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A TAILORED FINITE POINT METHOD FOR THE HELMHOLTZ EQUATION WITH HIGH WAVE NUMBERS IN HETEROGENEOUS MEDIUM*

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Abstract

In this paper, we propose a tailored-finite-point method for the numerical simulation of the Helmholtz equation with high wave numbers in heterogeneous medium. Our finite point method has been tailored to some particular properties of the problem, which allows us to obtain approximate solutions with the same behaviors as that of the exact solution very naturally. Especially, when the coefficients are piecewise constant, we can get the exact solution with only one point in each subdomain. Our finite-point method has uniformly convergent rate with respect to wave number k in L^2-norm.

Mathematics subject classification: 65N99, 74J05, 74J40.

Key words: Tailored finite point method, Helmholtz equation, Inhomogeneous media, High frequency wave.

1. Introduction

We are interested in the inhomogeneous Helmholtz equation in one-dimensional case:

$$\frac{d}{dx}\left(c^2(x)\frac{du}{dx}\right) + k^2 n^2(x) u = f(x), \quad \forall x \in \Omega = (a,b) \subset \mathbb{R}, \tag{1.1}$$

$$u(a) = 0, \quad (cu' - \mathrm{i}knu)(b) = 0, \tag{1.2}$$

$$u(x) \quad \text{and} \quad c^2(x)u'(x) \quad \text{are continuous on } \Omega, \tag{1.3}$$

where 'i' is the imaginary unit, $k > 0$, $f \in L^2(\Omega)$, $c(x)$ and $n(x)$ are two piecewise smooth functions which represent the local speed of sound and the index of refraction respectively and satisfy

$$0 < c_0 \le c(x) \le C_0 < \infty, \qquad 0 < n_0 \le n(x) \le N_0 < \infty. \tag{1.4}$$

The above boundary value problem of the Helmholtz equation arises in many physical fields, such as the acoustic wave propagation, the electromagnetic wave propagation, seismic wave propagation in geophysics, and so on. It is well known that the numerical simulation of the Helmholtz equation with high wave numbers in inhomogeneous medium is extremely difficult, see, e.g., [2, 13, 14, 15]. In the last ten years, there have been some efficient methods for this kind of problems with constant coefficients, including the discrete singular convolution method [4], hybrid numerical asymptotic method [10], spectral approximation method [20], element-free Galerkin method [21, 23], the so-called ultra weak variational formulation [12], etc. Generally speaking, one needs the restriction $kh = \mathcal{O}(1)$ for the mesh size h to achieve a satisfactory numerical result.

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For our model problem (1.1)-(1.3), if we let

$$y(x) = \int_a^x \frac{1}{c^2(\xi)} d\xi, \quad \tau = y(b), \quad m(y) \equiv c\big(x(y)\big)n\big(x(y)\big), \quad F(y) \equiv f\big(x(y)\big), \tag{1.5}$$

then the function $U(y) \equiv u\big(x(y)\big)$ will satisfy the following equivalent problem:

$$U''(y) + k^2 m^2(y)U(y) = F(y), \quad y \in I = (0, \tau), \tag{1.6}$$

$$U(0) = 0, \qquad U'(\tau) - \mathrm{i}km(\tau)U(\tau) = 0, \tag{1.7}$$

$$U(y) \quad \text{and} \quad U'(y) \quad \text{are continuous on } I. \tag{1.8}$$

In this paper, we propose an approach which is based on the properties of the localized approximate problem to solve our model problem (1.6)-(1.8). Our method can give a natural approximation of the original problem with its essential properties. In particular, we can give the *exact solution* when m is a piecewise-constant function.

The rest part of this paper is organized as follows. In Section 2, we fix notations and discuss the stability results for our model problem. In Section 3, we present our finite-point method for the inhomogeneous Helmholtz equation based on the properties of the solutions. We also give the stability analysis and error estimates for the proposed method. In Section 4, some numerical examples are given to show the efficiency of our method. Finally, we make a short conclusion in Section 5.

2. Stability Analysis for Analytical Solution

Without loss of generality, we assume that $I = (0, \tau) \equiv (0, 1)$. Let

$$L^2(I) = \left\{ v \,\middle|\, \int_I |v(y)|^2 dy < +\infty \right\}$$

denote the space of all square-integrable complex-valued functions equipped with the inner product

$$(v, w) := \int_I v(y)\bar{w}(y)dy$$

and the norm

$$\|v\|_{0,I} := \sqrt{(v, v)}.$$

We also introduce the standard Sobolev spaces, for $l \in \mathbb{N}$,

$$H^l(I) = \left\{ v \,\middle|\, v \in L^2(I),\ v^{(j)} \in L^2(I),\ j = 1, \cdots, l \right\},$$

