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THE OPTIMAL CONVERGENCE ORDER OF THE DISCONTINUOUS FINITE ELEMENT METHODS FOR FIRST ORDER HYPERBOLIC SYSTEMS*

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Abstract

In this paper, a discontinuous finite element method for the positive and symmetric, first-order hyperbolic systems (steady and nonsteady state) is constructed and analyzed by using linear triangle elements, and the $\mathcal{O}(h^2)$-order optimal error estimates are derived under the assumption of strongly regular triangulation and the H^3-regularity for the exact solutions. The convergence analysis is based on some superclose estimates of the interpolation approximation. Finally, we discuss the Maxwell equations in a two-dimensional domain, and numerical experiments are given to validate the theoretical results.

Mathematics subject classification: 65N30, 65M60.

Key words: First order hyperbolic systems, Discontinuous finite element method, Convergence order estimate.

1. Introduction

It is well known that for the k-th order finite element approximations to elliptic or parabolic problems, the optimal order error estimate in the L_2 norm is of order $\mathcal{O}(h^{k+1})$ with the exact solution u in $H^{k+1}(\Omega)$. However, for linear hyperbolic problems, it is still a completely unsolved problem whether or not the finite element solutions admit this optimal order estimate. Generally speaking, the convergence order of the Galerkin finite element method for hyperbolic problems is of order $\mathcal{O}(h^k)$, that is one order lower than the approximation order of the finite element space; see, e.g., [7,14]. In addition, in [7] Dupont gave a counterexample by using a third-order Hermite element to indicate that this convergence rate is sharp. Since then, in order to obtain the high accuracy and cope with the characteristics of hyperbolic problems, the discontinuous Galerkin method is proposed and used extensively in this area; see, e.g., [4,9,12,15,18,20].

Historically, the original discontinuous Galerkin finite element method was introduced by Reed and Hill [18] in 1973 to solve the linear neutron transport equation. Soon Lesaint and Raviart [15] gave its mathematical analysis and obtained the $\mathcal{O}(h^k)$-order error estimates when the k-th order discontinuous finite element spaces were used. Later on, Johnson and Pitkaranta [12] improved this convergence rate to $\mathcal{O}(h^{k+\frac{1}{2}})$, and Peterson [17] further proved that, under the quasi-uniform triangulation condition, the $\mathcal{O}(h^{k+\frac{1}{2}})$ convergence rate is sharp, namely, this is the optimal order error estimate for discontinuous Galerkin finite element approximations to first-order hyperbolic problems. On the superconvergence research, Lesaint and Raviart [15] first obtained estimate of the form $\|u - u_h\| \le Ch^{k+1}\|u\|_{k+2}$, for rectangular mesh finite elements (also see [16] for the piecewise constant approximation case), and Richter [19] did so

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for semi-uniform triangular meshes under the curious assumption that all element edges are bounded away from the characteristic direction of the hyperbolic equation, which is less significant in the practical case. However, all the papers mentioned above discuss the single scalar hyperbolic equation only. Indeed, the discontinuous Galerkin finite element methods have been extended successfully to linear systems of first-order hyperbolic equations and nonlinear problems. Now there is a lot of literature available in this area. For example, Johnson and Huang [13] studied this method for Friedrichs system of equations, and gained the $\mathcal{O}(h^{k+\frac{1}{2}})$-order error estimate, this result was extended to the initial-boundary value problems of positive and symmetric, linear systems of hyperbolic equations by Zhang in [20]. Huang also discussed discontinuous Galerkin finite element methods for mixed Tricomi equations and nonlinear vorticity transport equations [9,10]. Since the 1990s, Cockburn and Shu et al. systematically studied the discontinuous Galerkin finite element method for nonlinear convection laws and related problems. By using numerical flow of finite differences with higher resolution, TVB, and gradient limiters, some new type of discontinuous Galerkin finite element methods for various hyperbolic problems were designed, see, e.g., [1,2,3,4]. Furthermore, the Maxwell equations with periodic boundary conditions were also discussed by using the locally divergence-free discontinuous Galerkin method in [5]. For more literature, the reader is referred to Cockburn and Shu's review article [3] and the references therein.

In this paper, we will discuss the discontinuous linear finite element approximations to positive and symmetric linear hyperbolic systems (steady and nonsteady state). Under the assumption of strongly regular triangulation and H^3-regularity for the exact solutions, the $\mathcal{O}(h^2)$-order optimal error estimate is established. The theoretical tools for the error analysis are some superclose estimates of interpolation approximation that are also derived in this paper. In our discontinuous finite element method (see (2.7)-(2.8)), the approximations of the traces of the fluxes on the boundary of the elements (the so-called numerical fluxes) are different from that introduced by Reed and Hill [18] or Lesaint and Raviart [15] in the original DG method. Generally speaking, our method will lead to an implicit scheme, while those schemes in [15,18] are in an explicit fashion such that the discrete equations can be solved explicitly through an ordering, element by element. The advantage of our method is that it allows us to derive the optimal order error estimates. To the authors' knowledge, very few optimal order error estimates have been obtained for hyperbolic problems, even in one dimensional case. Hence, our research work in this paper is theoretically significant.

Let $\Omega \subset R^2$ be a polygonal domain, $J_h = \{e\}$ the finite element triangulation of the domain Ω parameterized by the mesh size h so that $\overline{\Omega} = \cup_{e \in J_h}\{\,\overline{e}\,\}$. Introduce the discontinuous linear finite element space S_h defined by

$$S_h = \{\, v \in L_2(\Omega) : v|_e \text{ is linear}, \ \forall e \in J_h \,\}.$$

We will use the standard notations for the Sobolev spaces $W_p^m(\Omega)$ with corresponding norms and seminorms, and when $p = 2$, $W_2^m(\Omega) = H^m(\Omega)$, $\|\cdot\|_{m,2} = \|\cdot\|_m$. Denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the standard inner product and norm in $L_2(\Omega)$. Let X be a Banach space. For constant $T > 0$, we will also use the space,

$$L_p(0,T;\, X) = \Big\{\, v(t) : (0,T) \to X :\ \|v\|_{L_p(X)} = \big(\int_0^T \|v(t)\|_X^p \, dt \big)^{\frac{1}{p}} < \infty \Big\}.$$

In this paper, the letter C represents a generic constant independent of the mesh size h.

