Journal of Computational Mathematics, Vol.26, No.5, 2008, 716–727.

UNIFORMLY A POSTERIORI ERROR ESTIMATE FOR THE FINITE ELEMENT METHOD TO A MODEL PARAMETER DEPENDENT PROBLEM*

Yiran Zhang and Jun Hu
LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, China
Email: zhangyiran@math.pku.edu.cn

Abstract

This paper proposes a reliable and efficient a posteriori error estimator for the finite element methods for the beam problem. It is proved that the error can be bounded by the computable error estimator from above and below up to multiplicative constants that do neither depend on the meshsize nor on the thickness of the beam.

Mathematics subject classification: 65N30, 65N15.
Key words: The beam problem, A posteriori error estimator, Finite element method.

1. Introduction

The beam model [1,2,12] considered here reads: Seek two functions $\varphi_d(x)$ and $\omega_d(x)$ defined in the unit interval $I=[0,1]$ such that

$$\begin{array}{ll} -\varphi_d''+d^{-2}(\varphi_d-\omega_d')=0, & \text{in } (0,1),\\ d^{-2}(\varphi_d-\omega_d')'=g, & \text{in } (0,1),\\ \varphi_d(0)=\varphi_d(1)=\omega_d(0)=\omega_d(1)=0. & \end{array} \tag{1.1}$$

Here and throughout the paper, the parameter d $(0<d<1)$ denotes the thickness of the beam. This model may be derived from the equations of plane linear elasticity by dimensional reduction, which means that an undisplaced plane body occupying the region $\{0\le x\le 1, -\frac{d}{2}\le y\le \frac{d}{2}\}$ be subject to a smooth vertical body force $-d^2g(x)$. Physically ω_d represents the vertical displacement of the midline, and φ_d the rotation of the cross section.

The corresponding variational formulation is as follows. Given $g\in L^2(I)$, find φ_d, $\omega_d\in H_0^1(I)$ such that

$$(\varphi_d',\psi')+d^{-2}(\varphi_d-w_d',\psi-\upsilon')=(g,\upsilon), \qquad \text{for all } \psi,\upsilon\in H_0^1(I), \tag{1.2}$$

with the shear force

$$\gamma_d=d^{-2}(\varphi_d-\omega_d'). \tag{1.3}$$

This paper is devoted to this beam problem which is difficult due to the small parameter related to the thickness of the beam. For the reason of a highly desirable quality of a numerical method, we hope to approximate the solution as accurately as possible for all values of this parameter. For a priori error estimate analysis of the beam problem, Arnold [1,2] investigates the robustness of two families of finite element methods with respect to the parameter d. He points out that a standard linear finite element is found to be not robust at all which

* Received August 31, 2007 / Revised version received November 28, 2007 / Accepted December 4, 2007 /

means the approximation errors do not converge to zero at the optimal rates uniformly in d. Although the same method does converge uniformly with respect to the parameter when the spaces of piecewise polynomials of order at least two are used, the approximation degenerates as the thickness of the beam decreases, resulting in a reduced uniform order of convergence. Comparatively, the mixed method he considers as the second method produces good results and the errors converge uniformly with respect to the parameter for (almost) any choice of finite element spaces for the original displacement without the degeneracy mentioned above. All aforementioned papers are only concerning the a priori error analysis of the beam problem. As for a posteriori error analysis of this problem, as far as we know, no work can be found in the literature.

It is worth mentioning that recently there are some progress on the a posteriori error estimates of the Reissner-Mindlin plate problem [7,8,11] and the unifying theory of a posteriori error analysis of finite element methods [5–9,11]. In [11], Hu and Huang introduce a sparse mixed formulation and establish the equivalence between the energy norms of errors and the dual norms of the residuals. They propose some sufficient conditions and provide a unified framework for the a posteriori error analysis of the finite element methods of the Reissner-Mindlin plate problem. This paper follows these ideas and establishes some residual representation which is closely related to the approximation errors, and presents a posteriori error estimates of the beam problem. Then we analyze the Arnold's discrete scheme of this problem [1,2] within this framework, and propose a reliable and efficient residual-based a posteriori error estimator. The related multiplicative constants do neither depend on the meshsize nor on the beam thickness.

The outline of the paper is as follows. In Section 2 we establish some equivalence between the norms of errors and the dual norms of some residuals. In Section 3 we present Arnold's discrete scheme for the beam problem. The main results of this paper will be also stated. We prove the results in Sections 4 and 5.

Throughout this paper, all function spaces will be formed with respect to the unit interval $I = [0,1]$. For functions $f(x)$ and $g(x)$ defined in $[0,1]$, we let (f,g) denote the inner productor $\int_0^1 f(x)g(x)dx$. The associated L^2-norm of the function is written as $\|f\|$, while $\|f\|_r$ denote the norm in the Sobolev space $H^r(I) : \|f\|_r^2 = \|f\|^2 + \|f^{'}\|^2 \ldots + \|f^{(r)}\|^2$, where $f^{(r)} = \frac{d^r f}{dx^r}$. The space $H_0^1(I) = \{f \in H^1(I) | f(0) = f(1) = 0\}$, on which the norm $\|f^{'}\|$ is equivalent to the H^1 norm. The space $H^{-1}(I)$ is dual to $H_0^1(I)$ equipped with the norm

$$\|g\|_{-1} = \sup_{f \in H_0^1} \frac{(f,g)}{\|f'\|}, \qquad \forall g \in H^{-1}(I).$$

In this paper the generic constant $C > 0$ independent of the beam thickness d below may be different at different occurrences. An inequality $a \preceq b$ replaces $a \leq Cb$, $a \approx b$ abbreviates $a \preceq b \preceq a$.