where $v^{(j)}$ are the derivatives of order j in the distribution sense. By $|v|_{l,I} := \|v^{(l)}\|_{0,I}$ a semi-norm is given in $H^l(I)$. A norm of the space $H^l(I)$ is defined as

$$\|v\|_{l,I} = \left(\sum_{j=0}^{l} |v|_{j,I}^2 \right)^{1/2}.$$

From now on, if not stated otherwise, all constants C, or C_j, with $j \in \mathbb{N}$, are assumed to be independent of all parameters of the given estimate, and having, in general, different meanings in different contexts. Furthermore, we suppose that the piecewise smooth function $m(y)$ is also piecewise monotone, *i.e.*, there are some points y_j $(j = 0, 1, \cdots, J)$ such that

$$0 = y_0 < y_1 < \cdots < y_J = 1, \;\; I_j = (y_{j-1}, y_j),$$
$$m|_{I_j} \in C^1(\bar{I}_j) \text{ and } m|_{I_j} \text{ is monotone}, \;\; j = 1, \cdots, J.$$

Let

$$\mathcal{M}_j^0 \equiv \|m\|_{\infty,I_j}, \quad \mathcal{M}_j^1 \equiv \|m'\|_{\infty,I_j}, \qquad j = 1, \cdots, J.$$

Clearly, we have

$$c_0 n_0 \le \max_{1\le j\le J} \mathcal{M}_j^0 \le C_0 N_0, \qquad \mathcal{M}_1 \equiv \max_{1\le j\le J} \|m'\|_{\infty,I_j} < +\infty.$$

We then have the following estimates for the model problem (1.6)-(1.8) (*cf.* [13, 14, 20]).

Lemma 2.1. (Stability analysis for analytic solution) *Suppose $F \in L^2(I)$ in* (1.6)*, m is piecewise smooth and piecewise monotone, U is the solution of* (1.6)-(1.8)*. Then $U \in H^2(I) \bigcap C^1(\bar{I})$ and the following estimates*

$$|U|_{1,I} + k\|U\|_{0,I} \le C\|F\|_{0,I}, \quad |U|_{2,I} \le C(1+k)\|F\|_{0,I}, \tag{2.1}$$

hold for a positive constant C which is independent of F and k.

Proof. Multiplying (1.6) by $\bar{U}$ and integrating over I yields

$$U'\bar{U}\Big|_0^1 - \int_I \left(|U'|^2 - k^2 m^2 |U|^2\right) dy = \int_I F\bar{U}\, dy. \tag{2.2}$$

Taking the real and imaginary parts in (2.2), we arrive at, $\forall \varepsilon_1, \varepsilon_2 > 0$,

$$\Big||U|_{1,I}^2 - k^2\|mU\|_{0,I}^2\Big| = \left|\mathrm{Re}\left(\int_I F\bar{U} dy\right)\right| \le \frac{1}{2k^2\varepsilon_1}\|F\|_{0,I}^2 + \frac{k^2\varepsilon_1}{2}\|U\|_{0,I}^2, \tag{2.3}$$

$$km(1)|U(1)|^2 = \mathrm{Im}\left(\int_I F\bar{U}\, dy\right) \le \frac{1}{2k\varepsilon_2}\|F\|_{0,I}^2 + \frac{k\varepsilon_2}{2}\|U\|_{0,I}^2. \tag{2.4}$$

Let

$$z_1 = 1, \quad z_{j+1} = z_j \max\left(1, \frac{m^2(y_j^-)}{m^2(y_j^+)}\right), \quad j = 1\cdots, J-1, \tag{2.5}$$

$$z(y) = \begin{cases} z_j y, & \text{if } m(y) \text{ is increasing on } I_j \\ \frac{m^2(y_j^-)}{m^2(y)} z_j y, & \text{if } m(y) \text{ is decreasing on } I_j, \end{cases} \quad \text{for } \; y \in I_j, \quad j = 1\cdots, J, \tag{2.6}$$

where $y^\pm$ means the limit to y from the right/left side. Certainly, we have

$$z'(y) \ge z_j, \quad \left(zm^2\right)'(y) \ge z_j n_0^2, \quad \text{for } \; y \in I_j, \quad j = 1, \cdots, J.$$

As $z_j \ge 1$ $(j = 1, \cdots, J)$, we can obtain

$$z'(y) \ge 1, \quad \left(zm^2\right)'(y) \ge n_0^2, \quad 0 \le z(y) \le \left(\frac{N_0}{n_0}\right)^{4J}, \quad \text{for } \; y \in I.$$

