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CONVERGENCE OF A MIXED FINITE ELEMENT FOR THE STOKES PROBLEM ON ANISOTROPIC MESHES*

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Abstract

The main aim of this paper is to study the convergence properties of a low order mixed finite element for the Stokes problem under anisotropic meshes. We discuss the anisotropic convergence and superconvergence independent of the aspect ratio. Without the shape regularity assumption and inverse assumption on the meshes, the optimal error estimates and natural superconvergence at central points are obtained. The global superconvergence for the gradient of the velocity and the pressure is derived with the aid of a suitable postprocessing method. Furthermore, we develop a simple method to obtain the superclose properties which improves the results of the previous works.

Mathematics subject classification: 65N30.
Key words: Mixed finite element, Stokes problem, Anisotropic meshes, Superconvergence, Shape regularity assumption and inverse assumption.

1. Introduction

There have been many studies for the mixed finite elements approximation to the stationary Stokes problem [10, 15, 16, 21, 25] which satisfy the Babuška-Brezzi condition (see, e.g., [5, 11]). The optimal error estimates were obtained under the shape regularity assumption [9, 14] on the meshes. However, the solution of the Stokes problem may have anisotropic behavior in parts of the domain, for instance, the presence of boundary layers and other localized features. This means that the solution varies significantly in certain directions with less significant changes along the other ones. It is an obvious idea to reflect this anisotropy in the discretization by using anisotropic meshes with a small mesh size in the direction of the rapid variation of the solution and a larger mesh size in the other direction, where elements are aligned to follow (in some sense) the geometry of the solution. Compared with the standard isotropic techniques, the number of degrees of freedom required for a given accuracy may be considerably reduced.

Recently, some efforts have been made to develop stable mixed methods for the meshes that include elements of arbitrary high aspect ratio. For instance, Schötzau *et al.* [23, 24] for $\mathcal{Q}_{k+1}-\mathcal{Q}_{k-1}$ families, Becker and Rannacher [6, 7] for $\mathcal{Q}_1-\mathcal{Q}_0$, $\mathcal{Q}_1-\mathcal{Q}_1$, Apel and Nicaise [3] for $\tilde{\mathcal{Q}}_1-\mathcal{Q}_0$. By $\mathcal{Q}_1$ we denote, as usual, the space of bilinear functions, and by $\tilde{\mathcal{Q}}_1$ the rotated

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$\mathcal{Q}_1$ element. All the methods developed in the above works used high aspect ratio meshes, although most of them placed some restrictions on the meshes. For the stability of the method it is required that the discrete spaces satisfy the Babuška-Brezzi condition with a constant (inf-sup constant) independent of the aspect ratio of the elements. It has been reported by Russo that the mini element becomes instable on anisotropic meshes (cf. [22]). As to the estimate of the interpolation error under anisotropic meshes, Apel [1, 4, pp.35-38] presented a criterion to judge the anisotropy of an interpolation; Chen *et al.* developed an anisotropic interpolation theorem in [12, 13, 27, 28], which is much easier to use. In this work, we consider another familiar scheme which can be regarded as a low order Bernardi-Raugel element (cf. [8]) under anisotropic rectangular meshes. Recently, the stability of this scheme with the inf-sup constant independent of the aspect ratio has been discussed by Apel and Nicaise in [2]. We check the anisotropy of the interpolation of velocity, and then the optimal error estimates can be obtained by using the anisotropic interpolation theorem.

On the other hand, the superconvergence for the mixed elements is very effective in practice. Some superconvergence results for several mixed finite elements have been obtained when the meshes are sufficiently good. Lin and Pan in [20] and [18] proved $\mathcal{O}(h^2)$-superconvergence for the $\mathcal{Q}_1 - \mathcal{Q}_0$ element under square meshes and $\mathcal{O}(h^3)$-superconvergence for the biquadratic-linear element over uniform rectangular meshes, respectively. On quasi-uniform rectangular meshes, the $\mathcal{O}(h^2)$-superconvergence for the Bernardi-Raugel element was obtained by [18]. A key concept in their derivation is the integral identity technique which has been proven to be an efficient tool for the superconvergence analysis of rectangular finite elements (cf. [17, 19]). In this paper, a simple method is developed to obtain the superclose results. The basic tool employed by us is the well-known Bramble-Hilbert Lemma. Furthermore, compared with the previous works, our results can be worked without the shape regularity assumption and inverse assumption requirement on the meshes and can be applied to more general meshes.