The plan of this paper is as follows. In Section 2, the discontinuous finite element approximations are constructed for steady and nonsteady positive and symmetric hyperbolic systems, respectively. In Section 3, some superclose properties for interpolation are analyzed, and the optimal order error estimates are derived. Finally, in Section 4, we discuss the discontinuous finite element approximations to Maxwell equations with periodic boundary value condition in a two-dimensional domain, and numerical experiments are given to validate the theoretical results.

2. Problems and Discontinuous Finite Element Methods

2.1. Steady problems

Consider the following first-order hyperbolic problem:

$$\mathbf{A}\cdot\nabla\mathbf{u}+B\mathbf{u}=\mathbf{f}(x,y),\quad (x,y)\in\Omega, \tag{2.1}$$

$$N\mathbf{u}=\frac{1}{2}(M-D_n)\mathbf{u}=\mathbf{0},\quad (x,y)\in\partial\Omega. \tag{2.2}$$

Here, $\mathbf{A}=(A_1,A_2)$, $A_k=(a_{ij}^{(k)}(x,y)), k=1,2$, $B=(b_{ij}(x,y))$ and $M=(m_{ij}(x,y))$ are some given $m\times m$ matrices, $a_{ij}^{(k)}\in W_\infty^1(\Omega)$, $b_{ij},m_{ij}\in L_\infty(\Omega)$, $D_n=\mathbf{A}\cdot n$, $n=(n_x,n_y)$ is the outward unit normal vector on $\partial\Omega$, $\mathbf{u}=(u_1,\cdots,u_m)^T$ and $\mathbf{f}=(f_1,\cdots,f_m)^T$ are m-dimensional vector functions. We assume that problem (2.1)-(2.2) is a positive and symmetric hyperbolic system, namely (see [8])

$$A_1=A_1^T,\quad A_2=A_2^T,\quad (x,y)\in\Omega, \tag{2.3}$$

$$B+B^T-div\mathbf{A}\geq\sigma_0 I,\quad (x,y)\in\Omega, \tag{2.4}$$

$$M+M^T\geq 0,\quad (x,y)\in\partial\Omega, \tag{2.5}$$

$$Ker(M-D_n)+Ker(M+D_n)=R^m,\quad (x,y)\in\partial\Omega, \tag{2.6}$$

where the constant $\sigma_0>0$. Let ϕ be a piecewise smooth function on J_h, define its jump at a point $p_0\in\partial e$,

$$[\,\phi\,]=\phi^+-\phi^-,\quad \phi^+(p_0)=\lim_{p\to p_0,p\in e}\phi(p),\quad \phi^-(p_0)=\lim_{p\to p_0,p\notin e}\phi(p),$$

and we always set $\phi^-\big|_{\partial\Omega}=0$. Below we will use the notations,

$$(u,v)_h=\sum_{e\in J_h}(u,v)_e,\quad (u,v)_e=\int_e u\,v;\quad <u,v>_{\partial e}=\int_{\partial e}u\,v.$$

Introduce the bilinear form:

$$A(\mathbf{u},\mathbf{v})=(\mathbf{A}\cdot\nabla\mathbf{u},\mathbf{v})_h+(B\mathbf{u},\mathbf{v})+\frac{1}{2}\sum_{e\in J_h}<(M_e-D_e)[\mathbf{u}],\mathbf{v}>_{\partial e}, \tag{2.7}$$

where

$$D_e=D_n,\ M_e=M,\ (x,y)\in\partial e\cap\partial\Omega;\quad D_e=\mathbf{A}\cdot n_e,\ M_e=c_0h^{-1}I,\ (x,y)\in\partial e\backslash\partial\Omega\,,$$

and n_e is the outward unit normal vector on ∂e, $c_0>0$ (for example, $c_0=1$) is a constant independent of h, which may be adjusted according to the practical computation results.

Now we define the discontinuous finite element approximation to problem (2.1)-(2.2) by finding $\mathbf{u}_h\in[S_h]^m$ such that

$$A(\mathbf{u}_h,\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_h),\ \ \forall\,\mathbf{v}_h\in[S_h]^m. \tag{2.8}$$

Lemma 2.1. *Let* $\mathbf{w}$ *be any piecewise smooth vector function on* J_h*. Then we have the following identity,*

$$\begin{aligned} A(\mathbf{w},\,\mathbf{w}) =& \frac{1}{2}(\,(B+B^T-div\mathbf{A})\mathbf{w},\,\mathbf{w})_h + \frac{1}{2} < M\mathbf{w},\,\mathbf{w} >_{\partial\Omega} \\ &+ \frac{1}{2}\sum_{l\in\mathcal{B}} < M_e[\,\mathbf{w}\,],\,[\,\mathbf{w}\,] >_l, \end{aligned} \tag{2.9}$$

where $\mathcal{B}$ *represents the union of all interior boundary segments* $l \subset \partial e\backslash\partial\Omega$, $e \in J_h$.

