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ERROR ESTIMATES FOR THE TIME DISCRETIZATION FOR NONLINEAR MAXWELL'S EQUATIONS*

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Abstract

This paper is devoted to the study of a nonlinear evolution eddy current model of the type $\partial_t \boldsymbol{B}(\boldsymbol{H}) + \nabla \times (\nabla \times \boldsymbol{H}) = \mathbf{0}$ subject to homogeneous Dirichlet boundary conditions $\boldsymbol{H} \times \boldsymbol{\nu} = \mathbf{0}$ and a given initial datum. Here, the magnetic properties of a soft ferromagnet are linked by a nonlinear material law described by $\boldsymbol{B}(\boldsymbol{H})$. We apply the backward Euler method for the time discretization and we derive the error estimates in suitable function spaces. The results depend on the nonlinearity of $\boldsymbol{B}(\boldsymbol{H})$.

Mathematics subject classification: 65M15, 83C50.
Key words: Electromagnetic field, Nonlinear eddy current problem, Time discretization, Error estimate.

1. Introduction

Eddy current problems are described by quasistationary Maxwell's equations

$$\begin{aligned} \nabla \times \boldsymbol{H} &= \boldsymbol{J}, \\ \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} &= \mathbf{0}, \end{aligned} \tag{1.1}$$

where $\boldsymbol{H}$ denotes the magnetic field, $\boldsymbol{J}$ is the current density (current per unit area), $\boldsymbol{B}$ stands for the magnetic induction and $\boldsymbol{E}$ is the electric field. The quasistationary Maxwell equations can be obtained from the full Maxwell system omitting the term $\varepsilon_0 \partial_t \boldsymbol{E}$, where ε_0 is the permittivity of the free space, which has some strong physical interpretations.

The time dependent magnetic variables are related as follows

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \boldsymbol{B}(\boldsymbol{H}), \tag{1.2}$$

where μ_0 denotes the magnetic permeability of free space and $\boldsymbol{M}$ describes the magnetization. The relation between $\boldsymbol{B}$ and $\boldsymbol{H}$ is nonlinear in ferromagnetic materials. Neglecting the hysteresis effects, the relationship $\boldsymbol{B}(\boldsymbol{H})$ is strictly monotone and invertible. The usual form is $\boldsymbol{B} = b(|\boldsymbol{H}|)\boldsymbol{H}$ or $\boldsymbol{H} = \nu(|\boldsymbol{B}|)\boldsymbol{B}$.

Taking into account Ohm's law

$$\boldsymbol{J} = \sigma \boldsymbol{E},$$

where σ is the conductivity (which can be a tensor in anisotropic materials), and eliminating the electric field $\boldsymbol{E}$ we arrive at

$$\partial_t \boldsymbol{B}(\boldsymbol{H}) + \nabla \times \left(\sigma^{-1} \nabla \times \boldsymbol{H}\right) = \mathbf{0}. \tag{1.3}$$

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Another equivalent form expressed in terms of the vector potential $\boldsymbol{u}$ for the magnetic field density $\boldsymbol{B} = \nabla \times \boldsymbol{u}$ reads as

$$\sigma\partial_t \boldsymbol{u} + \nabla \times (\nu(|\nabla \times \boldsymbol{u}|)\nabla \times \boldsymbol{u}) = \boldsymbol{0}. \tag{1.4}$$

The formulation (1.4) along with the homogeneous Dirichlet boundary condition has been analyzed in [1] under the following assumptions for the continuous function $\nu : \mathbb{R}_0^+ \to \mathbb{R}^+$

$$0 < \nu_1 \leq \nu(s) \leq \nu_2 \qquad \forall s \in \mathbb{R}_0^+$$

and $\nu(s)s$ is strictly monotone and Lipschitz continuous. The authors addressed the solvability of the problem and they proposed a numerical treatment by the so-called multiharmonic approach.

The nonlinear PDE of the type (1.3) or (1.4) have some applications in superconductors – see [2,3]. It is well known that high-field (hard) type-II superconductors are not ideal conductors of electric current. They can be treated as electrically nonlinear conductors. The process of electromagnetic field penetration in such devices is the process of nonlinear diffusion. The equation describing the process can degenerate. For an overview of models with some hierarchy structure we refer the reader to [4, 5]. The magnetization of type-II superconductors in a nonstationary external magnetic field can also be formulated in terms of a scalar $p-$Laplacian equation if the magnetic field lies only in one direction. This situation has been studied in many papers. Authors in [6] showed that the limit as $p \to \infty$ for the scalar p-Laplacian is a solution to Bean's model. The Hölder continuity of solution has been analyzed in [7]. The 2-dimensional problem was studied in [8] using the theory of nonlinear semigroups.

Slodička in [9] applied the backward Euler scheme to (1.3) for the discretization in time and he derived the error estimates for a degenerate problem. A similar technique was used in [10] for an application in superconductors. The error estimates for the time-discretization in both papers were $\mathcal{O}(\tau^{\frac{1}{2}})$ – thus suboptimal.

Paper [11] was devoted to the fix-point approximation of a nonlinear steady-state problem, which arises from the backward Euler discretization of (1.3) at each time step. The authors proved the convergence of iterations for both, the Lipschitz continuous and the degenerate cases.

The main goal of this study is to derive the error estimates for the backward Euler scheme applied to (1.3) along with the homogeneous Dirichlet boundary condition and a given initial datum. We distinguish between the Lipschitz continuous nonlinearity and the degenerate case. Our results are formulated in Theorems 3.1-3.3. The convergence rate is optimal for the Lipschitz nonlinearity, i.e. $\mathcal{O}(\tau)$ – see Theorem 3.1. In the degenerate case we improved the error estimate from [9,10] for some type of nonlinearity – see Theorem 3.3. The last section is devoted to the numerical experiments to support our theoretical results.