The paper is organized as follows: we investigate the anisotropic interpolation properties of the Bernardi-Raugel element in Section 2. In Section 3, based on the stability of this scheme with the inf-sup constant independent of the aspect ratio, which has been obtained in [2], we get the optimal anisotropic error estimates. Without the shape regularity assumption and inverse assumption requirement on the meshes, the superclose result and global $\mathcal{O}(h^2)$-superconvergence of the Bernardi-Raugel element are obtained under rectangular meshes in Section 4 and Section 5, respectively. Finally, natural superconvergence at central points is derived in Section 6.

2. Some Notations and Basic Estimates

In this section, we introduce some notations and recall some estimates that are basic for our subsequent arguments.

For the sake of convenience, let $\Omega \subset \mathbf{R}^2$ be a convex polygon composed by a family of rectangular meshes J_h which need not satisfy the shape regular conditions. $\forall K \in J_h$, we denote the barycenter of the element K by (x_K, y_K), the length of edges parallel to x-axis and y-axis by $2h_{K1}, 2h_{K2}$ respectively, $h_K = \max\{h_{K1}, h_{K2}\}, h = \max_{K\in\mathcal{J}_h} h_K$, $h_K^\alpha = h_{K1}^{\alpha_1} h_{K2}^{\alpha_2}$. Assume that $\hat{K} = [-1,1]\times[-1,1]$ is the reference element, the four vertices are: $\hat{a}_1 = (-1,-1), \hat{a}_2 = (1,-1), \hat{a}_3 = (1,1), \hat{a}_4 = (-1,1)$, and its 4 sides are $\hat{l}_1 = \hat{a}_1\hat{a}_2, \hat{l}_2 = \hat{a}_2\hat{a}_3, \hat{l}_3 = \hat{a}_3\hat{a}_4, \hat{l}_4 = \hat{a}_4\hat{a}_1$. Then there exists a unique mapping $F_K : \hat{K} \to K$ defined as

$$\begin{cases} x = x_K + h_{K1}\xi, \\ y = y_K + h_{K2}\eta. \end{cases} \tag{2.1}$$

Throughout this paper, the positive constant C will be used as a generic constant, possibly different from place to place, but is independent of h_K and h_{K2}/h_{K1} (assume $h_{K2} \geq h_{K1}$). We will use the standard notations $\|\cdot\|_{m,K}, |\cdot|_{m,K}$ for the usual norms and seminorms on the Sobolev space $H^m(K)$ over K. For a function $\hat{w}$ defined on $\hat{K}$, we associate w defined on K in the usual way: $\hat{w} = w \circ F_K$. $\mathcal{Q}_{i,j} = P_i(x) \times P_j(y), P_k(x)$ is a polynomial as to the variable x with degree k.

For any $\vec{\hat{v}} = (\hat{v}^{[1]}, \hat{v}^{[2]}) \in [H^2(\hat{K})]^2$ and $\hat{q} \in L^2(\hat{K})$, define the interpolations $\vec{\hat{\Pi}} = (\hat{\Pi}^{[1]}, \hat{\Pi}^{[2]}) : [H^2(\hat{K})]^2 \to \mathcal{Q}_{1,2}(\hat{K}) \times \mathcal{Q}_{2,1}(\hat{K})$ and $\hat{I} : L^2(\hat{K}) \to \mathcal{Q}_0(\hat{K})$ on the reference element as follows:

$$\begin{cases} \vec{\hat{\Pi}}\vec{\hat{v}}(a_i) = \vec{\hat{v}}(a_i), \ i = 1,2,3,4, \\ \displaystyle\int_{\hat{l}_i} \hat{\Pi}^{[2]}\hat{v}^{[2]} d\xi = \int_{\hat{l}_i} \hat{v}^{[2]} d\xi, \ i = 1,3, \\ \displaystyle\int_{\hat{l}_i} \hat{\Pi}^{[1]}\hat{v}^{[1]} d\eta = \int_{\hat{l}_i} \hat{v}^{[1]} d\eta, \ i = 2,4, \\ \displaystyle\hat{I}\hat{q} = \frac{1}{|\hat{K}|}\int_{\hat{K}} \hat{q}\, d\xi d\eta. \end{cases} \tag{2.2}$$

The finite element space is defined as (cf. [8,16]):

$$\begin{cases} \vec{V}_h = \{\vec{v}_h \in [C^0(\Omega)]^2; \ \vec{v}_h|_K \in [\mathcal{Q}_{1,2}(\hat{K}) \times \mathcal{Q}_{2,1}(\hat{K})] \circ F_K^{-1}, K \in J_h, \vec{v}_h|_{\partial\Omega} = 0\}, \\ P_h = \{q_h \in L^2(\Omega); \ \ q_h|_K \in \mathcal{Q}_0(K), K \in J_h, \ \displaystyle\int_\Omega q_h dxdy = 0\}. \end{cases} \tag{2.3}$$