Multiplying (1.6) by $z\bar{U}'$ and integrating over I, then taking the real part yields,

$$\begin{aligned}
&\sum_{j=1}^{J}\left[\int_{I_j}\left(z'|U'|^2+(zm^2)'|kU|^2\right)dy+z(y_{j-1}^{+})\left(|U'(y_{j-1})|^2+|km(y_{j-1}^{+})U(y_{j-1})|^2\right)\right]\\
&=\sum_{j=1}^{J}z(y_j^{-})\left(|U'(y_j)|^2+|km(y_j^{-})U(y_j)|^2\right)-2\int_I \mathrm{Re}\left(zF\bar{U}'\right)\,dy\\
&\le\sum_{j=2}^{J+1}z(y_{j-1}^{-})\left(|U'(y_{j-1})|^2+|km(y_{j-1}^{-})U(y_{j-1})|^2\right)+\frac{1}{\delta}\|zF\|_{0,I}^2+\delta|U|_{1,I}^2,\quad\forall\delta>0.
\end{aligned}$$

From the definition of the function z, we have

$$z(y_{j-1}^{+})\ge z(y_{j-1}^{-}),\quad (zm^2)(y_{j-1}^{+})\ge(zm^2)(y_{j-1}^{-}),\quad j=2,\cdots,J.$$

Then we obtain

$$|U|_{1,I}^2+k^2n_0^2\|U\|_{0,I}^2\le 2z(1)|km(1)U(1)|^2+\frac{1}{\delta}\|zF\|_{0,I}^2+\delta|U|_{1,I}^2. \tag{2.7}$$

Taking $\varepsilon_2=\frac{n_0^2}{2z(1)m(1)}$ and $\delta=\frac{1}{2}$ in (2.4) and (2.7), we get

$$\frac{1}{2}\left(|U|_{1,I}^2+k^2n_0^2\|U\|_{0,I}^2\right)\le 2\|zF\|_{0,I}^2+2\left(\frac{m(1)z(1)}{n_0}\right)^2\|F\|_{0,I}^2\le C\|F\|_{0,I}^2.$$

That is the first inequality in (2.1). Furthermore, from (1.6), we have

$$|U|_{2,I}^2=\int_I|U''|^2dy=\int_I|F-k^2m^2U|^2dy\le(\|F\|_{0,I}+k^2N_0^2\|U\|_{0,I})^2.$$

Then we get the second inequality in (2.1). □

3. Finite Point Method

In this section, we propose a new approach to construct a discrete scheme for the Helmholtz equation (1.6). We call the new scheme a "**tailored finite point method**" (TFPM), because the finite point method has been tailored to some particular properties or solutions of the problem [11]. The finite point method [7, 16, 18, 19] is a development of finite difference method, in which the meshless technique is emphasized. There were also many work about the meshless methods for Helmholtz equation [1, 3, 6, 17]. For more related work, one can refer to two review papers [5, 22].

3.1. TFPM for Helmholtz equation in heterogeneous medium

We now want to construct a tailored finite-point scheme for the Helmholtz equation (1.6) with varying coefficients. First, we take a partition as:

$$0=\xi_0<\xi_1<\cdots<\xi_N=1,$$

with

$$h_j=\xi_j-\xi_{j-1},\quad j=1,2,\cdots,N,\quad\text{and}\quad h=\max_{1\le j\le N}h_j,$$

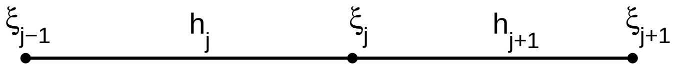


Fig. 3.1. The local mesh around points ξ_{j-1}, ξ_j, and ξ_{j+1}.

such that $m(y)$ is smooth and monotone on each subdomain $D_j = (\xi_{j-1}, \xi_j)$. Then we approximate the coefficient $m(y)$ by piecewise constant function, *i.e.*, we introduce an approximate function $m_h(y)$ for $m(y)$,

$$m_h(y) = m_j \equiv m(\xi_j^-), \quad \text{for } y \in D_j, \quad j = 1, \cdots, N.$$

Now we obtain an approximate problem of (1.6)-(1.8) for U_h,

$$U_h'' + k^2 m_h^2 U_h = F, \qquad \forall y \in D_j,\ j = 1, \cdots, N, \tag{3.1}$$

$$U_h(0) = 0, \qquad U_h'(1) - \mathrm{i}km_N U_h(1) = 0, \tag{3.2}$$

$$U_h(\xi_j^-) = U_h(\xi_j^+), \quad U_h'(\xi_j^-) = U_h'(\xi_j^+), \quad j = 1, \cdots, N-1, \tag{3.3}$$

For $y, s \in D_j$, $j = 1, \cdots, N$, let

$$G_j(y,s) = \frac{1}{km_j} \begin{cases} e^{\mathrm{i}km_j(s-\xi_{j-1})} \sin\big(km_j(y-\xi_{j-1})\big), & y \geq s, \\ e^{\mathrm{i}km_j(y-\xi_{j-1})} \sin\big(km_j(s-\xi_{j-1})\big), & s \geq y. \end{cases} \tag{3.4}$$