Proof. By Green's formula, from (2.7) we have

$$\begin{aligned} A(\mathbf{w},\,\mathbf{w}) =& \frac{1}{2}(\,(B+B^T-div\mathbf{A})\mathbf{w},\,\mathbf{w})_h + \frac{1}{2}\sum_{e\in J_h} < D_e\mathbf{w},\,\mathbf{w} >_{\partial e} \\ &+ \frac{1}{2}\sum_{e\in J_h} < (M_e - D_e)[\,\mathbf{w}\,],\,\mathbf{w}\, >_{\partial e} = \frac{1}{2}(\,(B+B^T-div\mathbf{A})\mathbf{w},\,\mathbf{w})_h \\ &+ \frac{1}{2}\sum_{e\in J_h} < M_e[\,\mathbf{w}\,],\,\mathbf{w}\, >_{\partial e} + \frac{1}{2}\sum_{e\in J_h} < D_e\mathbf{w}^-,\,\mathbf{w}^+\, >_{\partial e}. \end{aligned} \tag{2.10}$$

Note that

$$D_e = \mathbf{A}\cdot n_e, \quad D_e = D_e^T, \quad \mathbf{w}^-\big|_{\partial\Omega} = 0.$$

So we have

$$\sum_{e\in J_h} < D_e\mathbf{w}^-,\,\mathbf{w}^+ >_{\partial e} = 0.$$

Let $l = \partial e \cap \partial e'$, e and e' are two adjacent elements with common edge l. Since

$$\begin{aligned} &(\mathbf{w}^+ - \mathbf{w}^-)\mathbf{w}^+\big|_{l\in\partial e} + (\mathbf{w}^+ - \mathbf{w}^-)\mathbf{w}^+\big|_{l\in\partial e'} \\ =& (\mathbf{w}^+ - \mathbf{w}^-)\mathbf{w}^+\big|_{l\in\partial e} + (\mathbf{w}^- - \mathbf{w}^+)\mathbf{w}^-\big|_{l\in\partial e} = [\mathbf{w}][\mathbf{w}]\big|_{l\in\partial e}, \end{aligned}$$

we have

$$\sum_{e\in J_h} < M_e[\,\mathbf{w}],\,\mathbf{w} >_{\partial e} = < M\mathbf{w},\,\mathbf{w} >_{\partial\Omega} + \sum_{l\in\mathcal{B}} < M_e[\mathbf{w}],\,[\mathbf{w}] >_l.$$

Thus, from (2.10) we complete the proof of Lemma 2.1. □

From Lemma 2.1, (2.4)-(2.5) and the Cauchy inequality, it is easy to see that the solution $\mathbf{u}_h$ of problem (2.8) uniquely exists and satisfies the stability estimate

$$\sigma_0\|\mathbf{u}_h\|^2 + 2 < M\mathbf{u}_h, \mathbf{u}_h >_{\partial\Omega} + 2\sum_{l\in\mathcal{B}} < M_e[\mathbf{u_h}],\,[\mathbf{u_h}] >_l \le \frac{4}{\sigma_0}\|\,\mathbf{f}\,\|^2. \tag{2.11}$$

2.2. Nonsteady problems

Consider the time-dependent first-order hyperbolic problem:

$$\mathbf{u}_t + \mathbf{A}\cdot\nabla\mathbf{u} + B\mathbf{u} = \mathbf{f}(t), \quad (t,x,y) \in [0,T)\times\Omega, \tag{2.12}$$

$$N\mathbf{u} = \frac{1}{2}(M - D_n)\mathbf{u} = \mathbf{0}, \quad (t,x,y) \in [0,T)\times\partial\Omega, \tag{2.13}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad (x,y) \in \Omega, \tag{2.14}$$

where the notation representations in (2.12)-(2.14) are the same as those in (2.1)-(2.2).

Define the discontinuous finite element approximation for problem (2.12)-(2.14) by finding $\mathbf{u}_h : [0,T) \to [S_h]^m$ such that

$$(\mathbf{u}_{h,t}, \mathbf{v}_h) + A(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \ \forall \mathbf{v}_h \in [S_h]^m, \tag{2.15}$$

$$\mathbf{u}_h(0) \in [S_h]^m, \tag{2.16}$$

where the bilinear form $A(\mathbf{u}, \mathbf{v})$ is given by (2.7). Taking $\mathbf{v}_h = \mathbf{u}_h$ in (2.15), from (2.9) we obtain

$$\frac{d}{dt}\|\mathbf{u}_h(t)\|^2 + \sigma_0\|\mathbf{u}_h(t)\|^2 + < M\mathbf{u}_h, \mathbf{u}_h >_{\partial\Omega} + \sum_l < M_e[\mathbf{u_h}], [\mathbf{u_h}] >_l \le 2\|\mathbf{f}(t)\|\|\mathbf{u}_h\|. \tag{2.17}$$

This implies the stability estimate

$$\|\mathbf{u}_h(t)\| \le e^{-\sigma_0 t/2}\|\mathbf{u}_h(0)\| + \int_0^t \|\mathbf{f}(\tau)\| \, d\tau \, , \ t > 0, \tag{2.18}$$

which ensures that the ordinary differential system (2.15)-(2.16) has a unique solution.

3. Superclose Estimates and Error Analysis

In this section, we will give the error analysis for discontinuous finite element approximations to the steady and nonsteady first-order hyperbolic problems. Our analysis is based on some superclose results.

3.1. Superclose estimates

Definition 3.1. *Let* $e = \triangle p_1p_2p_3$ *and* $e' = \triangle p'_1p'_2p'_3$ *be two adjacent triangle elements sharing a common edge in* J_h*. The quadrilateral* $\overline{e} \cup \overline{e'}$ *is called an approximate parallelogram if* (*see Fig.* 3.1)

$$|\overrightarrow{p_1p_2} + \overrightarrow{p'_1p'_2}| = \mathcal{O}(h^2)\,, \qquad |\overrightarrow{p_2p_3} + \overrightarrow{p'_2p'_3}| = \mathcal{O}(h^2). \tag{3.1}$$

Definition 3.2. *A triangulation* J_h *is called strongly regular, if any two adjacent triangular elements in* J_h *form an approximate parallelogram* (*see Fig.* 3.1).

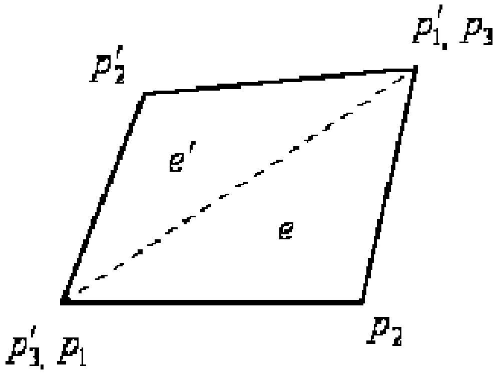

Fig. 3.1. An approximate parallelogram.