Then we can define the global interpolations $\vec{\Pi}_h : [H^2(\Omega)]^2 \to \vec{V}_h$ and $I_h : L^2(\Omega) \to P_h$ as

$$\vec{\Pi}_h|_K = \vec{\Pi}_K = \vec{\hat{\Pi}} \circ F_K, \ I_h|_K = I_K = \hat{I} \circ F_K. \tag{2.4}$$

Now, we begin to concentrate on some properties of the interpolation $\vec{\hat{\Pi}}$. We only consider $\hat{\Pi}^{[1]}\hat{v}^{[1]}|_{\hat{K}} \in \mathcal{Q}_{1,2}(\hat{K})$, the other component of $\vec{\hat{\Pi}}$ can be treated similarly.

On the reference element $\hat{K}$, by a direct calculation, we have

$$\begin{aligned} \hat{\Pi}^{[1]}\hat{v}^{[1]} = &\sum_{i=1}^{4} \hat{v}_i^{[1]} P_i + \frac{3}{8}(2\hat{v}_5^{[1]} + 2\hat{v}_6^{[1]} - \hat{v}_1^{[1]} - \hat{v}_2^{[1]} - \hat{v}_3^{[1]} - \hat{v}_4^{[1]})(1-\eta^2) \\ &+ \frac{3}{8}(2\hat{v}_5^{[1]} - 2\hat{v}_6^{[1]} + \hat{v}_1^{[1]} - \hat{v}_2^{[1]} - \hat{v}_3^{[1]} + \hat{v}_4^{[1]})\xi(1-\eta^2), \end{aligned} \tag{2.5}$$

where

$$\hat{v}_i^{[1]} = \hat{v}^{[1]}(a_i), \ i = 1,2,3,4, \quad \hat{v}_5^{[1]} = \frac{1}{2}\int_{\hat{l}_2} \hat{v}^{[1]} d\eta, \quad \hat{v}_6^{[1]} = \frac{1}{2}\int_{\hat{l}_4} \hat{v}^{[1]} d\eta,$$
$$P_1 = \frac{(1-\xi)(1-\eta)}{4}, \quad P_2 = \frac{(1+\xi)(1-\eta)}{4}, \quad P_3 = \frac{(1+\xi)(1+\eta)}{4}, \quad P_4 = \frac{(1-\xi)(1+\eta)}{4}.$$

Define the multi index $\alpha = (\alpha_1, \alpha_2), \hat{D}^\alpha \hat{v} = \frac{\partial^{|\alpha|}\hat{v}}{\partial\xi^{\alpha_1}\partial\eta^{\alpha_2}}$. We consider $\alpha = (1,0)$,

$$\hat{D}^\alpha \hat{\Pi}^{[1]}\hat{v}^{[1]} = \frac{\partial \hat{\Pi}^{[1]}\hat{v}^{[1]}}{\partial\xi} = \frac{1-\eta}{4}\beta_1(\hat{v}^{[1]}) + \frac{1+\eta}{4}\beta_2(\hat{v}^{[1]}) + \frac{3(1-\eta^2)}{8}\beta_3(\hat{v}^{[1]}), \tag{2.6}$$

where

$$\beta_1(\hat{v}^{[1]}) = -\hat{v}_1^{[1]} + \hat{v}_2^{[1]}, \quad \beta_2(\hat{v}^{[1]}) = \hat{v}_3^{[1]} - \hat{v}_4^{[1]},$$
$$\beta_3(\hat{v}^{[1]}) = \hat{v}_1^{[1]} - \hat{v}_2^{[1]} - \hat{v}_3^{[1]} + \hat{v}_4^{[1]} + 2\hat{v}_5^{[1]} - 2\hat{v}_6^{[1]}.$$

Obviously, $\{\frac{1-\eta}{4}, \frac{1+\eta}{4}, \frac{3(1-\eta^2)}{8}\}$ is a basis of $\hat{D}^\alpha \mathcal{Q}_{1,2}(\hat{K})$, and

$$\begin{cases} \beta_1(\hat{v}^{[1]}) = \displaystyle\int_{-1}^{1} \frac{\partial \hat{v}^{[1]}(\xi,-1)}{\partial \xi} d\xi \triangleq E_1(\hat{D}^\alpha \hat{v}^{[1]}), \\ \beta_2(\hat{v}^{[1]}) = \displaystyle\int_{-1}^{1} \frac{\partial \hat{v}^{[1]}(\xi,1)}{\partial \xi} d\xi \triangleq E_2(\hat{D}^\alpha \hat{v}^{[1]}), \\ \beta_3(\hat{v}^{[1]}) = -\displaystyle\int_{-1}^{1} [\frac{\partial \hat{v}^{[1]}(\xi,-1)}{\partial \xi} + \frac{\partial \hat{v}^{[1]}(\xi,1)}{\partial \xi}] d\xi \\ \qquad\qquad + \displaystyle\int_{-1}^{1}\int_{-1}^{1} \frac{\partial \hat{v}^{[1]}(\xi,\eta)}{\partial \xi} d\xi d\eta \triangleq E_3(\hat{D}^\alpha \hat{v}^{[1]}). \end{cases} \tag{2.7}$$