2. Preliminaries

Without loss of generality we assume that $\mu_0 = \sigma = 1$. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. We denote by $(\boldsymbol{w}, \boldsymbol{z})$ be the usual L_2-inner product of any real or vector-valued functions $\boldsymbol{w}$ and $\boldsymbol{z}$ in Ω, i.e.,

$$(\boldsymbol{w}, \boldsymbol{z}) = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{z}, \quad \|\boldsymbol{w}\| = \sqrt{(\boldsymbol{w}, \boldsymbol{w})}.$$

We will work in standard Hilbert spaces as follows (see [12–14])

$$\begin{aligned} &\mathbf{H}(\mathbf{curl};\Omega) = \{\boldsymbol{\varphi} \in \mathbf{L}_2(\Omega) : \ \nabla \times \boldsymbol{\varphi} \in \mathbf{L}_2(\Omega)\}, \\ &\mathbf{H}_0(\mathbf{curl};\Omega) = \{\boldsymbol{\varphi} \in \mathbf{H}(\mathbf{curl};\Omega) : \ \boldsymbol{\nu} \times \boldsymbol{\varphi} = \mathbf{0} \ \text{ on } \Gamma\}. \end{aligned} \tag{2.1}$$

The norm in $\mathbf{H}(\mathbf{curl};\Omega)$ is defined as

$$\|\boldsymbol{\varphi}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} = \|\boldsymbol{\varphi}\|^2 + \|\nabla \times \boldsymbol{\varphi}\|^2 .$$

The variational formulation of (1.3) along with the homogeneous Dirichlet boundary condition reads as

$$\begin{aligned} &(\partial_t \boldsymbol{B}(\boldsymbol{H}), \boldsymbol{\varphi}) + (\nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{\varphi}) = 0, \\ &\boldsymbol{H}(0) = \boldsymbol{H}_0 \end{aligned} \tag{2.2}$$

for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl};\Omega)$.

We use an equidistant partitioning of a finite time interval $[0,T]$ with a time step $\tau = \frac{T}{n}$, for any $n \in \mathbb{N}$. Thus, we split the time interval $[0,T]$ into n subintervals $[t_{i-1}, t_i]$ for $t_i = i\tau$. We will write

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}$$

for any function z.

The time discretization is based on backward Euler's method. The corresponding sequence of time-discrete problems reads as $(i = 1, \cdots, n)$

$$\begin{aligned} &\delta \boldsymbol{B}(\boldsymbol{h}_i) + \nabla \times \nabla \times \boldsymbol{h}_i = \mathbf{0} && \text{in } \Omega, \\ &\boldsymbol{\nu} \times \boldsymbol{h}_i = \mathbf{0} && \text{on } \Gamma, \\ &\boldsymbol{h}_0 = \boldsymbol{H}_0 \end{aligned} \tag{2.3}$$

with the variational formulation

$$(\delta \boldsymbol{B}(\boldsymbol{h}_i), \boldsymbol{\varphi}) + (\nabla \times \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) = 0 \tag{2.4}$$

for any $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl};\Omega)$.

Throughout the rest of the paper we assume that the vector field $\boldsymbol{B}$ is continuous and

$$\begin{aligned} &(\boldsymbol{B}(\boldsymbol{e}) - \boldsymbol{B}(\boldsymbol{f}))(\boldsymbol{e} - \boldsymbol{f}) \geq C_0 |\boldsymbol{e} - \boldsymbol{f}|^2 \qquad \forall \boldsymbol{e}, \boldsymbol{f} \in \mathbb{R}^3, \\ &\boldsymbol{B}(\mathbf{0}) = \mathbf{0}, \\ &|\boldsymbol{B}(\boldsymbol{e})| \leq C(1 + |\boldsymbol{e}|) \qquad \forall \boldsymbol{e} \in \mathbb{R}^3. \end{aligned} \tag{2.5}$$

The existence of a weak solution at each time step is guaranteed by the following lemma.

Lemma 2.1. *Assume* (2.5) *and* $\boldsymbol{H}_0 \in \mathbf{L}_2(\Omega)$. *Then there exists a uniquely determined* $\boldsymbol{h}_i \in \mathbf{H}_0(\mathbf{curl};\Omega)$ *solving* (2.3) *for any* $i = 1, \cdots, n$.

Proof. Let $\mathbf{H}^{-1}(\mathbf{curl};\Omega)$ be the dual space to $\mathbf{H}_0(\mathbf{curl};\Omega)$. We consider the nonlinear operator $\boldsymbol{A}(\boldsymbol{h}) : \mathbf{H}_0(\mathbf{curl};\Omega) \to \mathbf{H}^{-1}(\mathbf{curl};\Omega)$ defined as

$$\boldsymbol{A}(\boldsymbol{h}) := \frac{\boldsymbol{B}(\boldsymbol{h})}{\tau} + \nabla \times \nabla \times \boldsymbol{h}.$$

In virtue of (2.5) we can see that $\boldsymbol{A}(\boldsymbol{h})$ is strictly monotone, coercive and demicontinuous[1)]. We apply the theory of mono tone operators – see [15, 16] – to see that the equation $\boldsymbol{A}(\boldsymbol{h}) = \boldsymbol{f}$ has a unique solution for any $\boldsymbol{f} \in \mathbf{H}^{-1}(\mathbf{curl};\Omega)$, which concludes the proof. □

1) Demicontinuity of an operator A means that $x_n \to x$ implies $Ax_n \rightharpoonup Ax$.