2. Residual-based a Posteriori Error Control

Follows the ideas of [3,7,11], let $d^{-2} = \frac{3}{4} + \beta^{-2}$ and introduce an additional independent variable

$$\gamma_d^* = \beta^{-2}(\varphi_d - \omega_d'). \tag{2.1}$$

We obtain a new established mixed version of the beam problem which is equivalent to the weak formulation (1.2): Given $g \in L^2(I)$, find $(\varphi_d, \omega_d, \gamma_d^*) \in W \times \Theta \times Q = H_0^1(I) \times H_0^1(I) \times L^2(I)$

such that

$$B_\beta(\varphi_d,\omega_d,\gamma_d^*;\psi,\upsilon,\zeta)=(g,\upsilon),\qquad \forall(\psi,\upsilon,\zeta)\in W\times\Theta\times Q, \tag{2.2}$$

where the bilinear form $B_\beta(\cdot,\cdot)$ is defined by

$$B_\beta(\varphi_d,\omega_d,\gamma_d^*;\psi,\upsilon,\zeta)=a(\varphi_d,\omega_d;\psi,\upsilon)+b(\psi,\upsilon;\gamma_d^*)+b(\varphi_d,\omega_d;\zeta)-\beta^2(\zeta_d,\gamma_d^*), \tag{2.3}$$

with the bilinear forms

$$a(\varphi_d,\omega_d;\psi,\upsilon)=\int_0^1\varphi_d'(x)\psi'(x)dx+\frac{3}{4}\int_0^1\big(\varphi_d(x)-\omega_d'(x)\big)\big(\psi(x)-\upsilon'(x)\big)dx, \tag{2.4}$$

$$b(\psi,\upsilon;\gamma_d^*)=\int_0^1\big(\psi(x)-\upsilon'(x)\big)\gamma_d^*(x)dx. \tag{2.5}$$

We define the norms on the spaces $W\times\Theta$ as follows

$$|||(\varphi,\omega)|||=a(\varphi,\omega;\varphi,\omega)^{\frac12}=\{\|\varphi'\|^2+\frac{3}{4}\|\varphi-\omega'\|^2\}^{\frac12},\qquad \text{for all }\ \varphi,\omega\in H_0^1(I), \tag{2.6}$$

which is equivalent to the $(H_0^1(I))^2$-norm $\|(\varphi,\omega)\|_1=\{\|\varphi'\|^2+\|\omega'\|^2\}^{\frac12}$. In fact, by Poincare's inequality $\|\varphi\|\le\|\varphi'\|$, and the triangle inequality $\|\omega'\|\le\|\varphi-\omega'\|+\|\varphi\|$, we have

$$\frac13\{\|\varphi'\|^2+\|\omega'\|^2\}\le|||(\varphi,\omega)|||^2\le\frac52\{\|\varphi'\|^2+\|\omega'\|^2\},\ \ \text{for all }\ \varphi,\omega\in H_0^1(I). \tag{2.7}$$

Define the following parameter dependent norm for the space Q,

$$\|\zeta\|_Q=\sup_{(\psi,\upsilon)\in(H_0^1(I))^2\backslash\{0\}}\frac{(\psi-\upsilon',\zeta)}{|||(\psi,\upsilon)|||}+\|\beta\zeta\|. \tag{2.8}$$

It is in fact owing to this norm that we can obtain robust error estimates.

Remark 2.1. We set the space $Q_0=\{\zeta|\zeta\in H^{-1}(I),\zeta'\in H^{-1}(I)\}$ with the norm $\|\zeta\|_{Q,0}=\{\|\zeta\|_{-1}^2+\|\zeta'\|_{-1}^2\}^{\frac12}$. Then the norm $\|\zeta\|_{Q,0}$ is equivalent to $\sup_{(\psi,\upsilon)\in(H_0^1(I))^2\backslash\{0\}}\frac{(\psi-\upsilon',\zeta)}{|||(\psi,\upsilon)|||}$. In fact,

$$\begin{aligned}&\sup_{(\psi,\upsilon)\in(H_0^1(I))^2\backslash\{0\}}\frac{(\psi-\upsilon',\zeta)}{|||(\psi,\upsilon)|||}\\ \preceq&\sup_{\psi\in H_0^1(I)\backslash\{0\}}\frac{(\psi,\zeta)}{\|\psi'\|}+\sup_{\upsilon\in H_0^1(I)\backslash\{0\}}\frac{(\upsilon',\zeta)}{\|\upsilon'\|}\\ =&\|\zeta\|_{-1}+\|\zeta'\|_{-1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\|\zeta\|_{-1}+\|\zeta'\|_{-1}\\ =&\sup_{\psi\in H_0^1(I)\backslash\{0\}}\frac{(\psi,\zeta)}{\|\psi'\|}+\sup_{\upsilon\in H_0^1(I)\backslash\{0\}}\frac{(\upsilon,\zeta')}{\|\upsilon'\|}\\ \preceq&\sup_{\substack{\psi\in H_0^1(I)\backslash\{0\},\\ \upsilon=0}}\frac{(\psi-\upsilon',\zeta)}{|||(\psi,\upsilon)|||}+\sup_{\substack{\upsilon\in H_0^1(I)\backslash\{0\},\\ \psi=0}}\frac{(\psi-\upsilon',\zeta)}{|||(\psi,\upsilon)|||}\\ \preceq&\sup_{(\psi,\upsilon)\in(H_0^1(I))^2\backslash\{0\}}\frac{(\psi-\upsilon',\zeta)}{|||(\psi,\upsilon)|||}.\end{aligned}$$

Since the bilinear form $a(\cdot,\cdot)$ defined by (2.4) is coercive on the whole space $W\times\Theta$, we can conclude from Theorem 2 of [4] that

Theorem 2.1. *The bilinear form $B_\beta(\cdot,\cdot)$ defined by* (2.3) *provides an isomorphism between $W\times\Theta\times Q$ and its dual space of $W\times\Theta\times Q$, namely,*

$$\sup_{(\psi,\upsilon;\zeta)\in W\times\Theta\times Q\backslash\{0\}} \frac{B_\beta(\varphi,\omega,\gamma^*;\psi,\upsilon,\zeta)}{|||(\psi,\upsilon)|||+\|\zeta\|_Q}$$
$$\approx |||(\varphi,\omega)|||+\|\gamma^*\|_Q, \qquad \textit{for all } (\varphi,\omega,\gamma*)\in W\times\Theta\times Q. \tag{2.9}$$

Suppose $(\widetilde{\varphi}_h,\widetilde{\omega}_h,\widetilde{\gamma}_h^*)\in W\times\Theta\times Q$ is some approximation to $(\varphi_d,\omega_d,\gamma_d^*)$ over some partition Δ of I. Then we obtain the immediate corollary of Theorem 2.1.