By employing Cauchy-Schwarz inequality and the trace theorem, we can show that

$$|E_i(\hat{w})| \le C\|\hat{w}\|_{1,\hat{K}}, \quad i = 1,2,3,$$

i.e., $E_i, i = 1,2,3$, are bounded linear functionals on $H^1(\hat{K})$.

Then by employing the basic anisotropic interpolation theorem (cf. [12, 13]), we have

$$\|\hat{D}^\alpha(\hat{v}^{[1]} - \hat{\Pi}^{[1]}\hat{v}^{[1]})\|_{0,\hat{K}} \le C|\hat{D}^\alpha \hat{v}^{[1]}|_{1,\hat{K}}. \tag{2.8}$$

If $\alpha = (0,1)$, then

$$\hat{D}^\alpha \hat{\Pi}^{[1]}\hat{v}^{[1]} = \frac{\partial \hat{\Pi}^{[1]}\hat{v}^{[1]}}{\partial \eta} = \frac{1-\xi}{4}\beta_4(\hat{v}^{[1]}) + \frac{1+\xi}{4}\beta_5(\hat{v}^{[1]}) - \frac{3\eta}{4}\beta_6(\hat{v}^{[1]}) - \frac{3\xi\eta}{4}\beta_7(\hat{v}^{[1]}),$$

where

$$\beta_4(\hat{v}^{[1]}) = -\hat{v}_1^{[1]} + \hat{v}_4^{[1]}, \quad \beta_5(\hat{v}^{[1]}) = -\hat{v}_2^{[1]} + \hat{v}_3^{[1]},$$
$$\beta_6(\hat{v}^{[1]}) = -\hat{v}_1^{[1]} - \hat{v}_2^{[1]} - \hat{v}_3^{[1]} - \hat{v}_4^{[1]} + 2\hat{v}_5^{[1]} + 2\hat{v}_6^{[1]},$$
$$\beta_7(\hat{v}^{[1]}) = \hat{v}_1^{[1]} - \hat{v}_2^{[1]} - \hat{v}_3^{[1]} + \hat{v}_4^{[1]} + 2\hat{v}_5^{[1]} - 2\hat{v}_6^{[1]}.$$

It is easy to see that $\{\frac{1-\xi}{4}, \frac{1+\xi}{4}, -\frac{3\eta}{4}, -\frac{3\xi\eta}{4}\}$ is a basis of $\hat{D}^\alpha \mathcal{Q}_{1,2}(\hat{K})$, and

$$\beta_4(\hat{v}^{[1]}) = \int_{-1}^{1} \frac{\partial \hat{v}^{[1]}(-1,\eta)}{\partial \eta} d\eta \triangleq E_4(\hat{D}^\alpha \hat{v}^{[1]}),$$
$$\beta_5(\hat{v}^{[1]}) = \int_{-1}^{1} \frac{\partial \hat{v}^{[1]}(1,\eta)}{\partial \eta} d\eta \triangleq E_5(\hat{D}^\alpha \hat{v}^{[1]}),$$

$$\begin{aligned}
\beta_6(\hat{v}^{[1]}) &= \int_{-1}^{1}[\hat{v}^{[1]}(1,\eta)-\frac{1}{2}\hat{v}^{[1]}(1,-1)-\frac{1}{2}\hat{v}^{[1]}(1,1)]d\eta \\
&\quad+\int_{-1}^{1}[\hat{v}^{[1]}(-1,\eta)-\frac{1}{2}\hat{v}^{[1]}(-1,-1)-\frac{1}{2}\hat{v}^{[1]}(-1,1)]d\eta \\
&= \frac{1}{2}\int_{-1}^{1}[\int_{-1}^{\eta}\frac{\partial\hat{v}^{[1]}(1,t)}{\partial t}dt+\int_{1}^{\eta}\frac{\partial\hat{v}^{[1]}(1,t)}{\partial t}dt]d\eta \\
&\quad+\frac{1}{2}\int_{-1}^{1}[\int_{-1}^{\eta}\frac{\partial\hat{v}^{[1]}(-1,t)}{\partial t}dt+\int_{1}^{\eta}\frac{\partial\hat{v}^{[1]}(-1,t)}{\partial t}dt]d\eta \triangleq E_6(\hat{D}^{\alpha}\hat{v}^{[1]}), \\
\beta_7(\hat{v}^{[1]}) &= \frac{1}{2}\int_{-1}^{1}[\int_{-1}^{\eta}\frac{\partial\hat{v}^{[1]}(1,t)}{\partial t}dt+\int_{1}^{\eta}\frac{\partial\hat{v}^{[1]}(1,t)}{\partial t}dt]d\eta \\
&\quad-\frac{1}{2}\int_{-1}^{1}[\int_{-1}^{\eta}\frac{\partial\hat{v}^{[1]}(-1,t)}{\partial t}dt+\int_{1}^{\eta}\frac{\partial\hat{v}^{[1]}(-1,t)}{\partial t}dt]d\eta \triangleq E_7(\hat{D}^{\alpha}\hat{v}^{[1]}).
\end{aligned}$$