Lemma 2.2. *Assume* (2.5) *and* $\boldsymbol{H}_0 \in \mathbf{H}(\mathbf{curl};\Omega)$. *Then there exists a positive* C *such that (for any* $j = 1, \cdots, n$*)*

$$\sum_{i=1}^{j} \|\delta \boldsymbol{h}_i\|^2 \tau + \|\nabla \times \boldsymbol{h}_j\|^2 + \sum_{i=1}^{j} \|\nabla \times (\boldsymbol{h}_i - \boldsymbol{h}_{i-1})\|^2 \leq C.$$

Proof. Setting $\boldsymbol{\varphi} = \delta \boldsymbol{h}_i \tau$ in (2.4) and summing up for $i = 1, \cdots, j$ we have

$$\sum_{i=1}^{j} \left(\delta \boldsymbol{B}\left(\boldsymbol{h}_i\right), \delta \boldsymbol{h}_i\right) \tau + \sum_{i=1}^{j} \left(\nabla \times \boldsymbol{h}_i, \nabla \times (\boldsymbol{h}_i - \boldsymbol{h}_{i-1})\right) = 0.$$

Further we employ the monotonicity of $\boldsymbol{B}(\boldsymbol{h})$ – see (2.5) – together with the obvious algebraic identity

$$\sum_{i=1}^{j} (a_i - a_{i-1}) a_i = \frac{1}{2} \left[a_j^2 - a_0^2 + \sum_{i=1}^{j} (a_i - a_{i-1})^2 \right],$$

Cauchy's and Young's inequalities to get the desired result. □

Next result concerns the stability of $\boldsymbol{B}\left(\boldsymbol{h}_j\right)$ and $\delta \boldsymbol{B}\left(\boldsymbol{h}_j\right)$ in suitable function spaces.

Lemma 2.3. *Let the assumptions of Lemma* 2.2 *be fulfilled. Then there exists a positive* C *such that (for any* $j = 1, \cdots, n$*)*

(i) $\|\boldsymbol{h}_j\| + \|\boldsymbol{B}\left(\boldsymbol{h}_j\right)\| \leq C$,

(ii) $\displaystyle\sum_{i=1}^{j} \|\delta \boldsymbol{B}\left(\boldsymbol{h}_i\right)\|^2_{\mathbf{H}^{-1}(\mathbf{curl};\Omega)} \tau \leq C$,

(iii) *If* $\nabla \cdot \boldsymbol{B}(\boldsymbol{h}_0) \in L_2(\Omega)$, *then* $\|\nabla \cdot \boldsymbol{B}\left(\boldsymbol{h}_j\right)\| = \|\nabla \cdot \boldsymbol{B}(\boldsymbol{h}_0)\|$.

Proof. (i) The assertion can be readily obtained from (2.5), Lemma 2.2 and

$$\boldsymbol{h}_j = \boldsymbol{h}_0 + \sum_{i=1}^{j} \delta \boldsymbol{h}_i \tau.$$

(ii) This part follows from Lemma 2.2 and the definition of the norm in $\mathbf{H}^{-1}(\mathbf{curl};\Omega)$.
(iii) We set $\boldsymbol{\varphi} = \nabla \Phi$ in (2.4) for any $\Phi \in C_0^\infty(\Omega)$. We have

$$0 = \left(\delta \boldsymbol{B}\left(\boldsymbol{h}_i\right), \nabla \Phi\right) = \left(\nabla \cdot \delta \boldsymbol{B}\left(\boldsymbol{h}_i\right), \Phi\right).$$

Summing up for $i = 1, \cdots, j$ we obtain

$$\left(\nabla \cdot \boldsymbol{B}\left(\boldsymbol{h}_j\right), \Phi\right) = \left(\nabla \cdot \boldsymbol{B}(\boldsymbol{h}_0), \Phi\right).$$

The right-hand side can be seen as a linear bounded functional on $L_2(\Omega)$ because of $\nabla \cdot \boldsymbol{B}(\boldsymbol{h}_0) \in L_2(\Omega)$. The density of $C_0^\infty(\Omega)$ in $L_2(\Omega)$ together with Hahn-Banach theorem conclude the proof. □

3. Error Estimates

We introduce the piecewise linear in time vector fields $\boldsymbol{h}_n, \boldsymbol{b}_n$ given by

$$\begin{aligned}\boldsymbol{h}_n(0) &= \boldsymbol{h}_0,\\ \boldsymbol{h}_n(t) &= \boldsymbol{h}_{i-1} + (t-t_{i-1})\delta\boldsymbol{h}_i \qquad \text{for } t\in(t_{i-1},t_i], \quad i=1,\cdots,n\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{b}_n(0) &= \boldsymbol{B}\left(\boldsymbol{h}_0\right),\\ \boldsymbol{b}_n(t) &= \boldsymbol{B}\left(\boldsymbol{h}_{i-1}\right) + (t-t_{i-1})\delta\boldsymbol{B}\left(\boldsymbol{h}_i\right) \qquad \text{for } t\in(t_{i-1},t_i], \quad i=1,\cdots,n.\end{aligned}$$

Next, we define the step vector field $\overline{\boldsymbol{h}}_n$

$$\overline{\boldsymbol{h}}_n(0) \;=\; \boldsymbol{h}_0, \quad \overline{\boldsymbol{h}}_n(t) \;=\; \boldsymbol{h}_i, \quad \text{for } t\in(t_{i-1},t_i], \quad i=1,\cdots,n.$$

Using the new notation we rewrite (2.4) as

$$(\partial_t\boldsymbol{b}_n,\boldsymbol{\varphi}) + \left(\nabla\times\overline{\boldsymbol{h}}_n, \nabla\times\boldsymbol{\varphi}\right) = 0 \qquad \forall\boldsymbol{\varphi}\in\mathbf{H}_0(\mathbf{curl};\Omega). \tag{3.1}$$

Now, we are in a position to derive the error estimates for a Lipschitz continuous vector field $\boldsymbol{B}$.