Corollary 2.1. *The error $|||(\varphi_d-\widetilde{\varphi}_h,\omega_d-\widetilde{\omega}_h)|||+\|\gamma_d-\widetilde{\gamma}_h^*\|_Q$ is equivalent to the dual norms of residuals:*

$$\begin{aligned}
&|||(\varphi_d-\widetilde{\varphi}_h,\omega_d-\widetilde{\omega}_h)|||+\|\gamma_d-\widetilde{\gamma}_h^*\|_Q\\
\approx&\sup_{(\psi,\upsilon,\zeta)\in W\times\Theta\times Q\backslash\{0\}}\frac{B_\beta(\varphi_d-\widetilde{\varphi}_h,\omega_d-\widetilde{\omega}_h,\gamma_d^*-\widetilde{\gamma}_h^*;\psi,\upsilon,\zeta)}{|||(\psi,\upsilon)|||+\|\zeta\|_Q}\\
\approx&\sup_{(\psi,\upsilon)\in W\times\Theta\backslash\{0\}}\frac{Res_\Theta(\psi)+Res_W(\upsilon)}{|||(\psi,\upsilon)|||}+\sup_{\zeta\in Q\backslash\{0\}}\frac{Res_Q(\zeta)}{\|\zeta\|_Q},
\end{aligned}\tag{2.10}$$

with the residuals

$$Res_W(\upsilon)=(g,\upsilon)+\frac{3}{4}(\widetilde{\varphi}_h-\widetilde{\omega}_h',\upsilon')+(\upsilon',\widetilde{\gamma}_h^*), \tag{2.11}$$

$$Res_\Theta(\psi)=-(\widetilde{\varphi}_h',\psi')-\frac{3}{4}(\widetilde{\varphi}_h-\widetilde{\omega}_h',\psi)-(\psi,\widetilde{\gamma_h^*}), \tag{2.12}$$

$$Res_Q(\zeta)=-(\widetilde{\varphi}_h-\widetilde{\omega}_h'-\beta^2\widetilde{\gamma_h^*},\zeta). \tag{2.13}$$

Proof. We can immediately get the first "$\approx$" in (2.10) from (2.9) in Theorem 2.1. In order to get the second "$\approx$", we use the following two steps. First,

$$\begin{aligned}
&\sup_{(\psi,\upsilon,\zeta)\in W\times\Theta\times Q\backslash\{0\}}\frac{B_\beta(\varphi_d-\widetilde{\varphi}_h,\omega_d-\widetilde{\omega}_h,\gamma_d^*-\widetilde{\gamma}_h^*;\psi,\upsilon,\zeta)}{|||(\psi,\upsilon)|||+\|\zeta\|_Q}\\
=&\sup_{(\psi,\upsilon,\zeta)\in W\times\Theta\times Q\backslash\{0\}}\frac{Res_\Theta(\psi)+Res_W(\upsilon)+Res_Q(\zeta)}{|||(\psi,\upsilon)|||+\|\zeta\|_Q}\\
\le&\sup_{(\psi,\upsilon,\zeta)\in W\times\Theta\times Q\backslash\{0\}}\frac{Res_\Theta(\psi)+Res_W(\upsilon)}{|||(\psi,\upsilon)|||+\|\zeta\|_Q}+\sup_{(\psi,\upsilon,\zeta)\in W\times\Theta\times Q\backslash\{0\}}\frac{Res_Q(\zeta)}{|||(\psi,\upsilon)|||+\|\zeta\|_Q}\\
\le&\sup_{(\psi,\upsilon)\in W\times\Theta\backslash\{0\}}\frac{Res_\Theta(\psi)+Res_W(\upsilon)}{|||(\psi,\upsilon)|||}+\sup_{\zeta\in Q\backslash\{0\}}\frac{Res_Q(\zeta)}{\|\zeta\|_Q}.
\end{aligned}$$

Second,

$$\begin{aligned}
&\sup_{(\psi,\upsilon)\in W\times\Theta\backslash\{0\}}\frac{Res_\Theta(\psi)+Res_W(\upsilon)}{|||(\psi,\upsilon)|||}+\sup_{\zeta\in Q\backslash\{0\}}\frac{Res_Q(\zeta)}{\|\zeta\|_Q},\\
=&\sup_{\substack{(\psi,\upsilon,\zeta)\in W\times\Theta\times Q\backslash\{0\},\\ \zeta=0}}\frac{Res_\Theta(\psi)+Res_W(\upsilon)+Res_Q(\zeta)}{|||(\psi,\upsilon)|||+\|\zeta\|_Q}\\
&+\sup_{\substack{(\psi,\upsilon,\zeta)\in W\times\Theta\times Q\backslash\{0\},\\ (\psi,\upsilon)=0}}\frac{Res_\Theta(\psi)+Res_W(\upsilon)+Res_Q(\zeta)}{|||(\psi,\upsilon)|||+\|\zeta\|_Q}\\
\preceq&\sup_{(\psi,\upsilon,\zeta)\in W\times\Theta\times Q\backslash\{0\}}\frac{Res_\Theta(\psi)+Res_W(\upsilon)+Res_Q(\zeta)}{|||(\psi,\upsilon)|||+\|\zeta\|_Q}.
\end{aligned}$$

That ends the proof. $\square$

Remark 2.2. Here and throughout the paper, $\widetilde{\omega}_h, \widetilde{\varphi}_h, \widetilde{\gamma_h^*}$ are not necessarily the discrete functions. However, the subindex h refers to the fact that they might be closely related to some discrete functions $\varphi_h, \omega_h, \gamma_h^*$ and they are on our disposal. We will propose one simplest style of them in Section 3 just as $\widetilde{\omega}_h = \omega_h, \widetilde{\varphi}_h = \varphi_h, \widetilde{\gamma_h^*} = \gamma_h^*$. Nevertheless, we have different choices of these functions when we cope with different classes of methods which is one of the key ideas of uniform a posteriori error analysis in [5,6,11].