Then the solution of (3.1) can be expressed by

$$U_h(y) = A_j e^{\mathrm{i}km_j(y-\xi_{j-1})} + B_j e^{-\mathrm{i}km_j(y-\xi_{j-1})} + \int_{\xi_{j-1}}^{\xi_j} f(s) G_j(y,s) ds, \quad \text{for } y \in D_j, \tag{3.5}$$

with some constants $A_j, B_j \in \mathbb{C}$, $j = 1, \cdots, N$. From (3.2)-(3.3), we have

$$A_1 + B_1 + f_1^s = 0, \qquad -2\mathrm{i}B_N + f_N^e = 0, \tag{3.6}$$

$$\left.\begin{aligned} &A_j e^{\mathrm{i}km_jh_j} + B_j e^{-\mathrm{i}km_jh_j} + f_j^e \sin\big(km_jh_j\big) = A_{j+1} + B_{j+1} + f_{j+1}^s, \\ &A_j e^{\mathrm{i}km_jh_j} - B_j e^{-\mathrm{i}km_jh_j} - \mathrm{i}f_j^e \cos\big(km_jh_j\big) = \tfrac{m_{j+1}}{m_j}\left(A_{j+1} - B_{j+1} + f_{j+1}^s\right), \end{aligned}\right\} \tag{3.7}$$

for $1 \leq j \leq N-1$, where

$$f_j^s = \int_{\xi_{j-1}}^{\xi_j} \frac{f(y)}{km_j} \sin\big(km_j(y-\xi_{j-1})\big) dy,$$

$$f_j^c = \int_{\xi_{j-1}}^{\xi_j} \frac{f(y)}{km_j} \cos\big(km_j(y-\xi_{j-1})\big) dy, \quad f_j^e = f_j^c + \mathrm{i}f_j^s. \tag{3.8}$$

Now we obtain a linear system of $2N$ equations for all $2N$ unknowns $\{A_j, B_j,\ j = 1, \cdots, N\}$. Solving this linear system (3.6)-(3.7), we can get our approximate solution $U_h(y)$.

Remark 3.1. Certainly, in practice, we need some quadrature rules to get the integrals in (3.8). When the wave number k is large, it is not easy to get these integrals by standard quadrature rules. However, if we expand the function F by a series of piecewise trigonometric functions or if we approximate F by piecewise polynomials, we can get the approximation of these integrals explicitly with high accuracy.

3.2. Well-posedness of TFPM

It is not difficult to prove the following theorem about the well-posedness of our method.

Theorem 3.1. *The linear system* (3.6)-(3.7) *has the unique solvability.*

Proof. We need only to prove that the linear system has the zero solution when $F \equiv 0$. From (3.6)-(3.7), we have

$$\begin{pmatrix} A_j \\ B_j \end{pmatrix} = \begin{pmatrix} \alpha_j & \beta_j \\ \bar{\beta}_j & \bar{\alpha}_j \end{pmatrix} \begin{pmatrix} A_{j+1} \\ B_{j+1} \end{pmatrix}, \quad j = 1, \cdots, N-1, \tag{3.9}$$

with

$$\alpha_j = \frac{1}{2} e^{-\mathrm{i}km_jh_j} \left(1 + \frac{m_{j+1}}{m_j}\right), \qquad \beta_j = \frac{1}{2} e^{-\mathrm{i}km_jh_j} \left(1 - \frac{m_{j+1}}{m_j}\right).$$

Furthermore, we have

$$|\alpha_j|^2 - |\beta_j|^2 = \frac{m_{j+1}}{m_j} > 0, \quad j = 1, \cdots, N-1.$$

Therefore, we have two constants $\alpha, \beta \in \mathbb{C}$, such that

$$|\alpha|^2 - |\beta|^2 = \frac{m_N}{m_1} > 0,$$

and

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} A_N \\ B_N \end{pmatrix}. \tag{3.10}$$

If we have $F \equiv 0$, by (3.6), we get

$$A_1 + B_1 = 0, \quad \text{and} \quad B_N = 0.$$

Combining with (3.10), we have $A_1 = B_1 = A_N = B_N = 0$. By (3.9), we obtain $A_j = B_j = 0$, $j = 1, \cdots, N$, which implies our linear system (3.6)-(3.7) has the unique solvability. $\square$

Remark 3.2. From the proof of Theorem 3.1, we also give a procedure to solve our linear system (3.6)-(3.7) very easily. More precisely, from (3.6), we have

$$\begin{pmatrix} A_j \\ B_j \end{pmatrix} = \begin{pmatrix} \alpha_j & \beta_j \\ \bar{\beta}_j & \bar{\alpha}_j \end{pmatrix} \begin{pmatrix} A_{j+1} \\ B_{j+1} \end{pmatrix} + \begin{pmatrix} \mu_j \\ \nu_j \end{pmatrix}, \quad j = 1, \cdots, N-1,$$

with

$$\mu_j = \alpha_j f^s_{j+1} + \frac{\mathrm{i}}{2} f^e_j, \qquad \nu_j = \bar{\beta}_j f^s_{j+1} - \frac{\mathrm{i}}{2} f^e_{j+1}.$$