Theorem 3.1 (Lipschitz continuous case) *Let the assumptions of Lemma* 2.2 *be fulfilled. Moreover we assume that*

$$|\boldsymbol{B}(\boldsymbol{e})-\boldsymbol{B}(\boldsymbol{f})| \le C|\boldsymbol{e}-\boldsymbol{f}| \qquad \forall\boldsymbol{e},\boldsymbol{f}\in\mathbb{R}^3.$$

Then there exists a positive constant C such that

$$\int_0^T \left\|\overline{\boldsymbol{h}}_n-\boldsymbol{H}\right\|^2 + \left\|\int_0^T \nabla\times\left(\overline{\boldsymbol{h}}_n-\boldsymbol{H}\right)\right\|^2 \le C\tau^2.$$

Proof. First we subtract (2.2) from (3.1) and we integrate the result with respect to the time variable. Then we put $\boldsymbol{\varphi}=\overline{\boldsymbol{h}}_n-\boldsymbol{H}\in\mathbf{H}_0(\mathbf{curl};\Omega)$ and again integrate in time. We arrive at

$$\begin{aligned}&\int_0^T \left(\boldsymbol{B}(\overline{\boldsymbol{h}}_n)-\boldsymbol{B}(\boldsymbol{H}),\overline{\boldsymbol{h}}_n-\boldsymbol{H}\right) + \int_0^T\left(\int_0^t \nabla\times(\overline{\boldsymbol{h}}_n-\boldsymbol{H}),\nabla\times(\overline{\boldsymbol{h}}_n-\boldsymbol{H})(t)\right)\\ &= \int_0^T \left(\boldsymbol{B}(\overline{\boldsymbol{h}}_n)-\boldsymbol{b}_n,\overline{\boldsymbol{h}}_n-\boldsymbol{H}\right).\end{aligned}$$

For the first term on the left we use the coercivity assumption from (2.5) and we get

$$\int_0^T \left(\boldsymbol{B}(\overline{\boldsymbol{h}}_n)-\boldsymbol{B}(\boldsymbol{H}),\overline{\boldsymbol{h}}_n-\boldsymbol{H}\right) \ge C_0\int_0^T \left\|\overline{\boldsymbol{h}}_n-\boldsymbol{H}\right\|^2.$$

For the second term on the left can be rewritten as follows

$$\int_0^T\left(\int_0^t \nabla\times(\overline{\boldsymbol{h}}_n-\boldsymbol{H}),\nabla\times(\overline{\boldsymbol{h}}_n-\boldsymbol{H})(t)\right) = \frac{1}{2}\left\|\int_0^T \nabla\times\left(\overline{\boldsymbol{h}}_n-\boldsymbol{H}\right)\right\|^2.$$

Lipschitz continuity of the vector field $\boldsymbol{B}$ together with the stability result from Lemma 2.2 yield

$$\begin{aligned}\left|\int_0^T \left(\boldsymbol{B}(\overline{\boldsymbol{h}}_n) - \boldsymbol{b}_n, \overline{\boldsymbol{h}}_n - \boldsymbol{H}\right)\right| &\le C\int_0^T \left\|\boldsymbol{B}(\overline{\boldsymbol{h}}_n(t)) - \boldsymbol{B}(\overline{\boldsymbol{h}}_n(t-\tau))\right\| \left\|\overline{\boldsymbol{h}}_n(t) - \boldsymbol{H}(t)\right\| \\ &\le C\int_0^T \left\|\overline{\boldsymbol{h}}_n(t) - \overline{\boldsymbol{h}}_n(t-\tau)\right\| \left\|\overline{\boldsymbol{h}}_n(t) - \boldsymbol{H}(t)\right\| \\ &\le C\tau\int_0^T \left\|\partial_t \boldsymbol{h}_n\right\| \left\|\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right\| \\ &\le C_\varepsilon \tau^2 + \varepsilon\int_0^T \left\|\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right\|^2. \end{aligned} \tag{3.2}$$

We summarize the relations above and we obtain

$$(1-\varepsilon)\int_0^T \left\|\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right\|^2 + \left\|\int_0^T \nabla\times\left(\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right)\right\|^2 \le C_\varepsilon \tau^2. \tag{3.3}$$

Choosing a sufficiently small positive ε we conclude the proof. □

In some applications the vector field $\boldsymbol{B}$ may not be Lipschitz continuous. Then, according to the stability results we get weaker convergence rate.

Theorem 3.2 (general case) *Let the assumptions of Lemma* 2.2 *be fulfilled. Then there exists a positive constant C such that*

$$\int_0^T \left\|\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right\|^2 + \left\|\int_0^T \nabla\times\left(\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right)\right\|^2 \le C\tau.$$

Proof. The proof follows the same line as in Theorem 3.1. Therefore we point out the differences, only. The relation (3.2) is replaced by the following one

$$\begin{aligned}&\left|\int_0^T \left(\boldsymbol{B}(\overline{\boldsymbol{h}}_n) - \boldsymbol{b}_n, \overline{\boldsymbol{h}}_n - \boldsymbol{H}\right)\right| \\ &\le C\tau\sqrt{\int_0^T \left\|\partial_t \boldsymbol{b}_n\right\|^2_{\mathbf{H}^{-1}(\mathbf{curl};\Omega)}}\sqrt{\int_0^T \left\|\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}} \le C\tau.\end{aligned}$$

Thus, instead of (3.3) we will have

$$(1-\varepsilon)\int_0^T \left\|\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right\|^2 + \left\|\int_0^T \nabla\times\left(\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right)\right\|^2 \le C_\varepsilon \tau.$$

Finally, choosing a sufficiently small positive ε we conclude the proof. □

Theorem 3.2 presents the error estimates for a general continuous monotone vector field $\boldsymbol{B}$ with a linear growth condition. The same convergence rate has been established in [10] for an application in superconductors, where $\boldsymbol{B}(\boldsymbol{H}) = |\boldsymbol{H}|^{-\alpha}\boldsymbol{H}$ with $\alpha \in (0,1)$, and the nonlinearity has been linearized for large values of $|\boldsymbol{H}|$. The same result for $\boldsymbol{B}(\boldsymbol{H}) = \boldsymbol{H} + |\boldsymbol{H}|^{-\alpha}\boldsymbol{H}$ with $\alpha \in (0,1)$ was proved in [9] for a different application. The convergence rate $\mathcal{O}(\tau^{\frac{1}{2}})$ is not optimal. The following theorem improves this result for some values of $\alpha \in (0,1)$.