3. Discrete Schemes of the Beam Problem and the Main Results

For the purpose of discretization we shall use finite element spaces defined with reference to partitions of I. If $\triangle = \{x_0, x_0, \cdots, x_n\}$ $(0 = x_0 < x_1 < \cdots < x_n = 1)$ is a partition of I, let $I_{i,\triangle} = [x_{i-1}, x_i], h_{i,\triangle} = |x_i - x_{i-1}|$, $h_\triangle = \max\limits_i h_{i,\triangle}$. Without misapprehending, we often omit the subscript $\triangle$, just denote $I_{i,\triangle}$ by I_i, $h_{i,\triangle}$ by h_i, $h_\triangle$ by h. For $r \geq 0$ and the partition $\triangle$, $\mu_{-1}^r(\triangle)$ denotes the spaces of functions on I which restrict to polynomial functions of degree at most r on each subinterval $[x_{i-1}, x_i] = I_i$. The subscript -1 refers to the lack of continuity constraint. For $k \geq 0$, we let

$$\mu_k^r(\triangle) = \mu_{-1}^r(\triangle) \cap C^k(I), \quad \mu_{k,0}^r(\triangle) = \mu_k^r(\triangle) \cap H_0^1(I).$$

Let W_h, Θ_h, Q_h be finite dimensional subspaces of W, Θ, Q with respect to a partition $\triangle$ of I, and $\mu_{0,0}^1(\triangle) \subset W_h$, $\mu_{0,0}^1(\triangle) \subset \Theta_h$. The discretization scheme for the beam problem reads: Find $(\varphi_h, \omega_h, \gamma_h) \in W_h \times \Theta_h \times Q_h$ such that

$$\begin{cases} (\varphi_h', \psi_h') + (\gamma_h, \mathrm{R}_h(\psi_h - \upsilon_h)) = (g, \upsilon_h), & \forall (\psi_h, \upsilon_h) \in W_h \times \Theta_h, \\ (\mathrm{R}_h(\varphi_h - \omega_h'), \zeta_h) - d^2(\gamma_h, \zeta_h) = 0, & \forall \zeta_h \in Q_h, \end{cases} \tag{3.1}$$

with the discrete shear force

$$\gamma_h = d^{-2}\mathrm{R}_h(\varphi_h - \omega_h'). \tag{3.2}$$

Here the operator R_h is a reduction integration operator which can be regarded as a L^2 projection onto the space Q_h. Note that the discrete scheme is often not of the discrete counterpart of the mix formulation introduced in Section 2. This discrete scheme has been analyzed by Arnold in [1,2]. For (almost) any choice of finite element spaces for the original displacement variables, it is proved that there is a finite element space in which to approximate the shear stress variable such that the resulting mixed finite element method is stable with the constant independent of the beam thickness. A special case of the scheme will be provided as an example in this section.

We define

$$\gamma_h^* = \beta^{-2}\mathrm{R}_h(\varphi_h - \omega_h'). \tag{3.3}$$

and note that $\gamma_d^* = \beta^{-2}(\varphi_d - \omega_d')$ is defined by (2.1). We choose in (2.10)

$$\widetilde{\varphi}_h = \varphi_h, \quad \widetilde{\omega}_h = \omega_h, \quad \widetilde{\gamma}_h^* = \gamma_h^*. \tag{3.4}$$

Then we will prove in Sections 4 and 5 the following conclusion.

Theorem 3.1. *The error $|||(\varphi_d - \varphi_h, \omega_d - \omega_h)||| + \|\gamma_d^* - \gamma_h^*\|_Q$ can be bounded from above and below by the estimator η_h in the sense that*

$$|||(\varphi_d - \varphi_h, \omega_d - \omega_h)||| + \|\gamma_d^* - \gamma_h^*\|_Q \approx \eta_h + osc(g), \tag{3.5}$$

where the estimator and the oscillation read, respectively,

$$\eta_h^2 = \sum_i h_i^2 \|g - \gamma_h'\|^2_{L^2(I_i)} + \sum_i h_i^2 \|\gamma_h - \varphi_h''\|^2_{L^2(I_i)} + \|(I - R_h)(\varphi_h - \omega_h')\|^2, \tag{3.6}$$

$$osc^2(g) = \sum_i h_i^2 \|g - g_h\|^2_{L^2(I_i)}. \tag{3.7}$$

Here g_h *is the* L^2 *projection of* g *onto the space* $\mu^1_{-1}(\triangle)$.

Example 3.1 ([2]) Consider a special case of the discrete scheme (3.1). Take $W_h = \Theta_h = \mu^r_{0,0}(\triangle)$, and $Q_h = \mu^{r-1}_{-1}(\triangle)$. A reduced integration finite element is given as follows: Find $(\varphi_h, \omega_h) \in W_h \times \Theta_h$ such that

$$(\varphi_h', \psi_h') + d^{-2}(\varphi_h - \omega_h', \psi_h - \upsilon_h')_\triangle = (g, \upsilon), \quad \text{for all } (\psi_h, \upsilon_h) \in W_h \times \Theta_h. \tag{3.8}$$

The reduced integration $(\cdot,\cdot)_\triangle$ here is defined as below.

$$(\zeta, \delta)_\triangle = \sum_{i=1}^n (x_i - x_{i-1}) \sum_{j=1}^r \rho_j (\zeta\delta)[x_{i-1} + (x_i - x_{i-1}) l_j], \quad \text{for all } \zeta, \delta \in \mu^r_{-1}(\triangle),$$

where $0 < l_1 < \cdots < l_r < 1$ and $\rho_1, \rho_2, \cdots, \rho_r$ denote respectively the points and weights of the r point Gauss quadrature rule on [0,1].