Employing the same argument yields

$$|E_i(\hat{w})| \leq C\|\hat{w}\|_{1,\hat{K}}, \quad i=4,5,6,7,$$

i.e., $E_i, i=4,5,6,7$, are bounded linear functionals on $H^1(\hat{K})$. Therefore, (2.8) still holds for $\alpha=(0,1)$.

Summarizing the above analysis we have the following result.

Lemma 2.1. *The interpolation of the famous Bernardi-Raugel element has the anisotropic properties, which can be expressed as follows:*

$$\|\hat{D}^{\alpha}(\vec{\hat{v}}-\vec{\hat{\Pi}}\vec{\hat{v}})\|_{0,\hat{K}} \leq C|\hat{D}^{\alpha}\vec{\hat{v}}|_{1,\hat{K}}, \quad \forall\ |\alpha|=1,\ \vec{\hat{v}}\in[H^2(\hat{K})]^2. \tag{2.9}$$

3. Error Estimates on Anisotropic Meshes

The Stokes problem reads as [10,16]: Find $(\vec{u},p)$, such that

$$\begin{cases} -\mu\Delta\vec{u}+\nabla p=\vec{f}, & \text{in } \Omega, \\ \mathrm{div}\vec{u}=0, & \text{in } \Omega, \\ \vec{u}=\vec{0}, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\vec{u}=(u^{[1]},u^{[2]})$ is the velocity of fluids, p is the pressure, $\vec{f}=(f^{[1]},f^{[2]})$ denotes a given external force.

The equivalent variational formulation to the problem (3.1) is

$$\begin{cases} \text{Find } (\vec{u},p)\in[H_0^1(\Omega)]^2\times L_0^2(\Omega), \text{ such that} \\ \quad a(\vec{u},\vec{v})+b(\vec{v},p)=f(\vec{v}), \ \forall\vec{v}\in[H_0^1(\Omega)]^2, \\ \quad b(\vec{u},q)=0, \qquad \forall q\in L_0^2(\Omega), \end{cases} \tag{3.2}$$

where

$$a(\vec{u},\vec{v})=\int_{\Omega}\mu\nabla\vec{u}\cdot\nabla\vec{v}dxdy, \quad b(\vec{v},p)=-\int_{\Omega}p\mathrm{div}\vec{v}dxdy, \quad f(\vec{v})=\int_{\Omega}\vec{f}\cdot\vec{v}dxdy.$$

Then the finite element solution of (3.2) is to find $\vec{u}_h\in\vec{V}_h,\ p_h\in P_h$, such that

$$\begin{cases} a(\vec{u}_h,\vec{v}_h)+b(\vec{v}_h,p_h)=f(\vec{v}_h), & \forall\vec{v}_h\in\vec{V}_h, \\ b(\vec{u}_h,q_h)=0, & \forall q_h\in P_h. \end{cases} \tag{3.3}$$

We define the inf-sup constant by

$$\gamma_h \triangleq \inf_{0\neq p_h\in P_h} \sup_{0\neq \vec{v}_h\in \vec{V}_h} \frac{b(\vec{v}_h,p_h)}{|\vec{v}_h|_{1,\Omega}\|p_h\|_{0,\Omega}}. \tag{3.4}$$

Then the finite element errors can be estimated as (refer to [10])

$$|\vec{u}-\vec{u}_h|_{1,\Omega} \leq C\mu^{-1}\Big(\inf_{\vec{v}_h\in\vec{Z}_h}|\vec{u}-\vec{v}_h|_{1,\Omega}+\inf_{q_h\in P_h}\|p-q_h\|_{0,\Omega}\Big), \tag{3.5}$$

$$\|p-p_h\|_{0,\Omega} \leq C\Big(\gamma_h^{-2}\mu\inf_{\vec{v}_h\in\vec{V}_h}|\vec{u}-\vec{v}_h|_{1,\Omega}+\gamma_h^{-1}\inf_{q_h\in P_h}\|p-q_h\|_{0,\Omega}\Big), \tag{3.6}$$

where

$$\vec{Z}_h=\{\vec{v}_h\in\vec{V}_h:\ b(\vec{v}_h,q_h)=0,\ \forall q_h\in P_h\}.$$