Remark 3.1. Strongly regular triangulations must be quasi-uniform. Some common domains (for example, convex quadrilateral and L-shaped domain, etc.) can be subdivided into strongly regular triangulation; see [6, Chapter 2, p.23-65] and [21].

Lemma 3.1. *Let the triangulation J_h be strongly regular, $e = \triangle p_1p_2p_3$ and $e' = \triangle p_1'p_2'p_3'$ be two adjacent triangle elements (see Fig. 3.1). Assume vectors $\vec{L} = \overrightarrow{p_1p_2}$ (or $\overrightarrow{p_2p_3}$, $\overrightarrow{p_3p_1}$) and $\vec{L}' = \overrightarrow{p_1'p_2'}$ (or $\overrightarrow{p_2'p_3'}$, $\overrightarrow{p_3'p_1'}$), the lengths $l = |\vec{L}|$, $l' = |\vec{L}'|$, and the unit direction vectors $n_l = \vec{L}/l$, $n_{l'} = \vec{L}'/l'$. Then*

$$|\,l - l'\,| = \mathcal{O}(h^2)\,, \qquad |\,n_l + n_{l'}\,| = \mathcal{O}(h). \tag{3.2}$$

Proof. From (3.1) we have

$$\begin{aligned}
|\,l - l'\,| &= \frac{1}{l+l'}|\vec{L}\cdot\vec{L} - \vec{L}'\cdot\vec{L}'| = \frac{1}{l+l'}|(\vec{L}+\vec{L}')\cdot(\vec{L}-\vec{L}')| \\
&\le |\vec{L}+\vec{L}'| = \mathcal{O}(h^2), \\
|\,n_l + n_{l'}\,| &= |\frac{1}{l}\vec{L} + \frac{1}{l'}\vec{L}'| = \frac{1}{l\,l'}|(l'-l)\vec{L} + l\,(\vec{L}+\vec{L}')| \\
&\le \frac{1}{l'}(\,|\,l'-l\,| + |\,\vec{L}+\vec{L}'|\,) = \mathcal{O}(h).
\end{aligned}$$

It is obvious that $l - l' = 0$, $n_l + n_{l'} = 0$ when $\vec{L} = \overrightarrow{p_3p_1}$, $\vec{L}' = \overrightarrow{p_3'p_1'}$. Thus, the proof is completed. $\square$

Now let the triangular element $e = \triangle p_1p_2p_3$ have three edge vectors $\vec{L}_1 = \overrightarrow{p_1p_2}$, $\vec{L}_2 = \overrightarrow{p_2p_3}$, $\vec{L}_3 = \overrightarrow{p_3p_1}$, $l_i = |\vec{L}_i|$ and $n_{l_i} = l_i^{-1}\vec{L}_i$ denote the lengths and unit direction vectors of $\vec{L}_i$ $(i = 1, 2, 3)$, respectively, and $D_i = n_{l_i}\cdot\nabla$ is the directional derivatives (see Fig. 3.2).

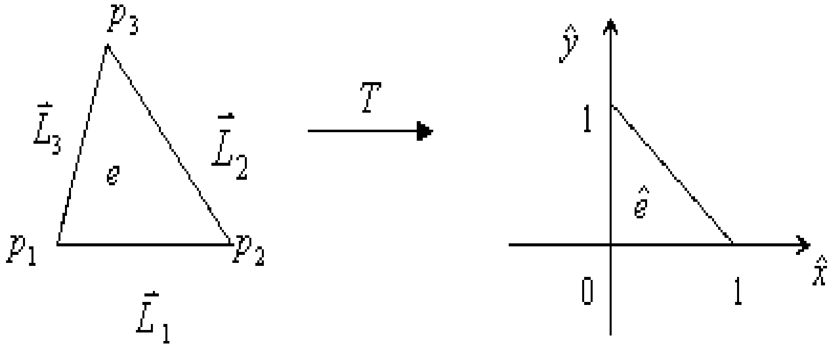

Fig. 3.2. Triangular element and unit triangular element.

Lemma 3.2. *Let $e = \triangle p_1p_2p_3$ in J_h, $w \in H^3(e)$, $\phi \in H^1(e)$. Then*

$$\int_e (w - w_I)\phi = -\frac{1}{24}\int_e \sum_{i=1}^{3} l_i^2 D_i^2 w\,\phi + \mathcal{O}(h^3)\Big(\|w\|_{2,e}\|\phi\|_{1,e} + \|w\|_{3,e}\|\phi\|_{0,e}\Big), \tag{3.3}$$

where $w_I \in C(\overline{\Omega}) \cap S_h$ is the piecewise linear interpolation approximation of the continuous function w.

Proof. Let $\hat{e}$ be the unit triangle with vertices $\hat{p}_1 = (0,0)$, $\hat{p}_2 = (1,0)$ and $\hat{p}_3 = (0,1)$ (see Fig. 3.2). Set $\hat{l}_1 = 1$, $\hat{l}_2 = \sqrt{2}$, $\hat{l}_3 = 1$, $\hat{D}_1 = \partial_x$, $\hat{D}_2 = (\partial_y - \partial_x)/\sqrt{2}$ and $\hat{D}_3 = -\partial_y$. By straightforward calculation, we can see that for any quadratic polynomial q,

$$\int_{\hat{e}} q = \frac{\hat{e}}{3}\sum_{i=1}^{3} q(\hat{p}_i) - \frac{1}{24}\int_{\hat{e}}\sum_{i=1}^{3}(\hat{l}_i)^2(\hat{D}_i)^2 q, \tag{3.4}$$

which is invariant under affine-linear transformations. Define the linear bounded functional F on $W_1^3(\hat{e})$ by

$$F(\hat{w}) = \int_{\hat{e}}(\hat{w} - \hat{w}_I) + \frac{1}{24}\int_{\hat{e}}\sum_{i=1}^{3}(\hat{l}_i)^2(\hat{D}_i)^2\hat{w}. \tag{3.5}$$