Let $\gamma_h = d^{-2}\mathrm{P}_h(\varphi_h - \omega_h')$ with the L_2 projection P_h onto the space $\mu^{r-1}_{-1}(\triangle) = Q_h$. We can show that

$$(\gamma_h, \mathrm{P}_h(\psi_h - \upsilon_h')) = d^{-2}(\varphi_h - \omega_h', \psi_h - \upsilon_h')_\triangle. \tag{3.9}$$

To show this claim it suffices to prove that

$$(\mathrm{P}_h\zeta, \sigma) = (\zeta, \sigma)_\triangle, \quad \text{for} \quad \zeta, \sigma \in \mu^r_{-1}(\triangle), \tag{3.10}$$

which means the scheme (3.8) is just the scheme of (3.1) with $\mathrm{R}_h = \mathrm{P}_h$. For $\zeta \in \mu^r_{-1}(\triangle)$, let $\mathrm{I}_\triangle\zeta \in \mu^{r-1}_{-1}(\triangle)$ interpolate ζ at the Gauss points. Then, for $\eta \in \mu^{r-1}_{-1}(\triangle)$,

$$(\mathrm{I}_\triangle\zeta, \eta) = (\mathrm{I}_\triangle\zeta, \eta)_\triangle = (\zeta, \eta)_\triangle = (\zeta, \eta), \tag{3.11}$$

so $\mathrm{I}_\triangle\zeta = \mathrm{P}_h\zeta$. Hence, for $\sigma \in \mu^r_{-1}(\triangle)$,

$$(\mathrm{P}_h\zeta, \sigma) = (\mathrm{I}_\triangle\zeta, \sigma) = (\mathrm{I}_\triangle\zeta, \sigma)_\triangle = (\zeta, \sigma)_\triangle,$$

which proves (3.10). Consequently, the claim (3.9) is proved.

4. The Reliability of the Estimator

This section is devoted to prove that the dual norms of residual $\mathrm{Res}_W(\upsilon) + \mathrm{Res}_\Theta(\psi)$ and the residual $\mathrm{Res}_Q(\zeta)$ can be bounded by the estimator η_h.

Theorem 4.1. *It holds that*

$$\sup_{\psi,\upsilon \in H_0^1(I)\backslash\{0\}} \frac{Res_W(\upsilon) + Res_\Theta(\psi)}{|||(\psi, \upsilon)|||} \preceq \eta_h,$$

where the estimator η_h *is defined by* (3.6).

Proof. From the definitions of $\mathrm{Res}_W(\upsilon)$ and $\mathrm{Res}_\Theta(\psi)$ in (2.11) and (2.12), and the choice of (3.4), we have, for all $(\psi,\upsilon)\in W\times\Theta$,

$$\begin{aligned}
&\mathrm{Res}_W(\upsilon)+\mathrm{Res}_\Theta(\psi)\\
=&(g,\upsilon)+\frac{3}{4}(\varphi_h-\omega_h',\upsilon')+(\upsilon',\beta^{-2}\mathrm{R}_h(\varphi_h-\omega_h'))\\
&-(\varphi_h',\psi')-\frac{3}{4}(\varphi_h-\omega_h',\psi)-(\psi,\beta^{-2}\mathrm{R}_h(\varphi_h-\omega_h')\\
=&(g,\upsilon-J\upsilon)+(\gamma_h,(\upsilon-J\upsilon)')-(\gamma_h,\psi-J\psi)-(\varphi_h',(\psi-J\psi)')\\
&+(g,J\upsilon)-(\varphi_h',(J\psi)')-(\gamma_h,\mathrm{R}_hJ\psi-\mathrm{R}_h(J\upsilon)')+(\gamma_h-\frac{3}{4}(\varphi_h-\omega_h')\\
&-\beta^{-2}\mathrm{R}_h(\varphi_h-\omega_h'),\psi-\upsilon')+(\gamma_h,(\mathrm{R}_h-\mathrm{I})(J\psi-(J\upsilon)'))\\
=&I_1+I_2+I_3+I_4+I_5,
\end{aligned}\tag{4.1}$$

where $J: H_0^1(I)\longrightarrow\mu_{0,0}^1(\triangle)$ is the usual Lagrange interpolation operator:

$$J\psi(x_i)=\psi(x_i),\quad x_i\in\triangle=\{x_0,x_1,\cdots,x_n\}\ (0=x_0<x_1<\cdots<x_n=1).\tag{4.2}$$

From this interpolation, one has

$$\|\psi-J\psi\|_{L^2(I_i)}\preceq h_i\|\psi'\|_{L^2(I_i)},\quad \psi\in H_0^1(I).\tag{4.3}$$

Estimate for I_1. Using an integration by parts and the Lagrange interpolation of (4.2) yields

$$\begin{aligned}
I_1=&(g,\upsilon-J\upsilon)+(\gamma_h,(\upsilon-J\upsilon)')\\
=&(g-\gamma_h',\upsilon-J\upsilon)+\sum_i\Big\{\gamma_h(x_i)(\upsilon-J\upsilon)(x_i)-\gamma_h(x_{i-1})(\upsilon-J\upsilon)(x_{i-1})\Big\}\\
\preceq&\sum_i\|g-\gamma_h'\|_{L^2(I_i)}\|\upsilon-J\upsilon\|_{L^2(I_i)}\\
\preceq&\sum_i h_i\|g-\gamma_h'\|_{L^2(I_i)}\|\upsilon'\|_{L^2(I_i)}\\
\preceq&\{\sum_i h_i^2\|g-\gamma_h'\|^2_{L^2(I_i)}\}^{\frac12}|||(\psi,\upsilon)|||.
\end{aligned}\tag{4.4}$$