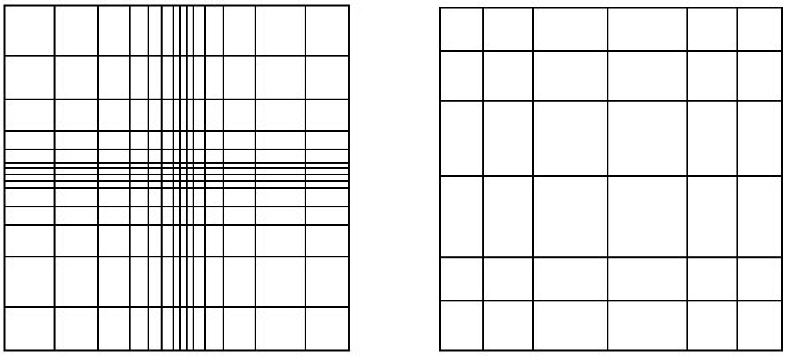

Fig. 3.1. Mesh J_h (left) and mesh T_H (right).

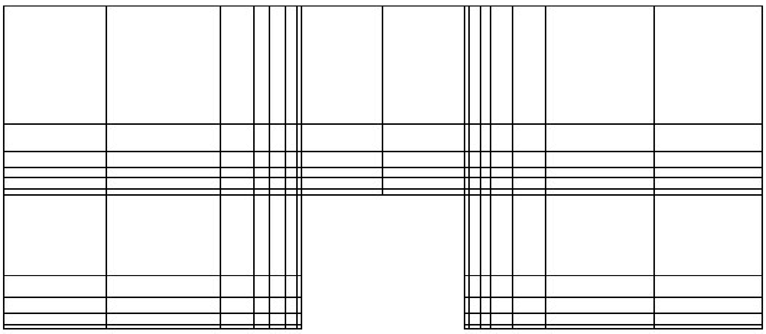

Fig. 3.2. Anisotropic mesh around corners.

For the sake of simplicity, we assume that the anisotropic rectangular meshes J_h are constructed in the following way: first divide Ω by a union of macro rectangular elements T_H, i.e., $\overline{\Omega}=\bigcup_{M\in T_H}M$, and further assume that T_H is a shape regular partition of Ω. Then the anisotropic meshes J_h are obtained by dividing a macroelement M into micro rectangular elements K along two opposite edges of M, i.e., $M=\sum_{K\subset M}K$, refer to the illustration in Fig. 3.1. Note that the micro triangulation J_h may not satisfy the shape regular assumption. Such meshes can also be designed around corners, see Fig. 3.2.

Based on Theorem 1 of [2], we have the following stability theorem.

Theorem 3.1. *There exists a constant $\gamma^* > 0$ independent of both h_K and the aspect ratio, such that*

$$\inf_{0\neq q_h\in P_h}\sup_{0\neq\vec{v}_h\in\vec{V}_h}\frac{b(\vec{v}_h,q_h)}{|\vec{v}_h|_{1,\Omega}\|q_h\|_{0,\Omega}}\geq\gamma^*, \tag{3.7}$$

i.e., $b(\cdot,\cdot)$ satisfies the Babuška-Brezzi condition over $\vec{V}_h\times P_h$ under anisotropic meshes.

Then we can carry out the error analysis for the anisotropic Bernardi-Raugel element approximation to the Stokes problem.

Theorem 3.2. *Under the above hypothesis, let $(\vec{u},p)$ and $(\vec{u}_h,p_h)$ be the exact solution of the Stokes problem (3.2) and the finite element solution of (3.3) respectively, $(\vec{u},p)\in[H^2(\Omega)\cap H_0^1(\Omega)]^2\times H^1(\Omega)$. Then there hold the following estimates*

$$|\vec{u}-\vec{u}_h|_{1,\Omega}\leq C\left(\sum_{K\in J_h}\sum_{|\alpha|=1}\sum_{|\beta|=1}h_K^{2\beta}\|D^{\alpha+\beta}\vec{u}\|_{0,K}^2\right)^{\frac12}+C\left(\sum_{K\in J_h}\sum_{|\beta|=1}h_K^{2\beta}\|D^{\beta}p\|_{1,K}^2\right)^{\frac12}, \tag{3.8}$$

$$\|p-p_h\|_{0,\Omega}\leq C\left(\sum_{K\in J_h}\sum_{|\alpha|=1}\sum_{|\beta|=1}h_K^{2\beta}\|D^{\alpha+\beta}\vec{u}\|_{0,K}^2\right)^{\frac12}+C\left(\sum_{K\in J_h}\sum_{|\beta|=1}h_K^{2\beta}\|D^{\beta}p\|_{1,K}^2\right)^{\frac12}. \tag{3.9}$$

