all $T\in\mathcal{T}_h(\Omega)$,*

$$J_r^T(v^h)\lesssim\sum_{T'\in S_h(T)}|v^h|^2_{r,T'},\quad r=0,1,2.\tag{3.5}$$

Proof. Let F be a side of T. When $F\not\subset\partial\Omega$, there exists a $T'\in\mathcal{T}_h$ such that $F=T'\cap T$. For $|\alpha|=2$, by the inverse inequality, we have that

$$\big\|\big[\partial^\alpha_{\nu s}v^h\big]^T_{J,F}\big\|_F^2\lesssim\big\|\partial^\alpha_{\nu s}v^h|_T\big\|_F^2+\big\|\partial^\alpha_{\nu s}v^h|_{T'}\big\|_F^2\lesssim h_T^{-1}\big(\big|v^h\big|^2_{2,T}+\big|v^h\big|^2_{2,T'}\big).$$

When $F\subset\partial\Omega$, for $|\alpha|=2$, $\alpha_1<2$, by the inverse inequality, we have

$$\big\|\partial^\alpha_{\nu s}v^h\big\|_F^2\lesssim h_T^{-1}\big|v^h\big|^2_{2,T}.$$

Then (3.5) is proved for $r=2$. Similarly we can prove the other cases. $\square$

Lemma 3.2. *Let $v^h \in H^2(\mathcal{T}_h)$ be weakly continuous and satisfy the weakly homogeneous boundary condition. Then we have the following estimate:*

$$J_0^T(v^h) \lesssim h_T^2 J_1^T(v^h). \tag{3.6}$$

Furthermore, if v^h is weakly C^1 continuous and satisfies the weakly C^1 homogeneous boundary condition, then it holds that

$$J_1^T(v^h) \lesssim h_T^2 J_2^T(v^h). \tag{3.7}$$

Proof. Since v^h is weakly continuous and satisfies the weak homogeneous condition, for each side $F \subset \partial T$, there exists $x_0 \in F$, such that v^h is continuous at x_0 when $F \in \partial\mathcal{T}_h^i$ and v_h vanishes at x_0 when $F \in \partial\mathcal{T}_h^b$. Therefore we have that

$$\begin{aligned}\|[v^h]_{J,F}^T\|_F^2 &= \int_F \Big([v^h]_{J,F}^T\Big)^2 \mathrm{d}s \leqslant \int_F \Big(\int_{\overline{x_0 x}} \Big|\frac{\partial}{\partial s}[v^h]_{J,F}^T\Big|\mathrm{d}t\Big)^2 \mathrm{d}s \\ &\leqslant \Big(\int_F \Big(\int_F \Big|\frac{\partial}{\partial s}[v^h]_{J,F}^T\Big|^2 \mathrm{d}t\Big)^{1/2} \mathrm{d}s\Big)^2 \eqsim h_T^2 \int_F \Big(\Big[\frac{\partial}{\partial s}v^h\Big]_{J,F}^T\Big)^2 \mathrm{d}s,\end{aligned}$$

where by s we denote the unit vector along F. The second inequality follows the generalized Minkowski inequality.

From the definition of J_0^T and J_1^T, we proved (3.6). When v^h is weakly C^1 continuous and satisfies the weakly C^1 homogeneous boundary condition, we can prove (3.7) similarly. □

The following lemmas can be found in [31], and we refer there for the constructive proofs.

Lemma 3.3. *For arbitrary $v^h \in P_{ch}(\Omega)$, there exists $w \in H^1(\Omega)$ such that, for all $T \in \mathcal{T}_h(\Omega)$,*

$$\begin{aligned}&\|v - w\|_T^2 + |T|\,|w|_{1,T}^2 \\ &\lesssim |T|\,|v|_{1,T}^2 + \sum_{T' \in S_h(T)} \sum_{F \in \partial\mathcal{T}_h^i, F \subset \partial T'} |F|\,\big\|[v]_{J,F}^{T'}\big\|_F^2.\end{aligned} \tag{3.8}$$

Lemma 3.4. *There exists a linear interpolation operator Q_h from $P_{ch}(\Omega)$ to $V_{C0}^h(\Omega)$, thus $H_0^2(\Omega)$, such that for all $p \in P_{ch}(\Omega)$,*

$$\|p - Q_h p\|_T^2 \lesssim \sum_{T' \in S_h(T)} \Big\{J_0^{T'}(p) + h_{T'}^2 J_1^{T'}(p) + h_{T'}^4 J_2^{T'}(p)\Big\}, \quad \forall\, T \in \mathcal{T}_h(\Omega). \tag{3.9}$$

For ease of exposition, for any vector functions $\mathbf{w} = (w_1, w_2)^\top, \mathbf{v} = (v_1, v_2)^\top \in \big(H^2(\mathcal{T}_h)\big)^2$, we introduce a bilinear form as follows:

$$A_{\varepsilon,h}(\mathbf{w}, \mathbf{v}) = \sum_{T \in \mathcal{T}_h(\Omega)} \Big\{\varepsilon^2 \int_T \sum_{i,j=1}^2 \partial_{ij} w_1 \partial_{ij} v_1 + \int_T \sum_{l=1}^2 \partial_l w_2 \partial_l v_2\Big\}. \tag{3.10}$$

Further we define a linear expansion operator $\mathbf{I}$ from $L^2(\Omega)$ to $\big(L^2(\Omega)\big)^2$:

$$\mathbf{I}w = (w, w)^\top, \quad \forall\, w \in L^2(\Omega);$$

a linear interpolation operator $\mathbf{L}$ from $V^h(\Omega)$ to $\big(H^2(\mathcal{T}_h)\big)^2$:

$$\mathbf{L}v^h = (v^h, L_h v^h)^\top, \quad \forall\, v^h \in V^h(\Omega);$$

and a linear functional $\mathbf{F}$ on $\big(L^2(\Omega)\big)^2$:

$$\mathbf{F}(\mathbf{w}) = (f, w_2), \quad \forall\, \mathbf{w} = (w_1, w_2)^\top \in \big(L^2(\Omega)\big)^2.$$

Note that $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ here.

Then (WP) can be rewritten as to find $u \in H_0^2(\Omega)$ such that

$$A_{\varepsilon,h}(\mathbf{I}u, \mathbf{I}v) = \mathbf{F}(\mathbf{I}v), \qquad \forall\, v \in H_0^2(\Omega);$$

and (DWP) is to find $u^h \in V_0^h(\Omega)$ such that

$$A_{\varepsilon,h}(\mathbf{L}u^h, \mathbf{L}v^h) = \mathbf{F}(\mathbf{L}v^h), \quad \forall\, v^h \in V_0^h(\Omega).$$

Now we are to establish the global estimator, which is a main result of this paper.

Theorem 3.1. *Suppose Assumptions A1-A4 hold. Let u and u^h be the solutions of (WP) and (DWP), respectively. Then it holds that*

$$\varepsilon^2 |u - u^h|_{2,h}^2 + |u - L_h u^h|_{1,h}^2 \lesssim \sum_{T\in\mathcal{T}_h} R^T(u^h). \tag{3.11}$$

Proof. Define for $\mathbf{w} = (w_1, w_2)^\top \in \big(H^2(\mathcal{T}_h)\big)^2$ the functional $\|\cdot\|_{\varepsilon,h}$ to be

$$\|\mathbf{w}\|_{\varepsilon,h} := \big(\varepsilon^2 |w_1|_{2,h}^2 + |w_2|_{1,h}^2\big)^{1/2}. \tag{3.12}$$

It follows from the assumptions that $\|\cdot\|_{\varepsilon,h}$ is a norm on $\mathbf{I}H_0^2(\Omega) + \mathbf{L}V_0^h(\Omega)$ based on $A_{\varepsilon,h}(\cdot,\cdot)$, and the left-hand side of (3.11) is $\|\mathbf{I}u - \mathbf{L}u^h\|_{\varepsilon,h}^2$. It is easy to check that the bilinear form $A_{\varepsilon,h}$ and the norm $\|\cdot\|_{\varepsilon,h}$ satisfy the assumptions of Lemma 2.2. Therefore, by the lemma,

$$\begin{aligned}
&\varepsilon^2 |u - u^h|_{2,h}^2 + |u - L_h u^h|_{1,h}^2 \\
\eqsim &\sup_{v\in H_0^2(\Omega)} \frac{\big|\mathbf{F}(\mathbf{I}v) - A_{\varepsilon,h}(\mathbf{L}u^h, \mathbf{I}v)\big|}{\|\mathbf{I}v\|_{\varepsilon,h}} + \inf_{w\in H_0^2(\Omega)} \|\mathbf{L}u^h - \mathbf{I}w\|_{\varepsilon,h}.
\end{aligned}$$

We only need to estimate the two terms on the right-hand side.