Estimate for I_2. Again using an integration by parts and the Lagrange interpolation of (4.2) leads to

$$\begin{aligned}
I_2=&-(\gamma_h,\psi-J\psi)-(\varphi_h',(\psi-J\psi)')\\
=&(-\gamma_h+\varphi_h'',\psi-J\psi)-\sum_i\Big\{\varphi_h'(x_i)(\psi-J\psi)(x_i)-\varphi_h'(x_{i-1})(\psi-J\psi)(x_{i-1})\Big\}\\
\preceq&\sum_i h_i\|-\gamma_h+\varphi_h''\|_{L^2(I_i)}\|\psi'\|_{L^2(I_i)}\\
\preceq&\{\sum_i h_i^2\|-\gamma_h+\varphi_h''\|^2_{L^2(I_i)}\}^{\frac12}|||(\psi,\upsilon)|||.
\end{aligned}\tag{4.5}$$

Estimate for I_3. Note that

$$\mu_{0,0}^1(\triangle)\subset W_h\quad\textit{and}\quad\mu_{0,0}^1(\triangle)\subset\Theta_h.\tag{4.6}$$

It follows from the discrete Problem (3.1) with $\upsilon_h = J\upsilon$ and $\psi_h = J\psi$ that

$$I_3 = (g, J\upsilon) - (\varphi_h', (J\psi)') - (\gamma_h, \mathrm{R}_h J\psi - \mathrm{R}_h(J\upsilon)') = 0. \tag{4.7}$$

Estimate for I_4. By $\gamma_h = (\frac{3}{4} + \beta^{-2})\mathrm{R}_h(\varphi_h - \omega_h')$, we derive that

$$\begin{aligned} I_4 &= \Big(\gamma_h - \frac{3}{4}(\varphi_h - \omega_h') - \beta^{-2}\mathrm{R}_h(\varphi_h - \omega_h'), \psi - \upsilon'\Big) \\ &\preceq \|(\mathrm{I} - \mathrm{R}_h)(\varphi_h - w_h')\|_{L^2(I)} |||(\psi, \upsilon)|||. \end{aligned} \tag{4.8}$$

Estimate for I_5. Since R_h is a L^2 projection onto Q_h, we have

$$I_5 = (\gamma_h, (\mathrm{R}_h - \mathrm{I})(J\psi - (J\upsilon)')) = 0. \tag{4.9}$$

Combining (4.1)-(4.9) ends the proof of the theorem. $\square$

We now prove that the dual norm of the residual Res_Q can be bounded by the estimator η_h.

Theorem 4.2. *It holds that*

$$\sup_{\zeta \in Q \backslash \{0\}} \frac{Res_Q(\zeta)}{\|\zeta\|_Q} \preceq \eta_h.$$

Before proving Theorem 4.2, we need the following lemma.

Lemma 4.1. *Given $\zeta \in L^2(I)$, there exist $c_0 \in R$ and $\sigma_0 \in L^2(I)$ such that $\zeta = c_0 + \sigma_0$ and*

$$\sup_{\psi, \upsilon \in H_0^1(I) \backslash \{0\}} \frac{(\psi - \upsilon', \zeta)}{|||\psi, \upsilon|||} = \Big(\|c_0\|_{-1}^2 + \frac{4}{3}\|\sigma_0\|^2\Big)^{\frac{1}{2}}.$$

Proof. Given $\zeta \in L^2(I)$, let $(\varphi, \omega) \in W \times \Theta$ solves the following auxiliary problem:

$$(\varphi', \psi') + \frac{3}{4}(\varphi - \omega', \psi - \upsilon') = -(\zeta, \psi - \upsilon'), \quad \forall (\psi, \upsilon) \in V. \tag{4.10}$$

The fact that $a(\cdot, \cdot)$ from (2.4) is coercive on the whole space $W \times \Theta$ implies that the unique existence of $(\varphi, \omega) \in W \times \Theta$ for the auxiliary problem (4.10). We choose $(\psi, \upsilon) = (\varphi, \omega)$ in (4.10). Consequently,

$$|||(\varphi, \omega)||| = \frac{-(\varphi - \omega', \zeta)}{|||(\varphi, \omega)|||} \leq \sup_{\psi, \upsilon \in H_0^1(I) \backslash \{0\}} \frac{(\psi - \upsilon', \zeta)}{|||(\psi, \upsilon)|||}. \tag{4.11}$$

Setting $\psi = 0$ in (4.10), we have

$$(\frac{3}{4}(\varphi - \omega') + \zeta, \upsilon') = 0, \quad \forall \upsilon \in H_0^1(I),$$

which implies that $\frac{3}{4}(\varphi - \omega') + \zeta$ is a constant on I. Let

$$c_0 = \zeta + \frac{3}{4}(\varphi - \omega'), \quad \sigma_0 = \frac{3}{4}(\omega' - \varphi). \tag{4.12}$$

Thus, $\zeta = c_0 + \sigma_0$. Furthermore, (4.10) with $\upsilon = 0$ shows $(c_0, \psi) = -(\varphi', \psi')$, which gives

$$\|c_0\|_{-1} = \sup_{\psi \in \in H_0^1(I) \backslash \{0\}} \frac{(c_0, \psi)}{\|\psi'\|} = \sup_{\psi \in \in H_0^1(I) \backslash \{0\}} \frac{(\varphi', \psi')}{\|\psi'\|} = \|\varphi'\|. \tag{4.13}$$

Combining (4.11)-(4.13) yields

$$\begin{aligned}\Big(\|c_0\|_{-1}^2+\frac{4}{3}\|\sigma_0\|^2\Big)^{\frac12} &= \Big(\|\varphi'\|^2+\frac{3}{4}\|\varphi-\omega'\|^2\Big)^{\frac12}\\ &= |||(\varphi,\omega)||| \le \sup_{\psi,\upsilon\in H_0^1(I)\backslash\{0\}}\frac{(\psi-\upsilon',\zeta)}{|||(\psi,\upsilon)|||}. \qquad (4.14)\end{aligned}$$