Then we can get

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} A_N \\ B_N \end{pmatrix} + \begin{pmatrix} \mu \\ \nu \end{pmatrix},$$

by iteration. Combining with (3.6), we can get A_1, B_1, A_N, B_N immediately. Finally, we can get A_j, B_j $(j = 2, \cdots, N-1)$ by recursion. The total costs for solving the linear system (3.6)-(3.7) are $\mathcal{O}(N)$.

3.3. Convergence analysis of TFPM

We now turn to the convergence analysis of our method. From the definition of m_h, we have

$$\left|m^2(y) - m_h^2(y)\right| \le 2C_0 N_0 \mathcal{M}_1 h, \qquad \forall y \in I. \tag{3.11}$$

Suppose that U is the solution of problem (1.6)-(1.8), and U_h is the solution of (3.1)-(3.3). Let us denote by

$$E(y) \equiv U(y) - U_h(y), \qquad R_h \equiv \left[k^2(m_h^2 - m^2)U\right]. \tag{3.12}$$

We then obtain

$$E''(y) + k^2 m_h^2 E(y) = R_h(y), \quad y \in I, \tag{3.13}$$

$$E(0) = 0, \qquad E'(1) - \mathrm{i}k m_N E(1) = 0, \tag{3.14}$$

$$E \quad \text{and} \quad E' \quad \text{are continuous on } I. \tag{3.15}$$

Lemma 3.1. (Stability analysis for TFPM) *Suppose $F \in L^2(I)$, $m(y)$ is piecewise smooth and piecewise monotone on I. Then we have $E \in H^2(I) \bigcap C^1(\bar{I})$ and the following estimates*

$$|E|_{1,I} + k\|E\|_{0,I} \le C\|R_h\|_{0,I}, \qquad |E|_{2,I} \le C(1+k)\|R_h\|_{0,I}, \tag{3.16}$$

with a constant C independent of R_h and k.

Proof. As $F \in L^2(I)$ and $m(y)$ is piecewise smooth and piecewise monotone on I, we know $R_h \in L^2(I)$, and m_h is piecewise monotone on I. Now the auxiliary function $z(y)$ in (2.5)-(2.6) becomes

$$z_1 = 1, \quad z_{j+1} = z_j \max\left(1, \frac{m_j^2}{m_{j+1}^2}\right), \quad j = 1 \cdots, N-1;$$
$$z(y) = z_j y, \quad \text{for } \ y \in D_j, \quad j = 1 \cdots, N.$$

We still have

$$z(y_{j-1}^+) \ge z(y_{j-1}^-), \quad (zm_h^2)(y_{j-1}^+) \ge (zm_h^2)(y_{j-1}^-), \quad j = 2, \cdots, N,$$
$$z(y) \ge 1, \quad \left(zm_h^2\right)'(y) \ge n_0^2, \quad 0 \le z(y) \le \left(\frac{N_0}{n_0}\right)^{4J}, \quad \text{for } \ y \in I.$$

Then we obtain (3.16) from Lemma 2.1 immediately. □

Theorem 3.2. (Error estimate for TFPM) *The following error estimates*

$$|E|_{1,I} + k\|E\|_{0,I} \le C\mathcal{M}_1 kh\|F\|_{0,I}, \qquad |E|_{2,I} \le C\mathcal{M}_1 k^2 h\|F\|_{0,I}, \tag{3.17}$$

hold with a constant C independent of h and k.

Proof. From Lemmas 3.1 and 2.1, (3.11) and (3.12), we can get (3.17). □

Remark 3.3. From Theorem 3.2, we know that, $E(y) \equiv 0$ if $m(y) = c(y)n(y)$ is a piecewise constant function, which implies our method can get the exact solution in this case.

Remark 3.4. We can extend our idea in this paper to higher dimensional problems. But the algorithms and the stability analysis will be more complicate. We will study the higher dimensional problems later.

4. Numerical Experiments

In this section, we give some numerical examples to demonstrate the efficiency of our new method. Especially, we want to test whether the pollution effect of the finite element solutions (*cf.* [2, 9]) occurs in our finite-point solutions.