Set $v \in H_0^2(\Omega)$. For $v^h \in V_0^h(\Omega)$, we have that

$$\mathbf{F}(\mathbf{I}v) - A_{\varepsilon,h}(\mathbf{L}u^h, \mathbf{I}v) = (f, v - L_h v^h) - A_{\varepsilon,h}(\mathbf{L}u^h, \mathbf{I}v - \mathbf{L}v^h).$$

Using integrating by parts, we have that, for $i, j, l \in \{1, 2\}$,

$$\int_T \partial_{ij} u^h \partial_{ij}(v - v^h) = -\int_T \partial_{iij} u^h \partial_j (v - v^h) + E_{ij}^T, \tag{3.13}$$

$$\int_T \partial_l L_h u^h \partial_l (v - L_h v^h) = -\int_T \partial_{ll} L_h u^h (v - L_h v^h) + E_l^T, \tag{3.14}$$

where

$$\begin{aligned}
E_{ij}^T = &\int_T (\partial_{ij} u^h - v_{ij}) \partial_{ij}(v - v^h) + \int_T \Big(v_{ij}\partial_{ij}(v - v^h) + \partial_i v_{ij} \partial_j (v - v^h)\Big) \\
&+ \int_T \partial_i (\partial_{ij} u^h - v_{ij}) \partial_j (v - v^h), \\
E_l^T = &\int_T (\partial_l L_h u^h - v_l) \partial_l (v - L_h v^h) + \int_T \Big(v_l \partial_l (v - L_h v^h) + \partial_l v_l (v - L_h v^h)\Big) \\
&+ \int_T \partial_l (\partial_l L_h u^h - v_l)(v - L_h v^h),
\end{aligned}$$

for any v_{ij}, $v_l \in H^1(\Omega)$. By the divergence theorem, (3.13) and (3.14), we have

$$
\begin{aligned}
&(f, v - L_h v^h) - A_{\varepsilon,h}(\mathbf{L}u^h, \mathbf{I}v - \mathbf{L}v^h) \\
=& \sum_{T\in\mathcal{T}_h(\Omega)} \int_T (f - \varepsilon^2\Delta^2 u^h + \Delta L_h u^h)(v - L_h v^h) + \sum_{T\in\mathcal{T}_h(\Omega)} \int_T \varepsilon^2 \Delta^2 u^h (v^h - L_h v^h) \\
&+ \sum_{T\in\mathcal{T}_h(\Omega)} \int_{\partial T} \varepsilon^2 \frac{\partial}{\partial \nu}\Delta u^h (v - v^h) + \sum_{T\in\mathcal{T}_h(\Omega)} \Big\{ \sum_{i,j=1}^{2} \varepsilon^2 E_{ij}^T + \sum_{l=1}^{2} E_l^T \Big\} \\
:=& I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

From the assumption A1, we can choose $v^h \in V_0^h(\Omega)$ such that

$$
\sum_{j=0}^{t+1} h_T^{-2(t+1-j)} |v - v^h|_{j,T}^2 \lesssim \sum_{T'\in S_h(T)} |v|_{t+1,T'}^2, \ \forall T \in \mathcal{T}_h(\Omega), \quad t = 0, 1; \tag{3.15}
$$

the property of the interpolation operator L_h and the inverse inequalities imply that,

$$
h_T^2 |v^h - L_h v^h|_{1,T}^2 + \|v^h - L_h v^h\|_T^2 \lesssim \sum_{T'\in S_h(T)} h_{T'}^2 |v^h|_{1,T'}^2. \tag{3.16}
$$

Hence by (3.15) and (3.16), we can obtain that

$$
|v - L_h v^h|_{1,T}^2 + h_T^{-2} \|v - L_h v^h\|_T^2 \lesssim \sum_{T'\in S_h(T)} h_{T'}^{2t} |v|_{1+t,T'}^2, \quad t = 0, 1. \tag{3.17}
$$

Therefore, by the Schwartz inequality and the inverse inequality,

$$
\begin{aligned}
|I_1| \lesssim& \sum_{T\in\mathcal{T}_h(\Omega)} \Big\{ \|f - \varepsilon^2\Delta^2 u^h + \Delta L_h u^h\|_T \|v - L_h v^h\|_T \Big\} \\
\lesssim& \sum_{T\in\mathcal{T}_h(\Omega)} \Big\{ \|f - \varepsilon^2\Delta^2 u^h + \Delta L_h u^h\|_T \\
&\times \min\big(h_T, \frac{h_T^2}{\varepsilon}\big) \Big(\sum_{T'\in S_h(T)} \Big\{ \varepsilon^2 |v|_{2,T'}^2 + |v|_{1,T'}^2 \Big\} \Big)^{1/2} \Big\} \\
\lesssim& \Big(\sum_{T\in\mathcal{T}(\Omega)} \Big\{ \min\big(h_T^2, \frac{h_T^4}{\varepsilon^2}\big) \|f - \varepsilon^2\Delta^2 u^h + \Delta L_h u^h\|_T^2 \Big\} \Big)^{1/2} \|\mathbf{I}v\|_{\varepsilon,h}, \\
|I_2| \lesssim& \sum_{T\in\mathcal{T}_h(\Omega)} \Big\{ \varepsilon h_T |u^h|_{3,T} \big(\sum_{T'\in S_h(T)} \varepsilon^2 |v|_{2,T'}^2 \big)^{1/2} \Big\} \lesssim \Big(\sum_{T\in\mathcal{T}_h(\Omega)} \varepsilon^2 h_T^2 |u^h|_{3,T}^2 \Big)^{1/2} \|\mathbf{I}v\|_{\varepsilon,h}.
\end{aligned}
$$

We have used the shape regularity several times, and once again for

$$
\Big\| \frac{\partial}{\partial \nu} \Delta u^h \Big\|_{\partial T} \lesssim h_T^{-1/2} |u^h|_{3,T}.
$$

Applying the Schwartz inequality, we get

$$
|I_3| \lesssim \Big(\sum_{T\in\mathcal{T}_h(\Omega)} \varepsilon^2 h_T^2 |u^h|_{3,T}^2 \Big)^{1/2} \|\mathbf{I}v\|_{\varepsilon,h}.
$$

By (2.10) and (2.11) and the inverse inequalities, we have that

$$\begin{aligned}
\Big|\sum_{T\in\mathcal{T}_h(\Omega)} E_{ij}^T\Big| &\lesssim \sum_{T\in\mathcal{T}_h(\Omega)} \Big\{\big(\|\partial_{ij}u^h - v_{ij}\|_T + h_T|v_{ij}|_{1,T}\big)\\
&\qquad\times\big(|v-v^h|_{2,T} + h_T^{-1}|v-v^h|_{1,T}\big)\Big\}, \qquad (3.18)\\
\Big|\sum_{T\in\mathcal{T}_h(\Omega)} E_{l}^T\Big| &\lesssim \sum_{T\in\mathcal{T}_h(\Omega)} \Big\{\big(\|\partial_{l}L_hu^h - v_{l}\|_T + h_T|v_{l}|_{1,T}\big)\\
&\qquad\times\big(|v-L_hv^h|_{1,T} + h_T^{-1}\|v-L_hv^h\|_{T}\big)\Big\}. \qquad (3.19)
\end{aligned}$$

By Lemma 3.3, given v^h, thus L_hv^h, we can choose v_{ij}, $v_l\in H^1(\Omega)$ such that

$$\begin{aligned}
&\|\partial_{ij}v^h - v_{ij}\|_T^2 + |T|\,|v_{ij}|_{1,T}^2\\
\lesssim\ & |T|\,|\partial_{ij}v^h|_{1,T}^2 + \sum_{T'\in S_h(T)}\ \sum_{F\in\partial\mathcal{T}_h^i, F\subset\partial T'} |F|\,\big\|\big[\partial_{ij}v^h\big]_{J,F}^{T'}\big\|_F^2,\\
&\|\partial_{l}L_hv^h - v_{l}\|_T^2 + |T|\,|v_{l}|_{1,T}^2\\
\lesssim\ & |T|\,|\partial_{l}L_hv^h|_{1,T}^2 + \sum_{T'\in S_h(T)}\ \sum_{F\in\partial\mathcal{T}_h^i, F\subset\partial T'} |F|\,\big\|\big[\partial_{l}L_hv^h\big]_{J,F}^{T'}\big\|_F^2.
\end{aligned}$$