From (3.4) and noting that $(w - w_I)(p_i) = 0$ and $D_i^2 w_I = 0$, we obtain

$$F(q) = 0, \quad \forall\, q \in P_2(\hat{e}).$$

Then, by using the Bramble-Hilbert lemma,

$$|\, F(\hat{w}\,)| \le C|\hat{w}|_{3,1,\hat{e}}\,. \tag{3.6}$$

Combining (3.5) and (3.6) and using the affine-linear transformation, we have

$$\int_e (w - w_I) = -\frac{1}{24}\int_e\sum_{i=1}^{3} l_i^2 D_i^2 w + \mathcal{O}(h^3)|\, w\,|_{3,1,e}. \tag{3.7}$$

Let $\overline{\phi} = \dfrac{1}{|e|}\displaystyle\int_e \phi$. Writing

$$\int_e (w - w_I)\phi = \int_e (w - w_I)\overline{\phi} + \int_e (w - w_I)(\phi - \overline{\phi}),$$

it follows from (3.7) and the interpolation approximation properties that

$$\begin{aligned}\int_e (w - w_I)\phi &= -\frac{1}{24}\int_e\sum_{i=1}^{3} l_i^2 D_i^2 w(\overline{\phi} - \phi + \phi) + \mathcal{O}(h^3)|\, w\,|_{3,1,e}\overline{\phi} + \mathcal{O}(h^3)\|w\|_{2,e}\|\phi\|_{1,e}\\ &= -\frac{1}{24}\int_e\sum_{i=1}^{3} l_i^2 D_i^2 w\phi + \mathcal{O}(h^3)\Big(\|\, w\,\|_{2,e}\|\phi\|_{1,e} + \|w\|_{3,e}\|\phi\|_{0,e}\Big).\end{aligned}$$

The proof is completed. $\square$

Lemma 3.3. *Let the triangulation J_h be strongly regular, $\vec{\beta} \in [W_\infty^1(\Omega)]^2$, $w \in H^3(\Omega)$, $v \in S_h$. Then*

$$|\,(w - w_I, \vec{\beta}\cdot\nabla v\,)_h| \le Ch^2\|w\|_3\bigg(\|v\| + (\sum_{l\in\mathcal{B}} < M_e[v],[v] >_l + \int_{\partial\Omega}|\vec{\beta}\cdot nv|^2\,)^{\frac{1}{2}}\bigg).$$

Proof. By using Lemma 3.2 with $\phi = \vec{\beta}\cdot\nabla v$, the finite element inverse inequality and Green's formula, and noting that v is piecewise linear, we have

$$\begin{aligned}E_h &= \sum_{e\in J_h}\int_e (w - w_I)\vec{\beta}\cdot\nabla v = \sum_{e\in J_h} -\frac{1}{24}\int_e\sum_{i=1}^{3} l_i^2 D_i^2 w\vec{\beta}\cdot\nabla v + \mathcal{O}(h^2)\|\, w\,\|_3\|v\|\\ &= -\frac{1}{24}\sum_{e\in J_h}\int_{\partial e}\sum_{i=1}^{3} l_i^2 D_i^2 w\,\vec{\beta}\cdot n_e v + \mathcal{O}(h^2)\|\, w\,\|_3\|v\|\,.\end{aligned} \tag{3.8}$$

Now let the segment $l_e = l_{e'} = \partial e \cap \partial e'$ be the common edge of two adjacent triangular elements e and e' (see Fig. 3.1). Notice that, for any element boundary segment $l_e \not\subset \partial\Omega$, the integration in (3.8) is taken two times on $l_e = l_{e'}$, one is for $l_e \subset \partial e$ and the other for $l_{e'} \subset \partial e'$. Then, by using $[v] = v^+ - v^-$, $v(x \in e')^{\pm}|_{l_{e'}} = v(x \in e)^{\mp}|_{l_e}$ and $n_{e'} = -n_e$, we have from (3.8) that

$$\begin{aligned}
E_h =& -\frac{1}{24}\sum_{l_e\in\mathcal{B}}\int_{l_e}\sum_{i=1}^{3} l_i^2 D_i^2 w\,\vec{\beta}\cdot n_e v - \frac{1}{24}\sum_{l_{e'}\in\mathcal{B}}\int_{l_{e'}}\sum_{i=1}^{3}(l_i')^2(D_i')^2 w\vec{\beta}\cdot n_{e'} v \\
& -\frac{1}{24}\sum_{\partial e\in\partial\Omega}\int_{\partial e}\sum_{i=1}^{3} l_i^2 D_i^2 w\,\vec{\beta}\cdot n_e v + \mathcal{O}(h^2)\|\,w\,\|_3\|v\| \\
=& -\frac{1}{24}\sum_{l_e\in\mathcal{B}}\int_{l_e}\sum_{i=1}^{3}\Big(l_i^2 D_i^2 w - (l_i')^2(D_i')^2 w\Big)\vec{\beta}\cdot n_e v - \frac{1}{24}\sum_{l_{e'}\in\mathcal{B}}\int_{l_{e'}}\sum_{i=1}^{3}(l_i')^2(D_i')^2 w\vec{\beta}\cdot n_{e'}[\,v\,] \\
& -\frac{1}{24}\sum_{\partial e\in\partial\Omega}\int_{\partial e}\sum_{i=1}^{3} l_i^2 D_i^2 w\,\vec{\beta}\cdot n_e v + \mathcal{O}(h^2)\|\,w\,\|_3\|v\|, \qquad (3.9)
\end{aligned}$$

where l_i and l_i' ($i = 1, 2, 3$) are the edges of the two adjacent elements e and e', respectively, $D_i = n_{l_i} \cdot \nabla$, $D_i' = n_{l_i'} \cdot \nabla$, and n_{l_i} and $n_{l_i'}$ are the unit tangent vectors along l_i and l_i', respectively (see Lemmas 3.1 and 3.2). For example, as shown in Fig. 3.1, we may have

$$l_1 = |\overrightarrow{p_1p_2}|,\;\; l_2 = |\overrightarrow{p_2p_3}|,\;\; l_3 = |\overrightarrow{p_3p_1}|,\;\; l_1' = |\overrightarrow{p_1'p_2'}|,\;\; l_2' = |\overrightarrow{p_2'p_3'}|,\;\; l_3' = |\overrightarrow{p_3'p_1'}|.$$