Example 4.1. First, let us consider the problem

$$\left\{\begin{array}{ll} \left(c^2(x)u'(x)\right)' + k^2u(x) = 0, & x \in \Omega = (0,1), \\ u(0) = \frac{1}{4}e^{\frac{\mathrm{i}k}{4}}(3 + e^{\frac{\mathrm{i}k}{2}}), \quad u'(1) - \mathrm{i}ku(1) = 0; & \end{array}\right. \tag{4.1}$$

with piecewise constant coefficient:

$$c(x) = \left\{\begin{array}{ll} 2, & x \in (0, \frac{1}{2}); \\ 1, & x \in (\frac{1}{2}, 1). \end{array}\right.$$

The exact solution of problem (4.1) is

$$u(x) = \left\{\begin{array}{ll} \frac{1}{4}\left(3e^{\frac{\mathrm{i}k(1+2x)}{4}} + e^{\frac{\mathrm{i}k(3-2x)}{4}}\right), & x \in (0, \frac{1}{2}); \\ e^{\mathrm{i}kx}, & x \in (\frac{1}{2}, 1). \end{array}\right.$$

In this case, only three points, *i.e.*, $h = \frac{1}{2}$, can give the exact solution by using our tailored finite point method (*cf.* Table 4.1). In contrast to this, only as $kh \ll 1$ the standard finite volume scheme can yield some reasonable accuracy (*cf.* Table 4.1). Here

$$E^{FP}_{h,\infty} = \left\|u - u^{FP}_h\right\|_{L^\infty(\Omega)}, \quad E^{FV}_{h,\infty} = \left\|u - u^{FV}_h\right\|_{L^\infty(\Omega)},$$

where u^{FP}_h denotes the solution of our tailored finite point method with mesh size h, and U^{FV}_h denotes the solution of the standard finite volume scheme.

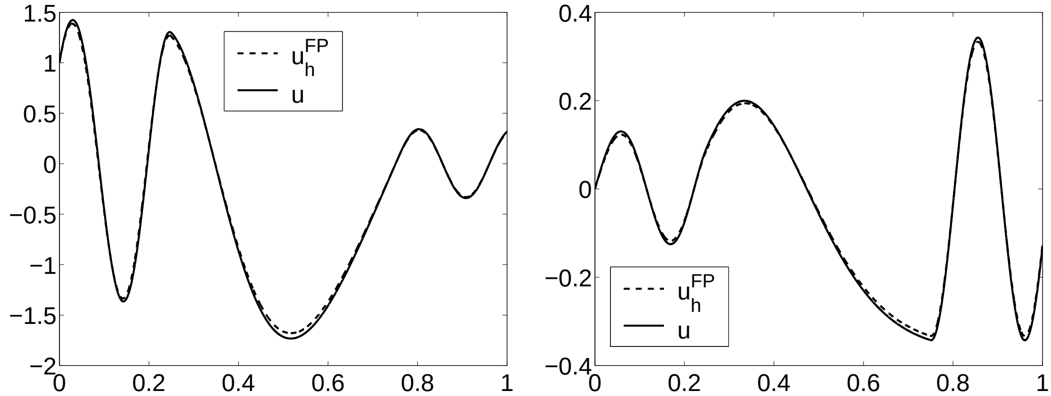
The above results suggest that for the traveling wave solutions, there is neither *pollution effect* nor *dispersion error* by our finite-point method. Furthermore, in our computation, we have no limitation of the wave number k by the so-called *Nyquist frequency* π/h. We can get any wave-number plane waves exactly with the same meshsize.

Table 4.1: Example 4.1: L^∞ errors of the numerical solutions for different methods with different wave numbers. For FV: $h = \frac{1}{8k}$, for TFPM: $h = \frac{1}{2}$.

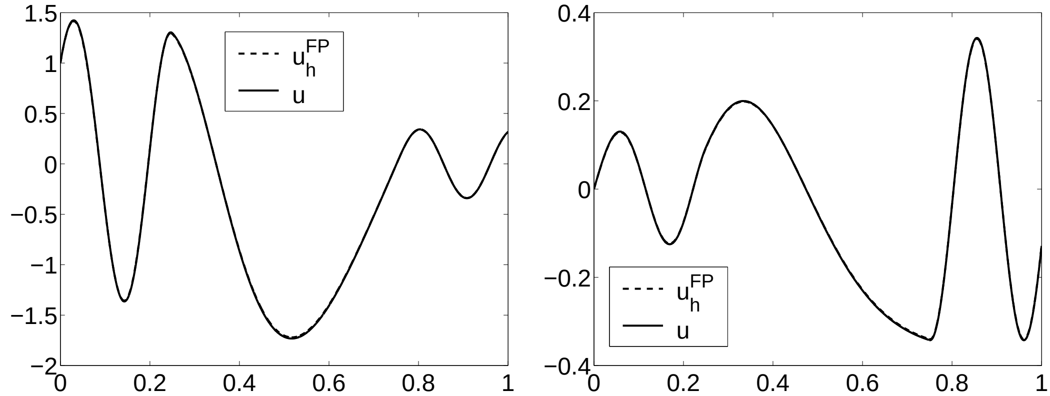
wave number k	1	5	25
$E^{FP}_{\frac{1}{2},\infty}$	1.07E-14	3.69E-14	2.02E-14
$E^{FV}_{h,\infty}$	8.30E-4	3.42E-3	5.21E-3