Therefore we derive that,

$$\begin{aligned}
|I_4| &= \Big|\sum_{T\in\mathcal{T}_h(\Omega)}\Big\{\varepsilon^2\sum_{i,j=1}^2 E_{ij}^T + \sum_{l=1}^2 E_l^T\Big\}\Big|\\
&\lesssim \sum_{T\in\mathcal{T}_h(\Omega)}\Big\{\varepsilon^2\big(|T|\,|u^h|_{3,T}^2 + \sum_{T'\in S_h(T)} J_2^{T'}(u^h)\big)^{1/2}\big(\sum_{T'\in S_h(T)}|v|_{2,T'}^2\big)^{1/2}\Big\}\\
&\quad + \sum_{T\in\mathcal{T}_h(\Omega)}\Big\{\big(|T|\,|L_hu^h|_{2,T}^2 + \sum_{T'\in S_h(T)} J_1^{T'}(L_hu^h)\big)^{1/2}\big(\sum_{T'\in S_h(T)}|v|_{1,T'}^2\big)^{1/2}\Big\}\\
&\lesssim \Big(\sum_{T\in\mathcal{T}_h(\Omega)}\Big\{\varepsilon^2h_T^2|u^h|_{3,T}^2 + \varepsilon^2J_2^T(u^h) + h_T^2|L_hu^h|_{2,T}^2 + J_1^T(L_hu^h)\Big\}\Big)^{1/2}\big\|\mathbf{I}v\big\|_{\varepsilon,h}.
\end{aligned}$$

Combining all the inequalities above, and noting the arbitrariness of $v\in H_0^2(\Omega)$, we can obtain by the Schwartz inequality that

$$\sup_{v\in H_0^2(\Omega)}\frac{\big|\mathbf{F}(\mathbf{I}v) - A_{\varepsilon,h}(\mathbf{L}u^h,\mathbf{I}v)\big|}{\|\mathbf{I}v\|_{\varepsilon,h}} \lesssim \Big(\sum_{T\in\mathcal{T}_h(\Omega)} R^T(u^h)\Big)^{1/2}. \qquad (3.20)$$

We are to estimate $\inf\limits_{w\in H_0^2(\Omega)}\|\mathbf{L}u^h - \mathbf{I}w\|_{\varepsilon,h}$. We only need to choose a proper $w^h\in H_0^2(\Omega)$ and to estimate the difference. Set $\widetilde{u}^h$ defined piecewisely as

$$\widetilde{u}^h|_T = \begin{cases} u^h|_T, & \text{if } h_T\leqslant\varepsilon,\\ L_hu^h|_T, & \text{otherwise.}\end{cases}$$

Let $w^h = Q_h\widetilde{u}^h$, where the operator Q_h follows that in Lemma 3.4. For $\forall\, T\in\mathcal{T}_h(\Omega)$, by the definition of $\widetilde{u}^h$ and (2.9), direct computation leads to that,

$$
\begin{aligned}
&\varepsilon^2|u^h-w^h|_{2,T}^2+|L_hu^h-w^h|_{1,T}^2\\
=&\begin{cases}\varepsilon^2|\widetilde{u}^h-w^h|_{2,T}^2+|L_hu^h-w^h|_{1,T}^2, \text{ if } h_T\leqslant\varepsilon,\\ \varepsilon^2|u^h-w^h|_{2,T}^2+|\widetilde{u}^h-w^h|_{1,T}^2, \quad \text{otherwise};\end{cases}\\
\lesssim&\begin{cases}\varepsilon^2|\widetilde{u}^h-w^h|_{2,T}^2+|\widetilde{u}^h-w^h|_{1,T}^2+|u^h-L_hu^h|_{1,T}^2, \quad \text{if } h_T\leqslant\varepsilon,\\ \varepsilon^2|\widetilde{u}^h-w^h|_{2,T}^2+|\widetilde{u}^h-w^h|_{1,T}^2+\varepsilon^2|u^h-L_hu^h|_{2,T}^2, \text{otherwise};\end{cases}\\
\lesssim&\begin{cases}\varepsilon^2|\widetilde{u}^h-w^h|_{2,T}^2+|\widetilde{u}^h-w^h|_{1,T}^2+h_T^2\sum\limits_{T'\in S_h(T)}|u^h|_{2,T'}^2, \text{ if } h_T\leqslant\varepsilon,\\ \varepsilon^2|\widetilde{u}^h-w^h|_{2,T}^2+|\widetilde{u}^h-w^h|_{1,T}^2+\varepsilon^2\sum\limits_{T'\in S_h(T)}|u^h|_{2,T'}^2, \text{ otherwise};\end{cases}\\
=&\,\varepsilon^2|\widetilde{u}^h-w^h|_{2,T}^2+|\widetilde{u}^h-w^h|_{1,T}^2+\min(h_T^2,\varepsilon^2)\sum_{T'\in S_h(T)}|u^h|_{2,T'}^2.
\end{aligned}
$$

We point out here that, since the mesh is shape-regular, that $T'\in S_h(T)$ (with $h_{T'}\leqslant\varepsilon\leqslant h_T$ or $h_T\leqslant\varepsilon\leqslant h_{T'}$) implies that $h_T\eqsim h_{T'}$ ($h_T\eqsim h_{T'}\eqsim\varepsilon$). Taking summation on all the elements, we have that

$$
\|\mathbf{L}u^h-\mathbf{I}w^h\|_{\varepsilon,h}^2\lesssim\sum_{T\in\mathcal{T}_h(\Omega)}\left\{\varepsilon^2|\widetilde{u}^h-w^h|_{2,T}^2+|\widetilde{u}^h-w^h|_{1,T}^2+\min(h_T^2,\varepsilon^2)|u^h|_{2,T}^2\right\}. \tag{3.21}
$$

By Lemma 3.4 and the inverse inequality, we have for all $T\in\mathcal{T}_h(\Omega)$ that

$$
|\widetilde{u}^h-w^h|_{r,T}^2\lesssim h_T^{-2r}\sum_{T'\in S_h(T)}\Big\{J_0^{T'}(\widetilde{u}^h)+h_{T'}^2J_1^{T'}(\widetilde{u}^h)+h_{T'}^4J_2^{T'}(\widetilde{u}^h)\Big\},\ r=0,1,2.
$$

Take summation, make use of the shape regularity and we will obtain that

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h(\Omega)}\left\{\varepsilon^2|\widetilde{u}^h-w^h|_{2,T}^2+|\widetilde{u}^h-w^h|_{1,T}^2\right\}\\
\lesssim&\sum_{T\in\mathcal{T}_h(\Omega)}\Big\{\big(h_T^{-2}+\varepsilon^2h_T^{-4}\big)\Big(J_0^T(\widetilde{u}^h)+h_T^2J_1^T(\widetilde{u}^h)+h_T^4J_2^T(\widetilde{u}^h)\Big)\Big\}.
\end{aligned}
$$

Utilizing Lemmas 3.2 and 3.1, we can derive from the definition that

$$
J_0^T(\widetilde{u}^h)\lesssim\begin{cases}h_T^4J_2^T(u^h)+\sum\limits_{T'\in S_h(T),h_{T'}>\varepsilon}h_{T'}^4|u^h|_{2,T'}^2, \quad \text{if } h_T\leqslant\varepsilon,\\ h_T^2J_1^T(L_hu^h)+\sum\limits_{T'\in S_h(T),h_{T'}\leqslant\varepsilon}h_{T'}^4|u^h|_{2,T'}^2, \quad \text{otherwise};\end{cases}
$$

$$
h_T^2J_1^T(\widetilde{u}^h)\lesssim\begin{cases}h_T^4J_2^T(u^h)+\sum\limits_{T'\in S_h(T),h_{T'}>\varepsilon}h_{T'}^4|u^h|_{2,T'}^2, \quad \text{if } h_T\leqslant\varepsilon,\\ h_T^2J_1^T(L_hu^h)+\sum\limits_{T'\in S_h(T),h_{T'}\leqslant\varepsilon}h_{T'}^4|u^h|_{2,T'}^2, \quad \text{otherwise};\end{cases}
$$

and

$$
h_T^4J_2^T(\widetilde{u}^h)\lesssim\begin{cases}h_T^4J_2^T(u^h)+\sum\limits_{T'\in S_h(T),h_{T'}>\varepsilon}h_{T'}^4|u^h|_{2,T'}^2, \quad \text{if } h_T\leqslant\varepsilon,\\ \sum\limits_{T'\in S_h(T)}h_{T'}^4|L_hu^h|_{2,T'}^2+\sum\limits_{T'\in S_h(T),h_{T'}\leqslant\varepsilon}h_{T'}^4|u^h|_{2,T'}^2, \quad \text{otherwise.}\end{cases}
$$

Therefore, on the elements where $h_T \leqslant \varepsilon$, we have that,

$$\begin{aligned}
&\big(h_T^{-2}+\varepsilon^2 h_T^{-4}\big)\Big(J_0^T\big(\widetilde{u}^h\big)+h_T^2 J_1^T\big(\widetilde{u}^h\big)+h_T^4 J_2^T\big(\widetilde{u}^h\big)\Big)\\
&\lesssim \big(h_T^{-2}+\varepsilon^2 h_T^{-4}\big)\Big(h_T^4 J_2^T(u^h)+\sum_{T'\in S_h(T),h_{T'}>\varepsilon} h_{T'}^4|u^h|_{2,T'}^2\Big)\\
&\lesssim \varepsilon^2 J_2^T(u^h)+\sum_{T'\in S_h(T),h_{T'}>\varepsilon}\min(h_{T'}^2,\varepsilon^2)|u^h|_{2,T'}^2.
\end{aligned}$$