Theorem 3.3. *Let the assumptions of Lemma* 2.2 *be fulfilled. We assume that*

$$\boldsymbol{B}(\boldsymbol{H}) = \boldsymbol{H} + |\boldsymbol{H}|^{-\alpha}\boldsymbol{H} \qquad \alpha \in (0,1).$$

Then there exists a positive constant C such that

$$\int_0^T \left\|\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right\|^2 + \left\|\int_0^T \nabla \times (\overline{\boldsymbol{h}}_n - \boldsymbol{H})\right\|^2 \le C\tau^{\max\{1,2(1-\alpha)\}}$$

for $\tau \le 1$.

Proof. The vector field $\boldsymbol{B}(\boldsymbol{H}) = \boldsymbol{H} + |\boldsymbol{H}|^{-\alpha}\boldsymbol{H}$ is not Lipschitz continuous. We will follow the same way as in the proof of Theorem 3.1, but we have to handle the right-hand side in a different way. We employ the idea from [17] and we approximate the nonlinearity by a Lipschitz continuous vector field keeping the approximation error under control.

First, we start with some preparatory work. For any $k > 0$ we define the function $b_k : \mathbb{R}_0^+ \to \mathbb{R}^+$ as follows

$$b_k(s) = \begin{cases} k & s \le k^{-\frac{1}{\alpha}}, \\ s^{-\alpha} & \text{else.} \end{cases}$$

We set

$$\boldsymbol{B}_k(\boldsymbol{s}) = \boldsymbol{s} + b_k\left(|\boldsymbol{s}|\right)\boldsymbol{s} \qquad \boldsymbol{s} \in \mathbb{R}^3.$$

The following inequalities hold true

$$|b_k(s)| \le k, \qquad |b_k(s)s - b_k(t)t| \le k|s-t| \qquad \forall s,t \in \mathbb{R}_0^+. \tag{3.4}$$

The first one is trivial. To show the second one, we see that for $0 \le s,t \le k^{-\frac{1}{\alpha}}$ we have $|b_k(s)s - b_k(t)t| = k|s-t|$. If $s,t \ge k^{-\frac{1}{\alpha}}$ we use the mean value theorem to get (for some θ between s and t)

$$\begin{aligned} |b_k(s)s - b_k(t)t| &= |s^{1-\alpha} - t^{1-\alpha}| = (1-\alpha)\theta^{-\alpha}|s-t| \\ &\le (1-\alpha)k|s-t| \le k|s-t|. \end{aligned}$$

If $s \le k^{-\frac{1}{\alpha}} \le t$ then

$$\begin{aligned} |b_k(s)s - b_k(t)t| &\le \left|b_k(s)s - b_k\left(k^{-\frac{1}{\alpha}}\right)k^{-\frac{1}{\alpha}}\right| + \left|b_k\left(k^{-\frac{1}{\alpha}}\right)k^{-\frac{1}{\alpha}} - b_k(t)t\right| \\ &\le k\left(k^{-\frac{1}{\alpha}} - s\right) + k\left(t - k^{-\frac{1}{\alpha}}\right) \\ &= k(t-s), \end{aligned}$$

which concludes the proof of (3.4).

We want to show the Lipschitz continuity of $\boldsymbol{B}_k$. We proceed similarly as in [18, Lemma 2.2]

$$\begin{aligned} &\left|b_k\left(|\boldsymbol{s}|\right)\boldsymbol{s} - b_k\left(|\boldsymbol{t}|\right)\boldsymbol{t}\right|^2 \\ &= \left|b_k\left(|\boldsymbol{s}|\right)|\boldsymbol{s}| - b_k\left(|\boldsymbol{t}|\right)|\boldsymbol{t}|\right|^2 + 2b_k\left(|\boldsymbol{s}|\right)b_k\left(|\boldsymbol{t}|\right)\left(|\boldsymbol{s}||\boldsymbol{t}| - \boldsymbol{s}\cdot\boldsymbol{t}\right) \\ &\le k^2\left||\boldsymbol{s}| - |\boldsymbol{t}|\right|^2 + 2k^2\left(|\boldsymbol{s}||\boldsymbol{t}| - \boldsymbol{s}\cdot\boldsymbol{t}\right) \\ &= k^2\left|\boldsymbol{s} - \boldsymbol{t}\right|^2. \end{aligned}$$

Therefore,

$$|\boldsymbol{B}_k(\boldsymbol{s}) - \boldsymbol{B}_k(\boldsymbol{t})| \le (1+k)|\boldsymbol{s}-\boldsymbol{t}|.$$

A simple calculation gives

$$\big||\boldsymbol{s}|^{-\alpha}\boldsymbol{s} - k\boldsymbol{s}\big| = \big||\boldsymbol{s}|^{1-\alpha} - k|\boldsymbol{s}|\big| \le k^{1-\frac{1}{\alpha}}\alpha(1-\alpha)^{\frac{1}{\alpha}-1}.$$

In fact,

$$\left(|\boldsymbol{s}|^{-\alpha}\boldsymbol{s} - k\boldsymbol{s}, |\boldsymbol{s}|^{-\alpha}\boldsymbol{s} - k\boldsymbol{s}\right) = \left(|\boldsymbol{s}|^{1-\alpha} - k|\boldsymbol{s}|, |\boldsymbol{s}|^{1-\alpha} - k|\boldsymbol{s}|\right).$$

Further one can examine the extremas of the real function $f(y) = y^{1-\alpha} - ky$.