On the other hand, with the decomposition $\zeta=c_0+\sigma_0$, we have for all $(\psi,\upsilon)\in W\times\Theta$,

$$\begin{aligned}(\psi-\upsilon',\zeta) &= (\psi-\upsilon',c_0+\sigma_0)\\ &= (\psi,c_0)+(\psi-\upsilon',\sigma_0)\\ &\le \|\psi'\|\|c_0\|_{-1}+\|\psi-\upsilon'\|\|\sigma_0\|\\ &\le |||(\psi,\upsilon)|||\Big(\|c_0\|_{-1}^2+\frac{4}{3}\|\sigma_0\|^2\Big)^{\frac12},\end{aligned}$$

which deduces

$$\sup_{\psi,\upsilon\in H_0^1(I)\backslash\{0\}}\frac{(\psi-\upsilon',\zeta)}{|||\psi,\upsilon|||}\le\Big(\|c_0\|_{-1}^2+\frac{4}{3}\|\sigma_0\|^2\Big)^{\frac12}.$$

This completes the proof. □

Remark 4.1. Due to the equivalence of the norms in 1-dimensional space, we conclude that

$$\|c\|_{-1}\approx\|c\|=\Big\{\int_0^1 c^2dx\Big\}^{\frac12}=|c|,\quad \text{for all } c\in R.$$

Remark 4.2. The norm $\|\cdot\|_Q$ and the L^2 norm $\|\cdot\|$ are equivalent, which can be easily verified by using the definition (2.8) and Lemma 4.1. In fact $|(\psi-\upsilon',\zeta)|\le|||(\psi,\upsilon)|||\|\zeta\|$ and $0<\beta<2$, so that

$$\|\zeta\|_Q=\sup_{\psi,\upsilon\in H_0^1(I)\backslash\{0\}}\frac{(\psi-\upsilon',\zeta)}{|||(\psi,\upsilon)|||}+\|\beta\zeta\|\preceq\|\zeta\|.$$

On the other hand, we conclude from Lemma 4.1 that

$$\begin{aligned}\|\zeta\| &= \sup_{\delta\in L^2}\frac{(\zeta,\delta)}{\|\delta\|}=\sup_{\delta\in L^2}\frac{(c_0+\sigma_0,\delta)}{\|\delta\|}\\ &\le \sup_{\delta\in L^2}\frac{(c_0,\delta)}{\|\delta\|}+\sup_{\delta\in L^2}\frac{(\sigma_0,\delta)}{\|\delta\|}\\ &\preceq \|c_0\|+\|\sigma_0\|\preceq\|c_0\|_{-1}+\|\sigma_0\|\preceq\|\zeta\|_Q.\end{aligned}$$

Proof of Theorem 4.2. With the representation of Res_Q in (2.13) and the choice in (3.4), together with Remarks 4.1 and 4.2, we have, for all $\zeta\in Q$,

$$\begin{aligned}\sup_{\zeta\in Q\backslash\{0\}}\frac{\mathrm{Res}_Q(\zeta)}{\|\zeta\|_Q} &= \sup_{\zeta\in Q\backslash\{0\}}\frac{(\varphi_h-\omega_h'-\beta^2\gamma_h^*,\zeta)}{\|\zeta\|_Q}\\ &\approx \sup_{\substack{\zeta\in Q\backslash\{0\},\zeta=c_0+\sigma_0\\ c_0\in R,\sigma_0\in L^2(I)}}\frac{((\mathrm{I}-\mathrm{R}_h)(\varphi_h-\omega_h'),c_0+\sigma_0)}{\|c_0\|_{-1}+\|\sigma_0\|+\|\beta(c_0+\sigma_0)\|}\\ &\preceq \|(\mathrm{I}-\mathrm{R}_h)(\varphi_h-\omega_h')\|,\end{aligned}$$

which ends the proof. □

From Theorems 4.1 and 4.2, and the equivalence in (2.9), we obtain the following result.

Theorem 4.3. *It holds that*

$$|||(\varphi_d-\varphi_h,\omega_d-\omega_h)|||+\|\gamma_d^*-\gamma_h^*\|_Q\preceq\eta_h.$$

5. The Efficiency of the Estimator

This section demonstrates the efficiency of the estimator η_h.

Theorem 5.1. *It holds that*

$$\eta_h \preceq |||(\varphi_d - \varphi_h, \omega_d - \omega_h)||| + \|\gamma_d - \gamma_h^*\|_Q + osc(g),$$

where η_h and $osc^2(g)$ are defined as in (3.6) *and* (3.7) *respectively.*

Proof. For each subinterval $I_i = [x_{i-1}, x_i]$, $\lambda_{1,i}$ and $\lambda_{2,i}$ are the barycentric co-ordinates on this subinterval, and $4\lambda_{1,i}\lambda_{2,i}$ is the quadratic bubble function on I_i. Let

$$b_{I_i}(x) = \begin{cases} 4\lambda_{1,i}\lambda_{2,i}, & x \in I_i, \\ 0, & x \in [0,1] \setminus I_i. \end{cases} \tag{5.1}$$

Obviously, $b_{I_i}(x) \in H_0^1(I)$ with

$$\operatorname{supp} b_{I_i}(x) = I_i, \quad 0 \le b_{I_i}(x) \le \max_{0\le x\le 1} b_{I_i}(x) = 1.$$

Firstly, by the definitions of γ_h and γ_d, we have

$$\begin{aligned} &\|\gamma_h - \gamma_d\| \\ &= \|\frac{3}{4}(\varphi_d - \varphi_h) - \frac{3}{4}(\omega_d - \omega_h)' + \frac{3}{4}(\mathrm{I} - \mathrm{R}_h)(\varphi_h - \omega_h') + \gamma_d^* - \gamma_h^*\|_{L^2(I)} \\ &\preceq |||(\varphi_d - \varphi_h, \omega_d - \omega_h)||| + \|\gamma_h^* - \gamma_d^*\|_Q + \|(\mathrm{I} - \mathrm{R}_h)(\varphi_h - \omega_h')\|. \end{aligned} \tag{5.2}$$