Similarly, on the elements where $h_T > \varepsilon$, we can show that

$$\begin{aligned}
&\big(h_T^{-2}+\varepsilon^2 h_T^{-4}\big)\Big(J_0^T\big(\widetilde{u}^h\big)+h_T^2 J_1^T\big(\widetilde{u}^h\big)+h_T^4 J_2^T\big(\widetilde{u}^h\big)\Big)\\
&\lesssim J_1^T\big(L_h u^h\big)+\sum_{T'\in S_h(T)} h_{T'}^2|L_h u^h|_{2,T'}^2+\sum_{T'\in S_h(T),h_{T'}\leqslant\varepsilon}\min(h_{T'}^2,\varepsilon^2)|u^h|_{2,T'}^2.
\end{aligned}$$

Taking summation on all the elements and noting the shape regularity of the elements, we can obtain the estimate from (3.21) that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h(\Omega)}\Big\{\varepsilon^2|u^h-w^h|_{2,T}^2+|L_h u^h-w^h|_{1,T}^2\Big\}\\
&\lesssim \sum_{T\in\mathcal{T}_h(\Omega)}\Big\{\varepsilon^2 J_2^T(u^h)+J_1^T(L_h u^h)+h_T^2|L_h u^h|_{2,T}^2+\min(h_T^2,\,\varepsilon^2)|u^h|_{2,T}^2\Big\}. \qquad (3.22)
\end{aligned}$$

Combining (3.20) and (3.22) yields the desired result. $\square$

Remark 3.2. If the interpolation operator is identity, i.e., $L_h = I$, and $V_0^h(\Omega)\subset H_0^1(\Omega)$, then the estimator can be simplified as

$$\begin{aligned}
&\varepsilon^2|u-u^h|_{2,h}^2+|u-u^h|_{1,h}^2\\
&\lesssim \sum_{T\in\mathcal{T}_h(\Omega)}\Big\{\varepsilon^2 h_T^2|u^h|_{3,T}^2+\varepsilon^2 J_2^T(u^h)+J_1^T(u^h)+\min\big(h_T^2,\frac{h_T^4}{\varepsilon^2}\big)\|f-\varepsilon^2\Delta^2 u^h+\Delta u^h\|_T^2\Big\}. \quad (3.23)
\end{aligned}$$

4. Efficiency of the Estimator

The reliability of the estimator has been proven, and it remains to show its efficiency. We will demonstrate that the error estimators can be controlled by the true error up to an extra term.

Lemma 4.1. *Let $w^h\in P_{ch}(\Omega)$. Then it holds for $w\in H^3(\Omega)\cap H_0^2(\Omega)$ that*

$$\begin{aligned}
R^T(w^h)\lesssim \sum_{T'\in S_h(T)}&\Big\{\big(\varepsilon^2|w-w^h|_{2,T'}^2+|w-L_h w^h|_{1,T'}^2\big)\\
&+h_{T'}^2\big(\varepsilon^2|w|_{3,T'}^2+|w|_{2,T'}^2\big)+\min\big(h_{T'}^2,\frac{h_{T'}^4}{\varepsilon^2}\big)\|f\|_{T'}^2\Big\}. \qquad (4.1)
\end{aligned}$$

Proof. We shall estimate each term in the estimator respectively. Since $w\in H^3(\Omega)\cap H_0^2(\Omega)$, by the interpolation theory, there exists $v^h\in P_{ch}(\Omega)$, such that

$$\sum_{j=0}^{2} h_T^{2j}|(w-v^h)|_{j,T}^2\lesssim h_T^{4+2s}|w|_{2+s,T}^2,\ \forall T\in\mathcal{T}_h(\Omega),\ s\in\{0,1\}. \qquad (4.2)$$

Therefore, by the inverse inequality, we obtain that

$$
\begin{aligned}
|w^h|_{3,T}^2 &\lesssim |w^h - v^h|_{3,T}^2 + |v^h|_{3,T}^2 \lesssim h_T^{-2}|w^h - v^h|_{2,T}^2 + |w|_{3,T}^2 \\
&\lesssim h_T^{-2}|w^h - w|_{2,T}^2 + h_T^{-2}|w - v^h|_{2,T}^2 + |w|_{3,T}^2 \\
&\lesssim h_T^{-2}|w^h - w|_{2,T}^2 + |w|_{3,T}^2. \qquad (4.3)
\end{aligned}
$$

Similarly, we can prove that

$$
|L_h w^h|_{2,T}^2 \lesssim h_T^{-2}|w - L_h w^h|_{1,T}^2 + |w|_{2,T}^2. \qquad (4.4)
$$

Hence, also by the inverse inequality, it is derived that

$$
\begin{aligned}
&\min\left(h_T^2, h_T^4/\varepsilon^2\right)\|f - \varepsilon^2\Delta^2 w^h + \Delta L_h w^h\|_T^2 \\
\lesssim\ &\min\left(h_T^2, h_T^4/\varepsilon^2\right)\|f\|_T^2 + h_T^4/\varepsilon^2\|\varepsilon^2\Delta^2 w^h\|_T^2 + h_T^2\|\Delta L_h w^h\|_T^2 \\
\lesssim\ &\min\left(h_T^2, h_T^4/\varepsilon^2\right)\|f\|_T^2 + h_T^2\varepsilon^2|w^h|_{3,T}^2 + h_T^2|L_h w^h|_{2,T}^2 \\
\lesssim\ &\min\left(h_T^2, h_T^4/\varepsilon^2\right)\|f\|_T^2 + \left(\varepsilon^2|w - w^h|_{2,T}^2 + |w - L_h w^h|_{1,T}^2\right) + h_T^2\left(\varepsilon^2|w|_{3,T}^2 + |w|_{2,T}^2\right). \qquad (4.5)
\end{aligned}
$$

Let F be a side of T. When $F \not\subset \partial\Omega$, there exists $T' \in \mathcal{T}_h$, such that $F = T' \cap T$. For $|\alpha| = 2$, by the inverse inequality, we have that

$$
\begin{aligned}
&\left\|\left[\partial_{\nu s}^\alpha w^h\right]_{J,F}^T\right\|_F^2 \\
=\ &\left\|\left[\partial_{\nu s}^\alpha (w^h - w)\right]_{J,F}^T\right\|_F^2 \lesssim \left\|\left[\partial_{\nu s}^\alpha (w^h - v^h)\right]_{J,F}^T\right\|_F^2 + \left\|\left[\partial_{\nu s}^\alpha (v^h - w)\right]_{J,F}^T\right\|_F^2 \\
\lesssim\ &h_T^{-1}|w^h - v^h|_{2,T\cup T'}^2 + h_T|w|_{3,T\cup T'}^2 \lesssim h_T^{-1}|w^h - w|_{2,T\cup T'}^2 + h_T|w|_{3,T\cup T'}^2
\end{aligned}
$$

with v^h here chosen to satisfy (4.2). When $F \subset \partial\Omega$, for $|\alpha| = 2$, $\alpha_1 < 2$, by the inverse inequality, noting that $\partial_{\nu s}^\alpha w\big|_F = 0$, we have

$$
\begin{aligned}
\left\|\partial_{\nu s}^\alpha w^h\right\|_F^2 &= \left\|\partial_{\nu s}^\alpha (w^h - w)\right\|_F^2 \lesssim \left\|\partial_{\nu s}^\alpha (w^h - v^h)\right\|_F^2 + \left\|\partial_{\nu s}^\alpha (v^h - w)\right\|_F^2 \\
&\lesssim h_T^{-1}\left|w^h - v^h\right|_{2,T}^2 + h_T\left|w\right|_{3,T}^2.
\end{aligned}
$$

Then

$$
J_2^T(w^h) \lesssim \sum_{T' \in S_h(T)} |w - w^h|_{2,T'}^2 + \sum_{T' \in S_h(T)} h_{T'}^2 |w|_{3,T'}^2. \qquad (4.6)
$$

Similarly, we can prove that

$$
J_1^T(L_h w^h) \lesssim \sum_{T' \in S_h(T)} |w - L_h w^h|_{1,T'}^2 + \sum_{T' \in S_h(T)} h_{T'}^2 |w|_{2,T'}^2. \qquad (4.7)
$$

As for the remaining term, we have the result that

$$
\begin{aligned}
\min(h_T^2, \varepsilon^2)|w^h|_{2,T}^2 &\leqslant \min(h_T^2, \varepsilon^2)(|w^h - w|_{2,T}^2 + |w|_{2,T}^2) \\
&\leqslant \varepsilon^2|w - w^h|_{2,T}^2 + h_T^2|w|_{2,T}^2. \qquad (4.8)
\end{aligned}
$$