If we put $k = \tau^{-\alpha}$ for $\tau < 1$ we can write

$$\begin{aligned} |\boldsymbol{B}_{\tau^{-\alpha}}(\boldsymbol{s}) - \boldsymbol{B}(\boldsymbol{s})| &\le \alpha(1-\alpha)^{\frac{1}{\alpha}-1}\tau^{1-\alpha}, \\ |\boldsymbol{B}_{\tau^{-\alpha}}(\boldsymbol{s}) - \boldsymbol{B}_{\tau^{-\alpha}}(\boldsymbol{t})| &\le 2\tau^{-\alpha}|\boldsymbol{s}-\boldsymbol{t}| \end{aligned} \tag{3.5}$$

for any $\boldsymbol{s}, \boldsymbol{t} \in \mathbb{R}^3$. Now, for $t \in [t_{i-1}, t_i]$ we can write

$$\begin{aligned} |\boldsymbol{B}(\overline{\boldsymbol{h}}_n(t)) - \boldsymbol{b}_n(t)| &\le 2\,|\boldsymbol{B}(\boldsymbol{h}_i) - \boldsymbol{B}(\boldsymbol{h}_{i-1})| \\ &\le 2\,|\boldsymbol{B}(\boldsymbol{h}_i) - \boldsymbol{B}_{\tau^{-\alpha}}(\boldsymbol{h}_i)| + 2\,|\boldsymbol{B}_{\tau^{-\alpha}}(\boldsymbol{h}_i) - \boldsymbol{B}_{\tau^{-\alpha}}(\boldsymbol{h}_{i-1})| \\ &\quad + 2\,|\boldsymbol{B}_{\tau^{-\alpha}}(\boldsymbol{h}_{i-1}) - \boldsymbol{B}(\boldsymbol{h}_{i-1})| \\ &\le C\,(1 + |\delta\boldsymbol{h}_i|)\,\tau^{1-\alpha}. \end{aligned}$$

We successively deduce

$$\begin{aligned} \left|\int_0^T \left(\boldsymbol{B}(\overline{\boldsymbol{h}}_n) - \boldsymbol{b}_n, \overline{\boldsymbol{h}}_n - \boldsymbol{H}\right)\right| &\le C_\varepsilon \int_0^T \left\|\boldsymbol{B}(\overline{\boldsymbol{h}}_n) - \boldsymbol{b}_n\right\|^2 + \varepsilon \int_0^T \left\|\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right\|^2 \\ &\le C_\varepsilon \tau^{2(1-\alpha)}\left(1 + \int_0^T \|\partial_t \boldsymbol{h}_n\|^2\right) + \varepsilon \int_0^T \left\|\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right\|^2 \\ &\le C_\varepsilon \tau^{2(1-\alpha)} + \varepsilon \int_0^T \left\|\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right\|^2. \end{aligned}$$

Choosing a sufficiently small positive ε we arrive at

$$\int_0^T \left\|\overline{\boldsymbol{h}}_n - \boldsymbol{H}\right\|^2 + \left\|\int_0^T \nabla\times(\overline{\boldsymbol{h}}_n - \boldsymbol{H})\right\|^2 \le C\tau^{2(1-\alpha)}.$$

Combining this result with Theorem 3.2 we readily conclude the proof. □

Theorem 3.3 improves the error estimates from Theorem 3.2 for $0 < \alpha < \frac{1}{2}$. Let us note that for $\alpha \to 0$ the nonlinear vector field $\boldsymbol{B}(\boldsymbol{H}) = \boldsymbol{H} + |\boldsymbol{H}|^{-\alpha}\boldsymbol{H}$ becomes linear and the convergence rate becomes optimal for the backward Euler scheme.

Let us note that Theorem 3.3 holds true also for the vector field $\boldsymbol{B}(\boldsymbol{H}) = |\boldsymbol{H}|^{-\alpha}\boldsymbol{H}$ with $\alpha \in (0,1)$, which is linearized for large values of $|\boldsymbol{H}|$.

4. Numerical Experiments

We study the performance of the method when applied to the test problem

$$\begin{aligned} &\partial_t(|\boldsymbol{H}|^{-\alpha}\boldsymbol{H} + \boldsymbol{H}) + \nabla\times\nabla\times\boldsymbol{H} = \boldsymbol{g} && (\boldsymbol{x}, t) \in \Omega\times[0,T], \\ &\boldsymbol{H}\times\boldsymbol{\nu} = \boldsymbol{H}_\Gamma\times\boldsymbol{\nu} && (\boldsymbol{x}, t) \in \partial\Omega\times[0,T], \\ &\boldsymbol{H}(\boldsymbol{x}, 0) = \begin{pmatrix} x_3 - x_2 \\ x_1 - x_3 \\ x_2 - x_1 \end{pmatrix} && \boldsymbol{x} \in \Omega, \end{aligned} \tag{4.1}$$

with Dirichlet boundary condition $\boldsymbol{H}_\Gamma$, where the right-hand side $\boldsymbol{g}$ is chosen such that

$$\boldsymbol{H}(\boldsymbol{x},t) = \begin{pmatrix} x_3 - x_2 \\ x_1 - x_3 \\ x_2 - x_1 \end{pmatrix} (t+1)$$

is the exact solution. Using backward Euler's method we can write

$$|\boldsymbol{H_i}|^{-\alpha}\boldsymbol{H_i} + \boldsymbol{H_i} + \tau\nabla\times\nabla\times\boldsymbol{H_i} = \bar{\boldsymbol{g}}, \tag{4.2}$$

where $\bar{\boldsymbol{g}} = \tau\boldsymbol{g} + |\boldsymbol{H_{i-1}}|^{-\alpha}\boldsymbol{H_{i-1}} + \boldsymbol{H_{i-1}}$ and τ is a time step.