Obviously the common edge $l_e = l_3 = l_3' = l_{e'}$. Now it follows from Lemma 3.1 that

$$\begin{aligned}
|\,l_i^2 D_i^2 w - (l_i')^2(D_i')^2 w\,| &= |\,(\,l_i^2 - (l_i')^2\,)D_i^2 w + (l_i')^2(\,D_i^2 - (D_i')^2\,)w\,| \\
&= |\,(l_i + l_i')(l_i - l_i')D_i^2 w + (l_i')^2(D_i + D_i')(D_i - D_i')w\,| \\
&\le Ch^3|D^2 w| + (l_i')^2|(n_{l_i} + n_{l_i'})\cdot\nabla(D_i - D_i')w\,| \le Ch^3|\,D^2 w\,|.
\end{aligned}$$

Substituting this into (3.9) yields

$$\begin{aligned}
|\,E_h\,| \le& Ch^3\sum_{l\in\mathcal{B}}\int_l |\,D^2 w\,||v| + Ch^2\sum_{l\in\mathcal{B}}\int_l |\,D^2 w\,||[\,v\,]| \\
& + Ch^2\|w\|_{2,\partial\Omega}\Big(\int_{\partial\Omega}|\vec{\beta}\cdot n\,v|^2\Big)^{\frac{1}{2}} + Ch^2\|w\|_3\|v\| \\
\le& Ch^2\|w\|_3\left[\;\|v\| + \Big(\sum_{l\in\mathcal{B}} < M_e[\,v\,],[\,v\,] >_l + \int_{\partial\Omega}|\vec{\beta}\cdot n\,v|^2\Big)^{\frac{1}{2}}\;\right],
\end{aligned}$$

where we have used the fact that $M_e = c_0 h^{-1} I$ and the following trace inequalities and inverse inequality:

$$\Big(\int_{\partial e} w^2\Big)^{\frac{1}{2}} \le Ch^{-\frac{1}{2}}\big(\,h\|\nabla w\|_{0,e} + \|w\|_{0,e}\big), \quad w \in H^1(e), \qquad (3.10)$$

$$\Big(\int_{\partial\Omega}|w|^2\Big)^{\frac{1}{2}} \le C(\Omega)\|w\|_1, \quad w \in H^1(\Omega); \quad \|v\|_{1.e} \le Ch^{-1}\|v\|_{0,e}, \quad v \in S_h. \qquad (3.11)$$

The proof is completed. □

3.2. Error analysis

First consider the steady-state problem. In order to do the error analysis here we assume a stronger condition than (2.5), which can be satisfied by some hyperbolic problems. There exists a constant $\sigma_1 > 0$ such that

$$(H) \qquad < (M + M^T)\mathbf{v}_h, \mathbf{v}_h >_{\partial\Omega} \geq \sigma_1 < \mathbf{v}_h, \mathbf{v}_h >_{\partial\Omega}, \ \forall \mathbf{v}_h \in [S_h]^m.$$

Theorem 3.1. *Let* $\mathbf{u}$ *and* $\mathbf{u}_h$ *be the solutions of problems* (2.1)-(2.2) *and* (2.8) *respectively,* $\mathbf{u} \in [H^3(\Omega)]^m$, *the triangulation* J_h *be strongly regular and hypothesis* (H) *hold. Then* $\mathbf{u}_h$ *satisfies the optimal order error estimate*

$$\begin{aligned}&\|\mathbf{u} - \mathbf{u}_h\| + \Big(< M(\mathbf{u} - \mathbf{u}_h), (\mathbf{u} - \mathbf{u}_h) >_{\partial\Omega} + \sum_{l \in \mathcal{B}} < M_e[\mathbf{u} - \mathbf{u}_h], [\mathbf{u} - \mathbf{u}_h] >_l \Big)^{\frac{1}{2}} \\ &\leq Ch^2 \|\mathbf{u}\|_3.\end{aligned}$$

Proof. First from (2.1)-(2.2) and (2.8), we have the error equation

$$A(\mathbf{u} - \mathbf{u}_h, \mathbf{v_h}) = 0, \ \ \forall \mathbf{v_h} \in [S_h]^m.$$

Then by using Green's formula and noting that $\mathbf{u} - \mathbf{u}_I \in C(\overline{\Omega})$, $D_e = \mathbf{A} \cdot n_e$, we obtain

$$\begin{aligned}& A(\mathbf{u}_h - \mathbf{u}_I, \mathbf{v}_h) = A(\mathbf{u} - \mathbf{u}_I, \mathbf{v}_h) \\ =& (\mathbf{A} \cdot \nabla(\mathbf{u} - \mathbf{u}_I), \mathbf{v}_h)_h + (B(\mathbf{u} - \mathbf{u}_I, \mathbf{v}_h)_h + \frac{1}{2} < (M - D_n)(\mathbf{u} - \mathbf{u}_I), \mathbf{v}_h >_{\partial\Omega} \\ =& - (\mathbf{u} - \mathbf{u}_I, \, \mathbf{A} \cdot \nabla \mathbf{v}_h)_h + ((B - div\mathbf{A})(\mathbf{u} - \mathbf{u}_I), \mathbf{v}_h)_h \\ &+ \sum_{e \in J_h} < D_e(\mathbf{u} - \mathbf{u}_I), \mathbf{v}_h >_{\partial e} + \frac{1}{2} < (M - D_n)(\mathbf{u} - \mathbf{u}_I), \mathbf{v}_h >_{\partial\Omega} \\ =& - (\mathbf{u} - \mathbf{u}_I, \, \mathbf{A} \cdot \nabla \mathbf{v}_h)_h + ((B - div\mathbf{A})(\mathbf{u} - \mathbf{u}_I), \mathbf{v}_h)_h \\ &+ \sum_{l \in \mathcal{B}} < D_e(\mathbf{u} - \mathbf{u}_I), [\mathbf{v}_h] >_l + \frac{1}{2} < (M + D_n)(\mathbf{u} - \mathbf{u}_I), \mathbf{v}_h >_{\partial\Omega}, \quad \forall \mathbf{v}_h \in [S_h]^m.\end{aligned}$$