Combining (4.3)-(4.8), we obtain the estimate (4.1). $\square$

Lemma 4.2. *Let $w^h \in P_{ch}(\Omega)$. Then for $w \in H^3(\Omega) \cap H_0^2(\Omega)$ and $w^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, it holds for all $T \in \mathcal{T}_h(\Omega)$ that*

$$
\begin{aligned}
R^T(w^h) \lesssim \sum_{T' \in S_h(T)} &\Big\{\left(\varepsilon^2|w - w^h|_{2,T'}^2 + |L_h w^h - w^0|_{1,T'}^2\right) \\
&+\left(\varepsilon^2|w|_{2,T'}^2 + h_{T'}^2|w^0|_{2,T'}^2 + h_{T'}^2\|f\|_{T'}^2\right)\Big\}. \qquad (4.9)
\end{aligned}
$$

Proof. By the inverse inequality,

$$\min\big(h_T^2, h_T^4/\varepsilon^2\big)\|f-\varepsilon^2\Delta^2 w^h+\Delta L_h w^h\|_T^2 \lesssim h_T^2\|f\|_T^2+\varepsilon^2|w^h|_{2,T}^2+h_T^2|L_h w^h|_{2,T}^2.$$

Similar to (4.7), we can show that

$$J_1^T(L_h w^h) \lesssim \sum_{T'\in S_h(T)} |w^0-L_h w^h|_{1,T'}^2+\sum_{T'\in S_h(T)} h_{T'}^2|w^0|_{2,T'}^2. \tag{4.10}$$

Using the same argument as (4.2) and (4.3), we can prove that

$$h_T^2|L_h w^h|_{2,T}^2 \lesssim |L_h w^h-w^0|_{1,T}^2+h_T^2|w^0|_{2,T}^2.$$

As for the remaining, we have

$$\varepsilon^2|w^h|_{2,T}^2 \leqslant \varepsilon^2|w^h-w|_{2,T}^2+\varepsilon^2|w|_{2,T}^2.$$

Combining all the inequalities above together and noting that

$$J_2^T(w^h) \lesssim \sum_{T'\in S_h(T)} |w^h|_{2,T'}^2,$$

we obtain the conclusion. $\square$

Define for $T\in\mathcal{T}_h(\Omega)$, $w\in H^3(T)$ and $w^0\in H^2(T)$,

$$E_1^T(w,w^0)=h_T^2\big(\varepsilon^2|w|_{3,T}^2+|w|_{2,T}^2\big)+\min\big(h_T^2,h_T^4/\varepsilon^2\big)\|f\|_T^2,$$
$$E_2^T(w,w^0)=\varepsilon^2|w|_{2,T}^2+h_T^2|w^0|_{2,T}^2+h_T^2\|f\|_T^2+|w-w^0|_{1,T}^2.$$

Combining the two lemmas above, we can obtain the following upper bounds for the estimator.

Theorem 4.1. *Let $w^h\in P_{ch}(\Omega)$. Then for $w\in H^3(\Omega)\cap H_0^2(\Omega)$ and $w^0\in H^2(\Omega)\cap H_0^1(\Omega)$ the following estimate holds*

$$\sum_{T\in\mathcal{T}_h(\Omega)} R^T(w^h) \lesssim \varepsilon^2|w-w^h|_{2,h}^2+|w-L_h w^h|_{1,h}^2+\sum_{T\in\mathcal{T}_h(\Omega)} E_k^T(w,w^0),\ k=1,2. \tag{4.11}$$

By the theorem, the efficiency is proved once we have shown the equivalence between the convergence of the true error and that of an upper bound of the estimator. In fact, the *a posteriori* estimator would be shown to be an optimal one if it could possess the same convergence rates as the error itself. In the way similar to that of [16, 25, 30], we can prove the following result.

Lemma 4.3. *Assume that Assumptions A1-A4 hold. Let u and u^h be the solutions of (WP) and (DWP) respectively. If $u\in H^3(\Omega)\cap H_0^2(\Omega)$, then*

$$\varepsilon^2\|u-u^h\|_{2,h}^2+\|u-L_h u^h\|_{1,h}^2 \lesssim h^2(\varepsilon^2|u|_{3,\Omega}^2+|u|_{2,\Omega}^2)+\min\big(h^2,\frac{h^4}{\varepsilon^2}\big)\|f\|_\Omega^2. \tag{4.12}$$

So far, we have not seen an *a priori* estimate with sharper convergence rate for the nonconforming finite element methods. Making use of the extra term E_1^T we have the following optimal estimation theorem by Theorem 4.1 and Lemma 4.3.

Theorem 4.2. *Assume that Assumptions A1-A4 hold. Let u and u^h be the solutions of (WP) and (DWP) respectively. Then if $u \in H^3(\Omega) \cap H_0^2(\Omega)$, we have*

$$\sum_{T \in \mathcal{T}_h(\Omega)} R^T(u^h) \lesssim h^2(\varepsilon^2 |u|_{3,\Omega}^2 + |u|_{2,\Omega}^2) + \min\big(h^2, \frac{h^4}{\varepsilon^2}\big)\|f\|_\Omega^2. \tag{4.13}$$

Remark 4.1. If Ω is convex, then we can obtain the following robust estimation through more careful arguments (see also [16, 25, 30]):

$$\varepsilon^2 \|u - u^h\|_{2,h}^2 + \|u - L_h u^h\|_{1,h}^2 \lesssim h\|f\|_\Omega^2. \tag{4.14}$$

Assume that u^h is derived via some finite element method and satisfies assumptions stronger than A1-A4, such as the modified Morley method and the modified Zienkiewicz method. Therefore, applying the extra terms E_1^T when $h \leqslant \varepsilon$ and E_2^T when $h > \varepsilon$, we get the estimation by (4.14), Theorem 4.1 and Lemma 2.1 that

$$\sum_{T \in \mathcal{T}_h(\Omega)} R^T(u^h) \lesssim h\|f\|_\Omega^2. \tag{4.15}$$

5. Numerical Examples

Several finite element methods of the type described in Section 2 have been applied to the elliptic perturbation problems, such as the modified Zienkiewicz method and the modified Morley method. The modified Zienkiewicz method is to make use of modified Zienkiewicz element; we refer to [29] for details. For the lower order part, the interpolation operator is the identity operator.

As for the modified Morley method, the finite element space follows the Morley element, see [24]. When utilized to discretize the model problem, the interpolation operator L_h for lower order part is no longer the identity. Instead, it uses the interpolation operator corresponding to the Courant triangle; see [30].

The convergences of the two methods are guaranteed by Lemma 4.3.

The behavior of the estimator is illustrated for a representative problem in this section. We shall solve the model problem numerically on an initial coarse mesh, and refine the mesh according to the *a posteriori* error estimation, and repeat this procedure. The sequence of meshes will be constructed adaptively by selecting for refinement all elements where the local error indicator exceeds a pre-given threshold value. To refine a triangle T when necessary, we divide it into four sub-triangles by connecting the midpoints of the edges of T, and eliminate the hanging points. The refinement strategy is shown in Fig. 5.1.

Example 5.1. The domain for computing is $[0,1] \times [0,1]$, and the model problem is

$$\varepsilon^2 \Delta^2 u - \Delta u = f.$$

We set the solution of the model problem to be

$$u = \big(\sin(\pi x)\sin(\pi y)\big)^2,$$

and f and the boundary conditions are chosen to satisfy the solution.

We make the parameter $\varepsilon > 0$ vary in the computations. The initial mesh is shown in Fig. 5.2. In this example, the threshold value of the error indicator on each element is 10^{-3}.

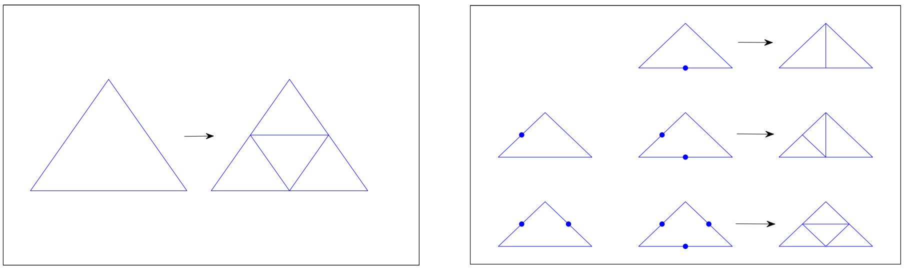

Fig. 5.1. The mesh-refine strategy. Left is the method to refine a chosen element, and right is the method to eliminate the hanging points.