To solve Eq. (4.2) we are going to use Newton's method. First we define a functional F in the following way

$$F(\boldsymbol{v}) = |\boldsymbol{v}|^{-\alpha}\boldsymbol{v} + \boldsymbol{v} + \tau\nabla\times\nabla\times\boldsymbol{v} - \bar{\boldsymbol{g}}.$$

For the weak formulation and its Fréchet derivative $DF(\boldsymbol{v})$ we can write

$$\begin{aligned}
&(F(\boldsymbol{v}),\boldsymbol{\varphi}_j) = (|\boldsymbol{v}|^{-\alpha}\boldsymbol{v} + \boldsymbol{v},\boldsymbol{\varphi}_j) + \tau(\nabla\times\boldsymbol{v},\nabla\times\boldsymbol{\varphi}_j) - (\bar{\boldsymbol{g}},\boldsymbol{\varphi}_j),\\
&(DF(\boldsymbol{v})\boldsymbol{\varphi}_i,\boldsymbol{\varphi}_j) = (-\alpha|\boldsymbol{v}|^{-\alpha-2}[\boldsymbol{v}\cdot\boldsymbol{\varphi}_i]\boldsymbol{v} + |\boldsymbol{v}|^{-\alpha}\boldsymbol{\varphi}_i + \boldsymbol{\varphi}_i,\boldsymbol{\varphi}_j) + \tau(\nabla\times\boldsymbol{\varphi}_i,\nabla\times\boldsymbol{\varphi}_j),
\end{aligned}$$

where $\boldsymbol{\varphi_i},\boldsymbol{\varphi_j} \in \mathbf{H}_0(\mathbf{curl};\Omega)$.

On each time layer now we can use Newton's method to solve

$$F(\boldsymbol{H}_i) = 0$$

by solving

$$DF(\boldsymbol{H_{i_m}})\boldsymbol{d_m} = F(\boldsymbol{H_{i_m}}) \tag{4.3}$$

and setting

$$\boldsymbol{H_{i_{m+1}}} = \boldsymbol{H_{i_m}} - \boldsymbol{d_m}.$$

Our computational domain is a unit cube, which we have split "quasi uniformly" into 3072 tetrahedrons. The mesh diameter was $\frac{1}{8}$. For the approximation of the field $\boldsymbol{H}$ we have used the lowest order Whitney's edge elements, cf. [12,13]. The computational error was calculated in the L_2-norm.

Our numerical scheme has been computed using "The Finite Element Toolbox Albert"[1)], which has been modified for the use of the lowest order Whitney's elements. The nonlinear algebraic system has been solved using the Newton method with the stopping criterion

$$\|\boldsymbol{d_m}\|_{\mathbf{H}(\mathbf{curl};\Omega)} < 1.0\cdot 10^{-6}.$$

We have used the GMRes solver for the linearized problem in Newton's algorithm.

Fig. 4.1 and Table 4.1 show results for the Lipschitz continuous case, where we have chosen $\alpha = -2$. The expected convergence rate ξ (according to Theorem 3.1) is 1. Our calculations report 0.9787, which corresponds to the expectations.

The results for the non-Lipschitz continuous case with $\alpha = 0.5$ are presented in Fig. 4.2 and Table 4.1. The calculated convergence rate $\xi = 0.8335$. This is even better than our theoretical results from Theorems 3.2 and 3.3. This also shows that the theoretical results from Theorems

[1)] ALBERT as well as it is successor ALBERTA can be downloaded from `http://www.alberta-fem.de/`. ALBERTA is an Adaptive multilevel finite element toolbox using Bisectioning refinement and Error control by Residual Techniques for scientific Applications.

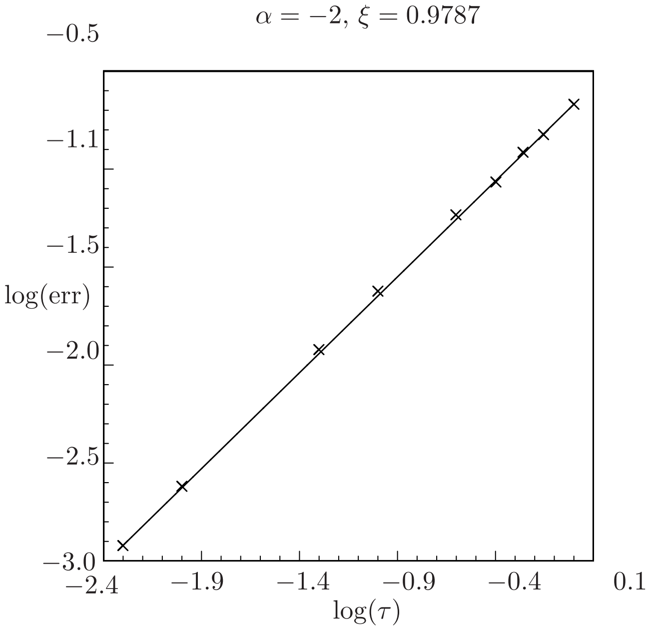

Fig. 4.1. Lipschitz continuous case.