Take $\mathbf{v}_h = \mathbf{u}_h - \mathbf{u}_I$. By using Lemmas 2.1 and 3.3, and the interpolation approximation property we obtain

$$\begin{aligned}& \frac{1}{2}\sigma_0 \|\mathbf{u}_h - \mathbf{u}_I\|^2 + \frac{1}{2} \sum_{l \in \mathcal{B}} < M_e[\mathbf{u}_h - \mathbf{u}_I], \, [\mathbf{u}_h - \mathbf{u}_I] >_l + \frac{1}{2} < M(\mathbf{u}_h - \mathbf{u}_I), \, \mathbf{u}_h - \mathbf{u}_I >_{\partial\Omega} \\ \leq& Ch^2 \|\mathbf{u}\|_3 \Big[\|\mathbf{u}_h - \mathbf{u}_I\| + \Big(\sum_{l \in \mathcal{B}} < M_e[\mathbf{u}_h - \mathbf{u}_I], [\mathbf{u}_h - \mathbf{u}_I] >_l + \int_{\partial\Omega} |D_n(\mathbf{u}_h - \mathbf{u}_I)|^2 \Big)^{\frac{1}{2}} \Big] \\ &+ Ch^2 \sum_{l \in \mathcal{B}} \int_l |\, D^2\mathbf{u}| \, |\, [\mathbf{u}_h - \mathbf{u}_I \,]| + Ch^2 \|\mathbf{u}\,\|_{2,\partial\Omega} \|\mathbf{u}_h - \mathbf{u}_I\|_{0,\partial\Omega} \\ \leq& Ch^2 \|\mathbf{u}\|_3 \Big[\|\mathbf{u}_h - \mathbf{u}_I\| + \Big(\sum_{l \in \mathcal{B}} < M_e[\mathbf{u}_h - \mathbf{u}_I], [\mathbf{u}_h - \mathbf{u}_I] >_l \\ &+ < M(\mathbf{u}_h - \mathbf{u}_I), (\mathbf{u}_h - \mathbf{u}_I) >_{\partial\Omega} \Big)^{\frac{1}{2}} \Big],\end{aligned}$$

where we have utilized hypothesis (H), $M_e = c_0 h^{-1} I$, and the trace inequalities (3.10)-(3.11). Thus, the proof is completed by using the continuity of $\mathbf{u}$ and $\mathbf{u}_I$. □

Remark 3.2. From Theorem 3.1 and noting that $M_e = c_0 hI$ we see that the jump of $\mathbf{u}_h$ on the element boundaries is of order $\mathcal{O}(h^{5/2})$. This is a superconvergence result.

Now we are in the position to discuss the nonsteady problem (2.12)-(2.14) and its discontinuous finite element approximation (2.15)-(2.16).

Theorem 3.2. *Let* $\mathbf{u}$ *and* $\mathbf{u}_h$ *be the solutions of problems* (2.12)-(2.14) *and* (2.15)-(2.16) *respectively,* $\mathbf{u}(0) \in [H^3(\Omega)]^m$, $\mathbf{u}_t(t) \in L_1(0,T;[H^3(\Omega)]^m)$, *the triangulation* J_h *be strongly regular and hypothesis* (H) *hold. Then, there exists a constant* C *independent of* $t \in [0,T)$ *such that*

$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\| \le e^{-\frac{\sigma_0}{2}t}\|\mathbf{u}(0) - \mathbf{u}_h(0)\| + Ch^2\Big(\|\mathbf{u}(0)\|_3 + \int_0^t \|\mathbf{u}_t(\tau)\|_3\, d\tau\Big), \quad t > 0.$$

Proof. First introduce the projection approximation of the solution $\mathbf{u}$ in $[S_h]^m$ by setting $R_h\mathbf{u}(t) : [0,T) \to [S_h]^m$ such that

$$A(\mathbf{u}(t) - R_h\mathbf{u}(t), \mathbf{v}_h) = 0, \quad \forall\, \mathbf{v}_h \in [S_h]^m.$$

From Theorem 3.1 we know that

$$\|D_t^j(\mathbf{u} - R_h\mathbf{u})(t)\| \le Ch^2\|D_t^j\mathbf{u}(t)\|_3, \quad t \in [0,T), \quad j = 0,1. \tag{3.12}$$

Now we write the error function as

$$\mathbf{u}(t) - \mathbf{u}_h(t) = \mathbf{u}(t) - R_h\mathbf{u}(t) + R_h\mathbf{u}(t) - \mathbf{u}_h(t) = \eta + \theta.$$

Then, from the equations satisfied by $\mathbf{u}(t)$, $\mathbf{u}_h(t)$ and $R_h\mathbf{u}(t)$, we see that $\theta \in [S_h]^m$ satisfies

$$(\theta_t, \mathbf{v}_h) + A(\theta, \mathbf{v}_h) = -(\eta_t, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in [S_h]^m. \tag{3.13}$$

Taking $\mathbf{v}_h = \theta$, similar to the argument of (2.18), and using the triangular inequality and (3.12), the proof is completed. $\square$

Remark 3.3. For nonsteady problems, the equation condition (2.4) is not necessary for our analysis. In fact, we may use the transformation: $\mathbf{u} = e^{\sigma t}\mathbf{w}$ with σ satisfying $\sigma > \|B + B^T - div\,\mathbf{A}\|_\infty$, so that (2.4) holds.

Remark 3.4. For some special cases, the hypothesis (H) in Theorems 3.1 and 3.2 can be removed, see Section 4 for an example.