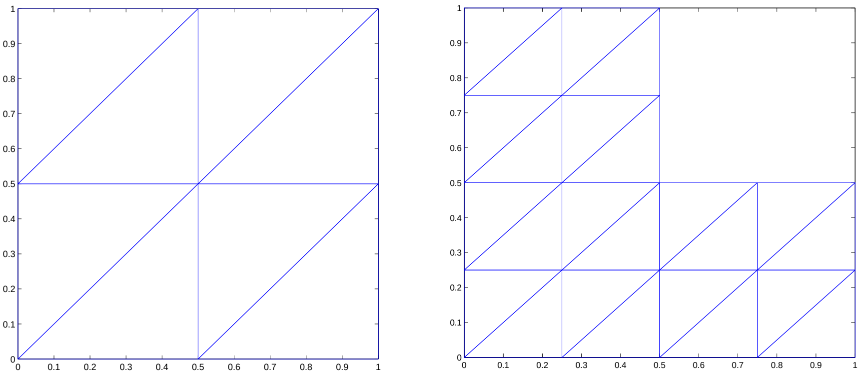

Fig. 5.2. Original triangulations before refinement. Left is on the unit square, and right is on the 'L-'shape domain.

On each mesh, we compute the quantities $\|u - u^h\|_{\varepsilon,h}$ and $R(u^h)$ at each element. A numerical quadrature rule using seven quadrature points per element was used. In particular, for those that are known to be piecewise polynomials, their numerical norms are computed exactly.

Firstly we make the parameter ε vary, and the performances of the estimator are shown in Table 5.1, where

$$ERR = \|u - u^h\|_{\varepsilon,h}, \quad \text{and} \quad EST = \big(\sum_{T\in\mathcal{T}_h} R^T(u^h)\big)^{1/2}.$$

All data are collected when the computation and refinements have stopped. We see that the proportion between the error and the estimator is robust in the parameter, even when ε is small. Fig. 5.3 shows the final meshes obtained with the modified Morley method and the modified Zienkiewicz method.

Example 5.2. The domain for computing is still $[0, 1] \times [0, 1]$, and the model problem is (5.1). We set the solution of the model problem to be

$$u = 1 - e^{-\frac{x+y}{\sqrt{2\varepsilon}}},$$

and f and the boundary conditions are chosen to satisfy the solution.

For this example, we set the threshold value to be 10^{-5}, and make $\varepsilon > 0$ vary. For each ε, we calculate ERR and EST on the final mesh and list them in Table 5.2. Fig. 5.4 shows the final meshes obtained using two finite element methods with $\varepsilon = 10^{-4}$.

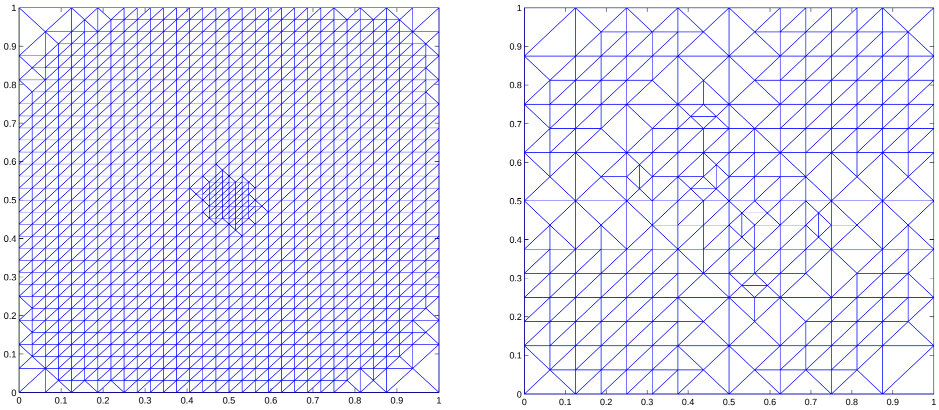

Fig. 5.3. Final meshes of Example 5.1. Left is via modified Morley method, and right is via modified Zienkiewicz method. $\varepsilon = 10^{-3}$.

Table 5.1: Performance of estimators with respect to ε in Example 5.1.

ε	modified Zienkiewicz method			modified Morley method		
	ERRor	ESTimator	EST/ERR	ERRor	ESTimator	EST/ERR
10^{-1}	0.0760	0.5531	7.28	0.1592	0.8790	5.52
10^{-2}	0.0275	0.1435	5.22	0.1384	0.7983	5.77
10^{-3}	0.0346	0.1606	4.64	0.1378	0.7837	5.69
10^{-4}	0.0348	0.1615	4.64	0.1378	0.7835	5.69

Table 5.2: Performance of estimators with respect to ε in Example 5.2.

ε	modified Zienkiewicz method			modified Morley method		
	ERRor	ESTimator	EST/ERR	ERRor	ESTimator	EST/ERR
10^{-1}	1.5117E-2	8.5072E-2	5.63	2.6503E-2	1.3489E-1	5.09
10^{-2}	1.4866E-2	8.3468E-2	5.61	2.5803E-2	1.3134E-1	5.09
10^{-3}	1.4964E-2	8.4590E-2	5.65	2.5635E-2	1.3048E-1	5.09
10^{-4}	1.4853E-2	8.4344E-2	5.68	2.6492E-2	1.3483E-1	5.09

Table 5.3: Performance of estimators with respect to ε in Example 5.3.

ε	modified Zienkiewicz method			modified Morley method		
	ERRor	ESTimator	EST/ERR	ERRor	ESTimator	EST/ERR
10^{-1}	1.9456E-2	1.3366E-1	6.87	4.4615E-2	2.2749E-1	5.77
10^{-2}	1.2334E-2	8.5135E-2	6.90	3.5617E-2	2.0948E-1	5.88
10^{-3}	8.4258E-3	4.7328E-2	5.62	3.3513E-2	1.9964E-1	5.96
10^{-4}	7.3541E-3	4.0581E-2	5.62	3.3112E-2	1.9113E-1	5.77

Example 5.3. In this example, the governing equation is (5.1) and the computing domain is the closure of $\big([0,1]\times[0,1]\big)\backslash\big([0.5,1]\times[0.5,1]\big)$. We let the solution of the model problem be

$$u = 1 - e^{-\frac{(x-0.5)^2+(y-0.5)^2}{\varepsilon}},$$

and choose f and boundary conditions correspondingly.

The initial mesh is shown in Fig. 5.2, and we follow the refinement strategy as in the previous two examples. We set the threshold value to be 10^{-5} and make $\varepsilon > 0$ vary. For each ε, we

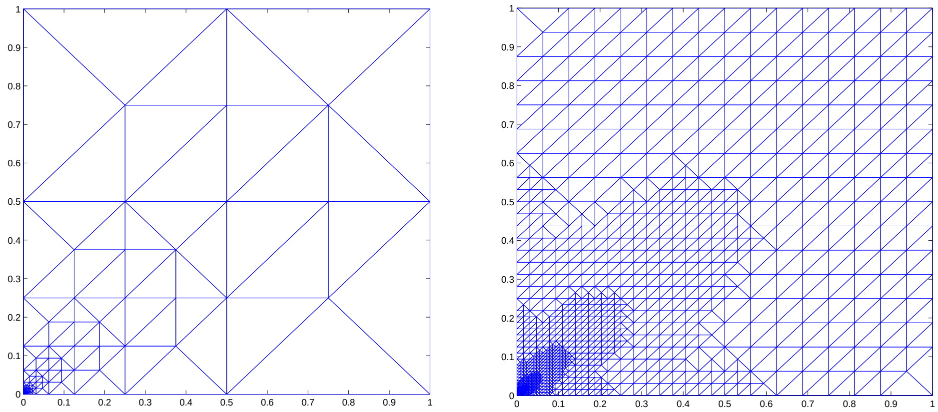

Fig. 5.4. Final meshes of Example 5.2. Left is via modified Morley method, and right is via modified Zienkiewicz method. $\varepsilon = 10^{-4}$.

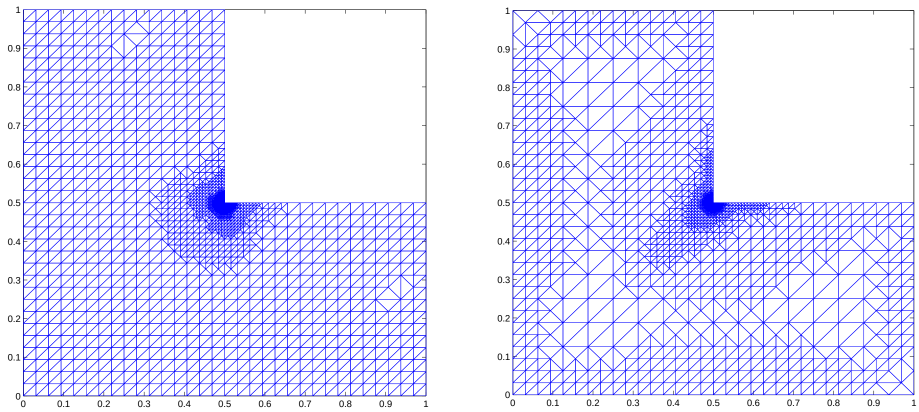

Fig. 5.5. Final meshes of Example 5.3. Left is via modified Morley method, and right is via modified Zienkiewicz method. $\varepsilon = 10^{-4}$.