Table 4.2: Example 4.2: L^∞ errors of the numerical solutions for different methods with different wave numbers. For FV: $h = \frac{1}{8k}$, for TFPM: $h = \frac{1}{4}$.

wave number k	10	50	100
$E^{FP}_{h,\infty}$	1.22E-14	2.76E-14	3.13E-14
$E^{FV}_{h,\infty}$	6.38E-3	9.75E-3	8.953E-2



Real part (left) and imaginary part (right) of solutions. ($h = \frac{1}{16}$)



Real part (left) and imaginary part (right) of solutions. ($h = \frac{1}{32}$)

Fig. 4.1. The numerical solutions for Example 4.3: $k = 5\pi$.

Example 4.2. Our second problem is

$$\begin{cases} \big(c^2(x)u'(x)\big)' + k^2 n^2(x)u(x) = 0, \quad x \in \Omega = (0,1), \\ u(0) = 1, \qquad u'(1) - \mathrm{i}ku(1) = 0 \end{cases} \tag{4.2}$$

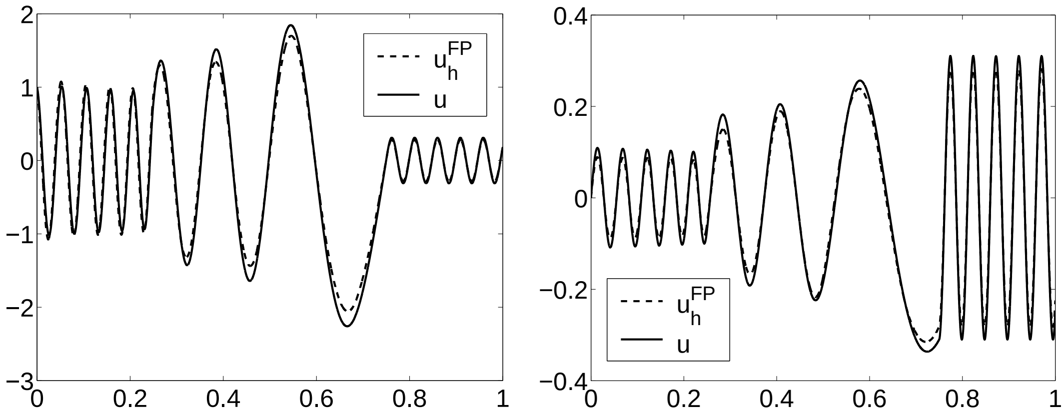
with piecewise smooth coefficient:

$$c(x) = \begin{cases} \frac{1}{1+\sin^2 x}, & 0 < x < 0.25; \\ \frac{1}{1+\cos^2 x}, & 0.25 < x < 0.75, \\ 1, & 0.75 < x < 1. \end{cases} \qquad n(x) = \frac{1}{c(x)}.$$

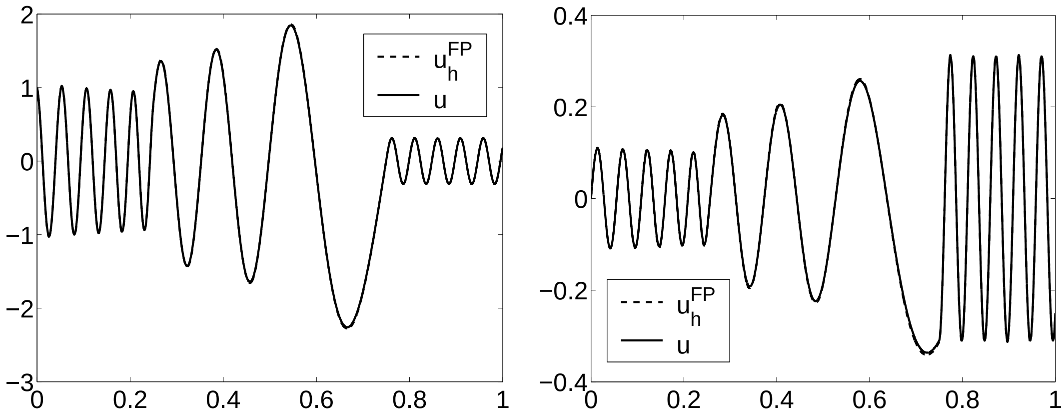
In this case, the exact solution $u(x)$ is easy to get by using (3.4)-(3.8). As $m(x) = c(x)n(x) \equiv 1$, the exact solution can be recovered by using the proposed tailored finite point method (*cf.* Table 4.2) whenever the wave number k is large or small.