Secondly, we consider the efficiency of $\{\sum\limits_i h_i^2\|g - \gamma_h'\|^2_{L_2(I_i)}\}^{\frac{1}{2}}$. Set $\delta_{I_i} = b_{I_i}(g_h - \gamma_h')$ with $g_h(x)$ the L^2 projection of $g(x)$ onto the space $\mu_{-1}^1(\triangle)$. Due to the equilibrium equation $g = \gamma_d'$ in (1.1) and the equivalence of the norms $\|\cdot b_{I_i}^{\frac{1}{2}}\|_{L^2(I_i)}$ and $\|\cdot\|_{L^2(I_i)}$ on a polynomial space, we have

$$\begin{aligned} h_i\|g_h - \gamma_h'\|^2_{L^2(I_i)} &\preceq h_i(g_h - \gamma_h', \delta_{I_i}) \\ &= h_i(g_h - g, \delta_{I_i}) + h_i(g - \gamma_h', \delta_{I_i}) \\ &= h_i(g_h - g, \delta_{I_i}) + h_i(\gamma_d' - \gamma_h', \delta_{I_i}) \\ &= h_i(g_h - g, \delta_{I_i}) - h_i(\gamma_d - \gamma_h, \delta_{I_i}') \\ &\preceq (h_i\|g_h - g\|_{L^2(I_i)} + \|\gamma_d - \gamma_h\|_{L^2(I_i)})\|\delta_{I_i}\|_{L^2(I_i)}. \end{aligned}$$

The last term above is reduced with the inverse inequality on polynomial space. Thus,

$$h_i\|g_h - \gamma_h'\|_{L^2(I_i)} \preceq h_i\|g_h - g\|_{L^2(I_i)} + \|\gamma_d - \gamma_h\|_{L^2(I_i)}.$$

Therefore,

$$\sum_i h_i^2\|g_h - \gamma_h'\|^2_{L^2(I_i)} \preceq \sum_i h_i^2\|g_h - g\|^2_{L^2(I_i)} + \|\gamma_d - \gamma_h\|^2. \tag{5.3}$$

Thirdly, we consider the efficiency of $\{\sum\limits_i h_i^2\|\gamma_h - \varphi_h''\|^2_{L^2(I_i)}\}^{\frac{1}{2}}$. Let $\xi_{I_i} = b_{I_i}(\gamma_h - \varphi_h'')$. Using the equivalence of the norms $\|\cdot b_{I_i}^{\frac{1}{2}}\|_{L^2(I_i)}$ and $\|\cdot\|_{L^2(I_i)}$ on a polynomial space over subinterval

I_i and the inverse inequality on the polynomial space, and with the equation $-\varphi_d'' + \gamma_d = 0$ of (1.1), we have

$$\begin{aligned}
& h_i\|\gamma_h - \varphi_h''\|_{L^2(I_i)}^2 \\
\preceq\ & h_i(\gamma_h - \varphi_h'', \xi_{I_i}) \\
=\ & h_i(\gamma_h - \gamma_d, \xi_{I_i}) + h_i(\varphi_d'' - \varphi_h'', \xi_{I_i}) \\
\preceq\ & h_i\|\gamma_h - \gamma_d\|_{L^2(I_i)}\|\xi_{I_i}\|_{L^2(I_i)} + h_i\|\varphi_d' - \varphi_h'\|_{L^2(I_i)}\|\xi_{I_i}'\|_{L^2(I_i)} \\
\preceq\ & (h_i\|\gamma_h - \gamma_d\|_{L^2(I_i)} + \|\varphi_d' - \varphi_h'\|_{L^2(I_i)})\|\xi_{I_i}\|_{L^2(I_i)} \\
\preceq\ & (h_i\|\gamma_h - \gamma_d\|_{L^2(I_i)} + \|\varphi_d' - \varphi_h'\|_{L^2(I_i)})\|\gamma_h - \varphi_h''\|_{L^2(I_i)}.
\end{aligned}$$

Therefore,

$$\sum_i h_i^2\|\gamma_h - \varphi_h''\|_{L^2(I_i)}^2 \preceq \|\gamma_h - \gamma_d\|^2 + |||(\varphi_d - \varphi_h, \omega_d - \omega_h)|||^2. \tag{5.4}$$

Fourthly, consider the efficiency of $\|(\mathrm{I} - \mathrm{R}_h)(\varphi_h - \omega_h')\|$:

$$\begin{aligned}
& \|(\mathrm{I} - \mathrm{R}_h)(\varphi_h - \omega_h')\| \\
\lesssim\ & \|(\varphi_h - \varphi_d) - (\omega_h - \omega_d)'\| + \|(\varphi_d - \omega_d') - \mathrm{R}_h(\varphi_h - \omega_h')\| \\
\preceq\ & |||(\varphi_h - \varphi_d, \omega_h - \omega_d)||| + \beta^2\|\gamma_d^* - \gamma_h^*\| \\
\preceq\ & |||(\varphi_h - \varphi_d, \omega_h - \omega_d)||| + \|\gamma_d^* - \gamma_h^*\|_Q,
\end{aligned} \tag{5.5}$$

where we have used Remark 4.2 in the last inequality above. Finally, from (5.2) and (5.3)-(5.5), we have

$$\eta_h \preceq |||(\varphi_d - \varphi_h, \omega_d^* - \omega_h)||| + \|\gamma_d - \gamma_h^*\|_Q + osc(g).$$

This ends the proof of the theorem. □

6. Conclusion

In this paper we extend the ideas of [3,7,11] to the a posteriori error analysis for the beam problem. In particular, for the discretization scheme of [1,2] we present a reliable and efficient local error estimator which is uniform with respect to the thickness of the beam. We believe that the framework of the present paper can be extended for other schemes of the beam problem (see Remark 2.2).

Acknowledgments. This research is supported by the fortieth postdoctoral foundation of China. The second author is supported by the NSFC under Grant 10601003, and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry, and A Foundation for the Author of National Excellent Doctoral Dissertation of PRC 200718.

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