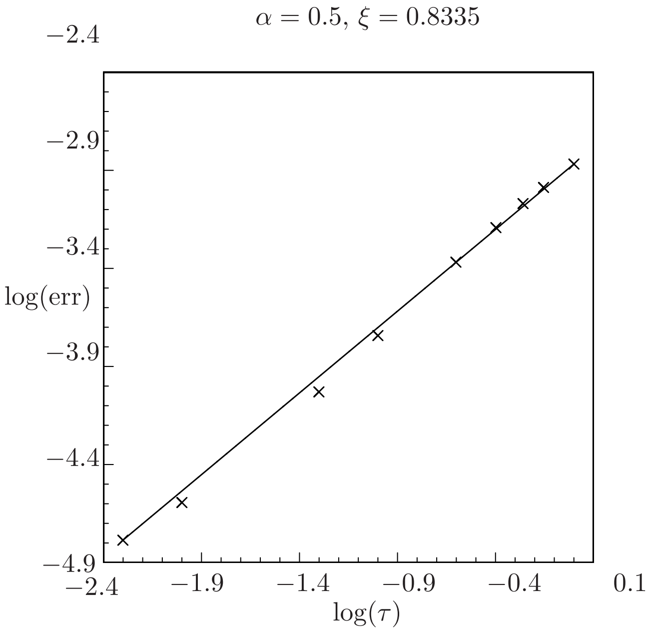

Fig. 4.2. Non-Lipschitz continuous case.

3.2 and 3.3 might not be optimal, which is a challenge for the future work to derive the optimal rates of convergence. Let us mention here an analogue situation from nonlinear diffusion, where the operator $\nabla \times \nabla \times$ is replaced by $-\Delta$. Here the optimal rate for the backward Euler method is 1.

Table 4.1: Errors for the time discretization.

		$\alpha = -2$		$\alpha = 0.5$	
τ	$\log(\tau)$	Absolute Err	log(Err)	Absolute Err	log(Err)
0.0050	-2.30103	0.0011992480	-2.92109	0.0000163743	-4.78584
0.0100	-2.00000	0.0024063610	-2.61864	0.0000254760	-4.59387
0.0500	-1.30103	0.0119922500	-1.92110	0.0000933973	-4.02967
0.1000	-1.00000	0.0238289900	-1.62289	0.0001809390	-3.74247
0.2500	-0.60206	0.0584279900	-1.23338	0.0004283891	-3.36816
0.4000	-0.39794	0.0859207100	-1.06590	0.0006425975	-3.19206
0.5500	-0.25964	0.1218973000	-0.91401	0.0008520724	-3.06952
0.8500	-0.07058	0.1411915000	-0.85019	0.0011139480	-2.95314
1.0000	0.00000	0.2142606000	-0.66906	0.0013554100	-2.86793

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References

[1] F. Bachinger, U. Langer and J. Schöberl, Numerical analysis of nonlinear multiharmonic eddy current problems, *Numer. Math.*, **100** (2005), 593–616.

[2] F. London, Superfluids. Vol. I.: Macroscopic theory of superconductivity, New York: John Wiley & Sons, Inc. London: Chapman & Hall, Ltd., New York, 1950.

[3] F. London, Superfluids. Vol. II. Macroscopic theory of superfluid helium, John Wiley & Sons, Inc., New York, 1954.

[4] S. Chapman, A hierarchy of models for type-II superconductors, *SIAM Rev.*, **42**:4 (2000), 555–598.

[5] M. Fabrizio and A. Morro, Electromagnetism of continuous media. Mathematical modelling and applications, Oxford University Press, Oxford, 2003.

[6] J. Barrett and L. Prigozhin, Bean's critical-state model as the $p \to \infty$ limit of an evolutionary p-Laplacian equation, *Nonlinear Anal-Theor.*, **42A**:6 (2000), 977–993.

[7] H.M. Yin, B.Q. Li and J. Zou, A degenerate evolution system modeling Bean's critical-state type-II superconductors, *Discrete Cont. Dyn. S.*, **8**:3 (2002), 781–794.

[8] W. Wei and H.M. Yin, Numerical solutions to Bean's critical-state model for type-II superconductors, *Int. J. Numer. Anal. Model.*, **2**:4 (2005), 479–488.

[9] M. Slodička, A time discretization scheme for a non-linear degenerate eddy current model for ferromagnetic materials, *IMA J. Numer. Anal.*, **26** (2006), 173–187.

[10] M. Slodička, Nonlinear diffusion in type-II superconductors, *J. Comput. Appl. Math.*, **215**:2 (2008), 568–576.

[11] E. Janíková and M. Slodička, Fix-point linearization schemes for nonlinear steady-state eddy current problems, *Math. Method. Appl. Sci.*, **30**:14 (2007), 1697–1704.

[12] A. Bossavit, Computational electromagnetism. Variational formulations, complementarity, edge elements, volume XVIII of *Electromagnetism*, Academic Press, Orlando, FL., 1998.

[13] M. Cessenat, Mathematical methods in electromagnetism. Linear theory and applications, volume 41 of *Series on Advances in Mathematics for Applied Sciences*, World Scientific Publishers, Singapore, 1996.

[14] P. Monk, Finite element methods for Maxwell's equations, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, 2003.

[15] L.C. Evans, Partial differential equations, volume 19 of *Graduate Studies in Mathematics*, American Mathematical Society, 1998.

[16] M. Vajnberg, Variational method and method of monotone operators in the theory of nonlinear equations., John Wiley & Sons, Chichester, 1973.

[17] M. Slodička, A robust linearization scheme for a nonlinear elliptic boundary value problem: Error estimates, *ANZIAM J.*, **46** (2005), 449–470.

[18] B. Heise, Analysis of a fully discrete finite element method for a nonlinear magnetic field problem, *SIAM J. Numer. Anal.*, **31**:3 (1994), 745–759.