Proof. By (3.5) and (3.6), we only need to prove that

$$\inf_{\vec{v}_h\in\vec{Z}_h}|\vec{u}-\vec{v}_h|_{1,\Omega}\leq C\left(\sum_{K\in J_h}\sum_{|\alpha|=1}\sum_{|\beta|=1}h_K^{2\beta}\|D^{\alpha+\beta}\vec{u}\|_{0,K}^2\right)^{\frac12}, \tag{3.10}$$

and

$$\inf_{q_h\in P_h}\|p-q_h\|_{0,\Omega}\leq C\left(\sum_{K\in J_h}\sum_{|\beta|=1}h_K^{2\beta}\|D^{\beta}p\|_{1,K}^2\right)^{\frac12}. \tag{3.11}$$

We will first prove (3.10). It is easy to see that $\vec{\Pi}_h\vec{u}\in\vec{Z}_h$. Then by Lemma 2.1, we have

$$\begin{aligned}\|\vec{u}-\vec{\Pi}_h\vec{u}\|_h&=\left(\sum_{K\in J_h}|\vec{u}-\vec{\Pi}_K\vec{u}|_{1,K}^2\right)^{\frac12}\\&=\left(\sum_{K\in J_h}\sum_{|\alpha|=1}\|D^{\alpha}(\vec{u}-\vec{\Pi}_K\vec{u})\|_{0,K}^2\right)^{\frac12}\\&=\left(\sum_{K\in J_h}\sum_{|\alpha|=1}h_K^{-2\alpha}(h_{K1}h_{K2})\|\hat{D}^{\alpha}(\vec{\hat{u}}-\vec{\hat{\Pi}}\vec{\hat{u}})\|_{0,\hat{K}}^2\right)^{\frac12}\\&\leq C\left(\sum_{K\in J_h}\sum_{|\alpha|=1}h_K^{-2\alpha}(h_{K1}h_{K2})|\hat{D}^{\alpha}\vec{\hat{u}}|_{1,\hat{K}}^2\right)^{\frac12}\\&\leq C\left(\sum_{K\in J_h}\sum_{|\alpha|=1}\sum_{|\beta|=1}h_K^{2\beta}\|D^{\alpha+\beta}\vec{u}\|_{0,K}^2\right)^{\frac12},\end{aligned} \tag{3.12}$$

which gives (3.10). For any $p \in L^2(\Omega)$, observe that

$$\inf_{q_h \in M_h} \|p - q_h\|_{0,\Omega} \le \|p - I_h p\|_{0,\Omega}, \tag{3.13}$$

and

$$\begin{aligned} \|p - I_h p\|_{0,\Omega}^2 &= \sum_{K \in J_h} \|p - I_K p\|_{0,K}^2 = \sum_{K \in J_h} \|\hat{p} - \widehat{I}\hat{p}\|_{0,\hat{K}}^2 h_{K1} h_{K2} \\ &\le C \sum_{K \in J_h} |\hat{p}|_{1,\hat{K}}^2 h_{K1} h_{K2} = C \sum_{K \in J_h} \sum_{|\beta|=1} \|\hat{D}^\beta \hat{p}\|_{0,\hat{K}}^2 h_{K1} h_{K2} \\ &\le C \sum_{K \in J_h} \sum_{|\beta|=1} h_K^{2\beta} \|D^\beta p\|_{1,K}^2. \end{aligned}$$

Then (3.11) follows. The proof of Theorem 3.2 is completed. □

4. Anisotropic Superclose Analysis

In this section, we will discuss the superconvergence for a stable Bernardi-Raugel scheme on anisotropic meshes. Unlike [17-20], the basic tool we employed here is the Bramble-Hilbert Lemma [9, 14] instead of the integral identity. Compared with [17-20], our analysis is simpler; and most importantly, we avoid the requirement of the shape regularity and inverse assumptions.