4. Maxwell Equations in a Two-Dimensional Domain

The Maxwell equations are a class of very important partial differential equations in electromagnetism. Various approximation methods have been proposed for solving the Maxwell equations numerically. In this paper, as an application of our discontinuous finite element method mentioned in Section 2, we will discuss the two-dimensional linear Maxwell equations in the following form:

$$\frac{\partial H_x}{\partial t} = -\frac{\partial E_z}{\partial y}, \quad \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x}, \quad \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}. \tag{4.1}$$

For convenience, we consider Eq. (4.1) with periodic boundary value condition on the rectangular domain $\Omega = [0,\, a\,] \times [0,\, b]$. Set the vector function $\mathbf{w} = (\, H_x, H_y, E_z\,)^T$. Then the problem mentioned above can be rewritten as the following first-order hyperbolic system:

$$\mathbf{w}_t + A_1 \partial_x \mathbf{w} + A_2 \partial_y \mathbf{w} = \mathbf{0}, \quad t > 0, \;\; (x,y) \in \Omega, \tag{4.2}$$

$$N\mathbf{w} = \frac{1}{2}(M - D_n)\mathbf{w} = \mathbf{0}, \quad t > 0, \;\; (x,y) \in \partial\Omega, \tag{4.3}$$

with given initial value $\mathbf{w}(0,x,y) = \mathbf{w}_0(x,y)$ and matrices

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad D_n = \begin{pmatrix} 0 & 0 & n_y \\ 0 & 0 & -n_x \\ n_y & -n_x & 0 \end{pmatrix}.$$

For the periodic boundary value problem, we may simply choose the boundary matrix $N = O$ or $M = D_n$. Introduce the periodic function space

$$[\,H_p\,]^3 = \{\,\mathbf{w} : \overline{\Omega} \to R^3\,;\mathbf{w}(0,y) = \mathbf{w}(a,y),\; \mathbf{w}(x,0) = \mathbf{w}(x,b),\; (x,y) \in \partial\Omega\,\}.$$

Notice that for any function $\mathbf{w} \in [\,H_p\,]^3$, we have $< D_n\mathbf{w}, \mathbf{w} >_{\partial\Omega} = 0$, hence for problem (4.2)-(4.3) with periodic boundary value condition, we have the stability estimate

$$\|\mathbf{w}(t)\| \le \|\mathbf{w}(0)\|, \quad t > 0. \tag{4.4}$$

Introduce the discontinuous finite element space $[S_{h,p}]^3 = [S_h]^3 \cap [H_p]^3$. Now the discontinuous finite element approximation to the Maxwell equations reads: Find $\mathbf{w}_h(t) : [0,\infty) \to [S_{h,p}]^3$ such that

$$(\mathbf{w}_{h,t}, \mathbf{v}_h) + A(\mathbf{w}_h, \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in [S_{h,p}]^3, \quad t > 0, \tag{4.5}$$

$$\mathbf{w}(0) \in [S_{h,p}]^3. \tag{4.6}$$

According to Theorem 3.2 and Remark 3.3 (note that $< D_n\mathbf{u}, \mathbf{v} >_{\partial\Omega} = < M\mathbf{u}, \mathbf{v} >_{\partial\Omega} = 0$, $\forall\, \mathbf{u},\, \mathbf{v} \in [H_p]^3$, then from the argument of Theorem 3.1, it is easy to see that the hypothesis (H) can be removed in Theorems 3.1 and 3.2), we immediately obtain the following theorem.

Theorem 4.1. *Let* $\mathbf{w}$ *and* $\mathbf{w}_h$ *be the periodic solutions of problems* (4.2)-(4.3) *and* (4.5)-(4.6) *respectively,* $\mathbf{w}(0) \in [H^3(\Omega)]^3$, $\mathbf{w}_t(t) \in L_1(0,T;[H^3(\Omega)]^3\,)$, *and let the triangulation* J_h *be strongly regular. Then*

$$\|\mathbf{w}(t) - \mathbf{w}_h(t)\| \le \|\mathbf{w}(0) - \mathbf{w}_h(0)\| + Ch^2\Big(\|\mathbf{w}(0)\|_3 + \int_0^t \|\mathbf{w}_t(\tau)\|_3\, d\tau\Big), \quad t > 0.$$

Finally, we give a numerical example to validate the theoretical analysis. Notice that the Maxwell equation (4.1) admit the following exact solution:

$$\begin{pmatrix} H_x \\ H_y \\ E_z \end{pmatrix} = \begin{pmatrix} -\beta \\ \alpha \\ 1 \end{pmatrix} f(\cos\omega(t + \alpha\, x + \beta\, y)),$$

where f is an arbitrary proper smooth function, α, β, ω are constants, and $\alpha^2 + \beta^2 = 1$. In our numerical experiment, we take $f(u) = e^u$ and the domain $\Omega = [\,0,\, 2\pi/|\alpha\omega|\,] \times [\,0,\, 2\pi/|\beta\omega|\,]$, and the periodic boundary condition is used. In the numerical experiments, we have taken $\beta = \alpha = 1/\sqrt{2}$, $\omega = 2\sqrt{2}\,\pi$ and the final time $t = 1$. For the time discretization, we use the backward Euler scheme with step size $\triangle t = ch^2$. A uniform triangulation is formed by means of first dividing the domain into a uniform square partition and then dividing each square into two right triangles in the same configuration. Now the discontinuous finite element methods described in Section 2 can be applied to solve the problem (4.2)-(4.3). The numerical results are given in Table 4.1, where L_2 and L_∞ errors are presented with different mesh sizes. The numerical convergence rates are computed by using the formula,

$$\alpha = \ln\left(\frac{e(h)}{e(h/2)}\right)/\ln 2,$$

where $e(h)$ represents the error in the L_2 norm with mesh size h.

The numerical example suggests that the proposed methods have 2nd-order convergence rate.

Table 4.1: L_2, L_∞ errors and convergence rates.

h	L_∞ error	L_2 error	convergence rate
1/10	6.63E-4	9.85E-4	
1/20	3.67E-4	3.42E-4	1.5261
1/40	1.86E-4	7.90E-5	2.1141
1/80	9.52E-5	1.81E-5	2.1259
1/160	4.84E-5	4.21E-6	2.1041

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