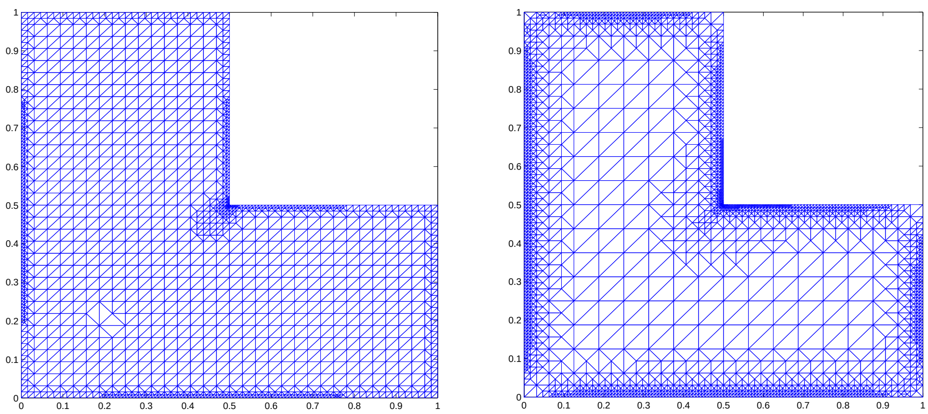

Fig. 5.6. Final meshes of Example 5.4. Left is via modified Morley method, and right is via modified Zienkiewicz method. $\varepsilon = 10^{-4}$.

calculate ERR and EST on the final mesh, which are listed in Table 5.3. Fig. 5.5 shows the final meshes obtained with $\varepsilon = 10^{-4}$.

Example 5.4. In this example, the computing domain is also the "L-"shape domain as the previous example. The problem is the model problem (5.1), and we set $f \equiv 1$ and choose homogeneous boundary conditions.

We set the threshold value to be 10^{-5} and make $\varepsilon > 0$ vary. Fig. 5.6 shows the final meshes obtained with each method, $\varepsilon = 10^{-4}$.

6. Analysis on the Local Behavior of the Estimator

In this section, we provide an analysis for the local behavior of the estimator. We mainly follow [32] and [31].

Given a function $\rho \in C^\infty(\overline{\Omega})$, we introduce the bilinear form

$$A_{h,\rho}(\mathbf{w},\mathbf{v}) = \sum_{T\in\mathcal{T}_h(\Omega)} \int_T \rho^2 \Big(\varepsilon^2 \sum_{i,j=1}^{2} \partial_{ij} w_1 \partial_{ij} v_1 + \sum_{l=1}^{2} \partial_l w_2 \partial_l v_2\Big),$$

where $\mathbf{w} = (w_1, w_2)^\top, \mathbf{v} = (v_1, v_2)^\top \in \big(H^2(\mathcal{T}_h)\big)^2$. For technical reasons, the following lemmas are needed.

Lemma 6.1. *Let A2 and A3 be true. Let G be a subdomain of Ω and $\varphi \in C^\infty(\overline{\Omega})$ with $\operatorname{supp}(\varphi) \subset\subset G$. Then for $i,j \in \{1,2\}$ and any $v, w \in H_0^2(\Omega) + V_0^h(\Omega)$,*

$$\Big|\sum_{T\in\mathcal{T}_h(\Omega)} \int_T \partial_{ij} v \varphi w\Big| \lesssim \sum_{T\in S_h(G)} \big(h_T|v|_{2,T} + |v|_{1,T}\big)\big(h_T\|w\|_{2,T} + \|w\|_{1,T}\big). \tag{6.1}$$

Proof. Let Π_h^C be the Clément interpolation operator from $L^2(\Omega)$ to $P_{1h}(\Omega)$, $P_{1h}(\Omega)$ the linear finite element space when the triangulation consists of triangles and the bilinear finite element space when each cell is a rectangle. For $w \in H_0^2(\Omega) + V_0^h(\Omega)$, $\Pi_h^C(\varphi w) \in H^1(\Omega)$, and the *weak continuity* assumption A3 gives

$$\sum_{j=0}^{s} h_T^{2j} |\varphi w - \Pi_h^C(\varphi w)|_{j,T}^2 \lesssim \sum_{T'\in S_h(T)} h_{T'}^{2s} |\varphi w|_{s,T'}^2, \quad s = 1,2, \qquad \forall T \in \mathcal{T}_h(\Omega).$$

Then from the *consistency* assumption A2, and approximation property, we obtain that

$$\begin{aligned}
\Big|\sum_{T\in\mathcal{T}_h(\Omega)} \int_T \partial_{ij} v \varphi w\Big| &= \Big|\sum_{T\in\mathcal{T}_h(\Omega)} \int_T \Big(\partial_{ij} v \Pi_h^C(\varphi w) + \partial_i v \partial_j \Pi_h^C(\varphi w)\Big) \\
&\quad - \sum_{T\in\mathcal{T}_h(\Omega)} \int_T \partial_i v \partial_j \Pi_h^C(\varphi w) + \sum_{T\in\mathcal{T}_h(\Omega)} \int_T \partial_{ij} v \Big(\varphi w - \Pi_h^C(\varphi w)\Big)\Big| \\
&\lesssim \sum_{T\in S_h(G)} \Big(h_T|v|_{2,T}\big|\Pi_h^C(\varphi w)\big|_{1,T} + |v|_{1,T}\big|\Pi_h^C(\varphi w)\big|_{1,T} + h_T^2|v|_{2,T}|\varphi w|_{2,T}\Big) \\
&\lesssim \sum_{T\in S_h(G)} \Big(h_T|v|_{2,T}\big(|\varphi w|_{1,T} + h_T|\varphi w|_{2,T}\big) \\
&\quad + |v|_{1,T}\big(|\varphi w|_{1,T} + h_T|\varphi w|_{2,T}\big) + h_T^2|v|_{2,T}|\varphi w|_{2,T}\Big).
\end{aligned}$$

Therefore the lemma follows. $\square$

Lemma 6.2. *Let A2 and A3 be true. Let Ω_0 be a subdomain of Ω, and $\rho \in C^{\infty}(\overline{\Omega})$ with* $\mathrm{supp}(\rho) \subset\subset \Omega_0$. *Then for* $\mathbf{v}$, $\mathbf{w} \in \big(H_0^2(\Omega) + V_0^h(\Omega)\big) \times \big(H_0^2(\Omega) + L_h V_0^h(\Omega)\big)$,

$$A_{h,\rho}(\mathbf{v},\mathbf{v}) \lesssim \big|A_h(\mathbf{v},\rho^2\mathbf{v})\big| + \sum_{T\in S_h(\Omega_0)} \{h_T^2\varepsilon^2\|v_1\|_{2,T}^2 + \varepsilon^2\|v_1\|_{1,T}^2 + \|v_2\|_T^2\}, \tag{6.2}$$

$$\begin{aligned}\big|A_h(\mathbf{v},\rho^2\mathbf{w})\big| &\lesssim \big|A_{h,\rho}(\mathbf{v},\mathbf{w})\big| + \big(A_{h,\rho}(\mathbf{v},\mathbf{v})\big)^{1/2}\Big(\sum_{T\in S_h(\Omega_0)} \varepsilon^2|w_1|_{1,T}^2 + \|w_2\|_T^2\Big)^{1/2} \\ &\quad + \sum_{T\in S_h(\Omega_0)} \Big\{\varepsilon^2(h_T|v_1|_{2,T} + |v_1|_{1,T})(h_T\|w_1\|_{2,T} + \|w_1\|_{1,T})\Big\}.\end{aligned} \tag{6.3}$$

Proof. Direct calculation leads to that

$$\begin{aligned}&\partial_{ij}v_1\partial_{ij}(\rho^2 w_1) = \rho^2\partial_{ij}v_1\partial_{ij}w_1 + \partial_{ij}v_1\partial_{ij}\rho^2 w_1 + 2\rho\partial_{ij}v_1(\partial_i\rho\partial_j w_1 + \partial_i w_1\partial_j\rho),\\ &\partial_l v_2\partial_l(\rho^2 w_2) = \rho^2\partial_l v_2\partial_l w_2 + 2\rho\partial_l v_2\partial_l\rho w_2.\end{aligned}$$