Example 4.3. Finally, we consider a more general problem

$$\begin{cases} \big(c^2(x)u'\big)' + k^2 n^2(x)u = f(x), \quad x \in \Omega = (0,1), \\ u(0) = 1, \quad u'(1) - \mathrm{i}k\frac{n}{c}u(1) = 0 \end{cases} \tag{4.3}$$



Real part (left) and imaginary part (right) of solutions. ($h = \frac{1}{16}$)



Real part (left) and imaginary part (right) of solutions. ($h = \frac{1}{32}$)

Fig. 4.2. The numerical solutions for Example 4.3: $k = 20\pi$.

with $f(x) = 1$,

$$c(x) = \begin{cases} 1 + x^2, & 0 < x < 0.25; \\ 1 - x^2, & 0.25 < x < 0.5, \\ 1, & 0.5 < x < 1. \end{cases} \qquad n(x) = \begin{cases} 1.75 + x, & 0 < x < 0.25; \\ 1.25 - x, & 0.25 < x < 0.5, \\ 2, & 0.5 < x < 1. \end{cases}$$

In this case, our tailored finite point method can still give good approximation solution, see Tables 4.3-4.4 and Figs. 4.1-4.2, even if we only use a small number of points per wavelength. The 'exact' solution $u(x)$ of problem (4.3) is obtained by using very fine mesh. Here

$$E_{h,0}^{FV} = \frac{\|u_h^{FV} - u\|_{0,\Omega}}{\|u\|_{0,\Omega}}, \quad E_{h,1}^{FV} = \frac{|u_h^{FV} - u|_{1,\Omega}}{|u|_{1,\Omega}},$$
$$E_{h,0}^{FP} = \frac{\|u_h^{FP} - u\|_{0,\Omega}}{\|u\|_{0,\Omega}}, \quad E_{h,1}^{FP} = \frac{|u_h^{FP} - u|_{1,\Omega}}{|u|_{1,\Omega}}.$$

From Tables 4.3-4.4, we also find that there is *no pollution effect* in our finite-point solutions. We achieve the correct convergence rate as shown in Theorem 3.2. In contrast to this, the finite

Table 4.3: Example 4.3: Relative errors in L^2-norm.

wave number k	1	50	100	200
$E_{h,0}^{FP}$ $(h=\frac{1}{64})$	5.01E-4	1.28E-3	1.38E-3	1.72E-3
$E_{h,0}^{FP}$ $(h=\frac{1}{128})$	2.50E-4	6.10E-4	6.06E-4	7.21E-4
$E_{h,0}^{FP}$ $(h=\frac{1}{256})$	1.24E-4	2.97E-4	2.93E-4	3.05E-4
$E_{h,0}^{FP}$ $(h=\frac{1}{512})$	6.10E-5	1.45E-4	1.43E-4	1.46E-4
$E_{h,0}^{FV}$ $(h=\frac{1}{512})$	2.27E-4	4.30E-2	1.55E-1	1.53

Table 4.4: Example 4.3: Relative errors in H^1-norm.

wave number k	1	50	100	200
$E_{h,1}^{FP}$ $(h=\frac{1}{64})$	1.12E-3	1.35E-3	1.70E-3	1.75E-3
$E_{h,1}^{FP}$ $(h=\frac{1}{128})$	5.61E-4	6.14E-4	6.50E-4	8.72E-4
$E_{h,1}^{FP}$ $(h=\frac{1}{256})$	2.79E-4	2.96E-4	2.99E-4	3.26E-4
$E_{h,1}^{FP}$ $(h=\frac{1}{512})$	1.37E-4	1.44E-4	1.44E-4	1.49E-4
$E_{h,1}^{FV}$ $(h=\frac{1}{512})$	4.58E-4	4.50E-2	1.87E-1	1.68

volume method gives poorer resolutions as the wave number k becomes larger. From Figs. 4.1 and 4.2, it is observed that the numerical solutions mimic the exact solutions very well even though the solutions are not smooth at $x = 0.25$ and $x = 0.5$.

5. Conclusion

In this paper, we present a tailored finite point method for the Helmholtz equation with high wave numbers in the heterogeneous medium. First, we approximate the coefficient $m(x) = c(x)n(x)$ with piecewise constants. Then we propose a *tailored finite point method* to solve the approximate problem numerically. The finite point method has been tailored to some particular properties of the problem. Therefore, the resulting approximate solution has the natural behaviors of the exact solution. In particular, when the coefficient $m(x)$ is piecewise-constant, we can recover the *exact solution* with only one point in each subdomain. Our convergence analysis also proves that the proposed method is *free of the pollution error*. Moreover, the error in L^2-norm has uniformly convergent rate with respect to the wave number k. The numerical results show that we can obtain accurate approximations even if a small number of points per wavelength are used for high frequency waves.

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