Then we have

$$\begin{aligned}A_h(\mathbf{v},\rho^2\mathbf{w}) = A_{h,\rho}(\mathbf{v},\mathbf{w}) + \sum_{T\in S_h(\Omega_0)} \Big\{\varepsilon^2\sum_{i,j=1}^{2}\int_T \partial_{ij}v_1\partial_{ij}\rho^2 w_1 \\ +\varepsilon^2\sum_{i,j=1}^{2}\int_T 2\rho\partial_{ij}v_1(\partial_i w_1\partial_j\rho + \partial_i\rho\partial_j w_1) + \sum_{i,j=1}^{2}\int_T 2\rho\partial_l v_2\partial_l\rho w_2\Big\}.\end{aligned}$$

Thus by Lemma 6.1 and Schwartz inequality, we obtain (6.3). The estimate (6.2) can be obtained in a similar way. □

Theorem 6.1. *Let A1-A4 be true. Let D be a subdomain of Ω, and Ω_0 be a subdomain of Ω such that $D \subset\subset \Omega_0$. If u and u^h are the solutions of (WP) and (DWP) respectively, then*

$$\begin{aligned}&\varepsilon^2|u-u^h|_{2,h,D}^2 + |u-L_h u^h|_{1,h,D}^2\\ \lesssim &\sum_{T\in S_h(\Omega_0)} \Big\{R^T(u^h) + \varepsilon^2 h_T^2\|u-u^h\|_{2,T}^2 + \varepsilon^2\|u-u^h\|_{1,T}^2 + \|u-L_h u^h\|_T^2\Big\}.\end{aligned} \tag{6.4}$$

Proof. Let $\rho \in C^{\infty}(\overline{\Omega})$, with $\mathrm{supp}\,(\rho) \subset\subset \Omega_0$, and $\rho|_D \equiv 1$. Then it is obvious that

$$\varepsilon^2|u-u^h|_{2,h,D}^2 + |u-L_h u^h|_{1,h,D}^2 \lesssim A_{h,\rho}(\mathbf{I}u-\mathbf{L}u^h, \mathbf{I}u-\mathbf{L}u^h).$$

It follows from Lemma 6.2 that

$$\begin{aligned}A_{h,\rho}(\mathbf{I}u-\mathbf{L}u^h, \mathbf{I}u-\mathbf{L}u^h) &\lesssim \big|A_h\big(\mathbf{I}u-\mathbf{L}u^h, \rho^2(\mathbf{I}u-\mathbf{L}u^h)\big)\big|\\ &+ \sum_{T\in S_h(\Omega_0)} \Big\{\varepsilon^2 h_T^2\|u-u^h\|_{2,T}^2 + \varepsilon^2\|u-u^h\|_{1,T}^2 + \|u-L_h u^h\|_T^2\Big\}.\end{aligned} \tag{6.5}$$

For $w \in H_0^2(\Omega)$, we have that

$$\begin{aligned}&A_h\big(\mathbf{I}u-\mathbf{L}u^h, \rho^2(\mathbf{I}u-\mathbf{L}u^h)\big)\\ &= A_h\big(\mathbf{I}u-\mathbf{L}u^h, \rho^2\mathbf{I}(u-w)\big) + A_h\big(\mathbf{I}u-\mathbf{L}u^h, \rho^2(\mathbf{I}w-\mathbf{L}u^h)\big).\end{aligned} \tag{6.6}$$

Similar to (3.12), we define $\|\cdot\|_{\varepsilon,T}$ and $\|\cdot\|_{\varepsilon,G}$, where G is a subdomain of Ω. As for the second term in the right-hand side of (6.6), by Lemma 6.2, we can derive that

$$\begin{aligned}
&\Big|A_h\big(\mathbf{I}u-\mathbf{L}u^h,\rho^2(\mathbf{I}w-\mathbf{L}u^h)\big)\Big| \\
\lesssim &\,\big|A_{h,\rho}(\mathbf{I}u-\mathbf{L}u^h,\mathbf{I}w-\mathbf{L}u^h)\big| \\
&+A_{h,\rho}(\mathbf{I}u-\mathbf{L}u^h,\mathbf{I}u-\mathbf{L}u^h)^{1/2}\Big(\sum_{T\in S_h(\Omega_0)}\big\{\varepsilon^2|w-u^h|_{1,T}^2+\|w-L_hu^h\|_T^2\big\}\Big)^{1/2} \\
&+\sum_{T\in S_h(\Omega_0)}\varepsilon^2\Big(h_T^2|u-u^h|_{2,T}^2+|u-u^h|_{1,T}^2+h_T^2\|w-u^h\|_{2,T}^2+\|w-u^h\|_{1,T}^2\Big) \\
\lesssim &\,A_{h,\rho}(\mathbf{I}u-\mathbf{L}u^h,\mathbf{I}u-\mathbf{L}u^h)^{1/2}\Big(\sum_{T\in S_h(\Omega_0)}\big\{\|\mathbf{I}w-\mathbf{L}u^h\|_{\varepsilon,T}^2 \\
&+\varepsilon^2|w-u^h|_{1,T}^2+\|w-L_hu^h\|_T^2\big\}\Big)^{1/2} \\
&+\sum_{T\in S_h(\Omega_0)}\varepsilon^2\Big(h_T^2|u-u^h|_{2,T}^2+|u-u^h|_{1,T}^2+h_T^2\|w-u^h\|_{2,T}^2+\|w-u^h\|_{1,T}^2\Big). \qquad (6.7)
\end{aligned}$$

Using the same argument as that in the derivation of (3.20), we have

$$\Big|A_h\big(\mathbf{I}u-\mathbf{L}u^h,\rho^2\mathbf{I}(u-w)\big)\Big|\lesssim\Big(\sum_{T\in S_h(\Omega_0)}R^T\big(u^h\big)\Big)^{1/2}\|\rho^2\mathbf{I}(u-w)\|_{\varepsilon,\Omega_0}. \qquad (6.8)$$

Note that

$$\begin{aligned}
\|\rho^2\mathbf{I}(u-w)\|_{\varepsilon,\Omega_0}^2\lesssim\, &A_{h,\rho}(\mathbf{I}u-\mathbf{L}u^h,\mathbf{I}u-\mathbf{L}u^h)+\varepsilon^2\|u-u^h\|_{1,h,\Omega_0}^2 \\
&+\|u-L_hu^h\|_{\Omega_0}^2+\varepsilon^2\|u^h-w\|_{2,h,\Omega_0}^2+\|w-L_hu^h\|_{1,h,\Omega_0}^2.
\end{aligned}$$

Hence combining (6.6)-(6.8) gives

$$\begin{aligned}
A_{h,\rho}(\mathbf{I}u-\mathbf{L}u^h,\mathbf{I}u-\mathbf{L}u^h)\lesssim\, &A_{h,\rho}(\mathbf{I}u-\mathbf{L}u^h,\mathbf{I}u-\mathbf{L}u^h)^{1/2}\Big(\sum_{T\in S_h(\Omega_0)}\Big\{R^T(u^h) \\
&+\|\mathbf{I}w-\mathbf{L}u^h\|_{\varepsilon,T}^2+\varepsilon^2|w-u^h|_{1,T}^2+\|w-L_hu^h\|_T^2\Big\}\Big)^{1/2} \\
&+\sum_{T\in S_h(\Omega_0)}\Big\{R^T(u^h)+\varepsilon^2h_T^2\|u-u^h\|_{2,T}^2+\varepsilon^2\|u-u^h\|_{1,T}^2+\|u-L_hu^h\|_T^2 \\
&+\varepsilon^2h_T^2|w-u^h|_{2,T}^2+\varepsilon^2|w-u^h|_{1,T}^2+\varepsilon^2\|w-u^h\|_{2,T}^2+\|w-L_hu^h\|_{1,T}^2\Big\}.
\end{aligned}$$

So by the arbitrariness of $w\in H_0^2(\Omega)$, as we have dropped the terms with higher order, we derive that

$$\begin{aligned}
&A_{h,\rho}(\mathbf{I}u-\mathbf{L}u^h,\mathbf{I}u-\mathbf{L}u^h) \\
\lesssim &\sum_{T\in S_h(\Omega_0)}\Big\{R^T(u^h)+\varepsilon^2h_T^2\|u-u^h\|_{2,T}^2+\varepsilon^2\|u-u^h\|_{1,T}^2+\|u-L_hu^h\|_T^2\Big\} \\
&+\inf_{w\in H_0^2(\Omega)}\Big(\varepsilon^2\|u^h-w\|_{2,h,\Omega_0}^2+\|L_hu^h-w\|_{1,h,\Omega_0}^2\Big). \qquad (6.9)
\end{aligned}$$

Again we set piecewise

$$\widetilde{u}^h|_T=\begin{cases}u^h, & \text{if } h_T\leqslant\varepsilon,\\ L_hu^h, & \text{otherwise,}\end{cases}$$

and $w^h = Q_h \widetilde{u}^h$ as in the proof of Theorem 3.1. Then via a similar argument, we can estimate the differences $u^h - w^h$ and $L_h u^h - w^h$ simultaneously, and derive the conclusion after dropping the higher order terms. $\square$

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