Lemma 4.1. *If $\vec{u} \in [H^3(\Omega)]^2$, then*

$$a(\vec{u} - \vec{\Pi}_h \vec{u}, \vec{v}_h) \le C h^2 |\vec{u}|_{3,\Omega} |\vec{v}_h|_{1,\Omega}, \quad \forall \vec{v}_h \in \vec{V}_h. \tag{4.1}$$

Proof. Let $\vec{u} = (u^{[1]}, u^{[2]}), \vec{v}_h = (v_h^{[1]}, v_h^{[2]})$. By the scaling argument and following the notations appeared in Section 2, we have

$$\int_\Omega \Big(u^{[1]} - \Pi_h^{[1]} u^{[1]}\Big)_x v_{hx}^{[1]} dxdy = \sum_{K \in J_h} h_{K1}^{-1} h_{K2} \int_{\hat{K}} \Big(\hat{u}_\xi^{[1]} - \sum_{i=1}^3 E_i(\hat{u}_\xi^{[1]}) \alpha_i(\eta)\Big) \hat{v}_{h\xi}^{[1]} d\xi d\eta, \tag{4.2}$$

where $\alpha_1(\eta) = (1-\eta)/4,\ \alpha_2(\eta) = (1+\eta)/4$ and $\alpha_3(\eta) = 3(1-\eta^2)/8$. Setting $\hat{w} = \hat{u}_\xi^{[1]}$, for any fixed $\hat{v}_{h\xi}^{[1]}$, we define the linear functional

$$B_1(\hat{w}) = \int_{\hat{K}} \Big(\hat{w} - \sum_{i=1}^3 E_i(\hat{w}) \alpha_i(\eta)\Big) \hat{v}_{h\xi}^{[1]} d\xi d\eta. \tag{4.3}$$

Obviously,

$$|B_1(\hat{w})| \le C \|\hat{v}_{h\xi}^{[1]}\|_{0,\hat{K}} \|\hat{w}\|_{2,\hat{K}}. \tag{4.4}$$

Hence $B_1 \in (H^2(\hat{K}))'$ and $\|B_1\| \le C \|\hat{v}_{h\xi}^{[1]}\|_{0,\hat{K}}$.

From (2.7), it can be seen easily that

$$\forall \hat{w} \in P_1(\hat{K}), \quad B_1(\hat{w}) = 0. \tag{4.5}$$

Then an application of the Bramble-Hilbert Lemma yields

$$|B_1(\hat{w})| \le C |\hat{w}|_{2,\hat{K}} \|\hat{v}_{h\xi}^{[1]}\|_{0,\hat{K}}. \tag{4.6}$$

The homogeneity argument and Cauchy-Schwarz inequality yield

$$\int_\Omega \left(u^{[1]} - \Pi_h^{[1]} u^{[1]}\right)_x v_{hx}^{[1]} dxdy \le Ch^2 |u^{[1]}|_{3,\Omega} |v_h^{[1]}|_{1,\Omega}. \tag{4.7}$$

By the similar argument, we have

$$\int_\Omega \left(u^{[1]} - \Pi_h^{[1]} u^{[1]}\right)_y v_{hy}^{[1]} dxdy \le Ch^2 |u^{[1]}|_{3,\Omega} |v_h^{[1]}|_{1,\Omega}, \tag{4.8}$$

$$\int_\Omega \left(u^{[2]} - \Pi_h^{[2]} u^{[2]}\right)_x v_{hx}^{[2]} dxdy \le Ch^2 |u^{[2]}|_{3,\Omega} |v_h^{[2]}|_{1,\Omega}, \tag{4.9}$$

$$\int_\Omega \left(u^{[2]} - \Pi_h^{[2]} u^{[2]}\right)_y v_{hy}^{[2]} dxdy \le Ch^2 |u^{[2]}|_{3,\Omega} |v_h^{[2]}|_{1,\Omega}. \tag{4.10}$$

Then a combination of the above four inequalities completes the proof. □

Lemma 4.2. *If $p \in H^2(\Omega)$, then*

$$b(\vec{v}_h, p - I_h p) \le Ch^2 |p|_{2,\Omega} |\vec{v}_h|_{1,\Omega}, \quad \forall \vec{v}_h \in \vec{V}_h. \tag{4.11}$$

Proof. We will concentrate on the linear functional B_2 defined as

$$B_2(\hat{p}) = \int_{\hat{K}} (\hat{p} - \hat{I}\hat{p}) \hat{v}_{h\xi}^{[1]} d\xi d\eta - \frac{1}{3} \int_{\hat{K}} \hat{p}_\eta \hat{v}_{h\xi\eta}^{[1]} d\xi d\eta. \tag{4.12}$$

Following the same lines of bounding $B_1(\cdot)$, one can check that

$$\forall\ \hat{p} \in P_1(\hat{K}), \quad B_2(\hat{p}) = 0, \tag{4.13}$$

which gives

$$|B_2(\hat{p})| \le C|\hat{p}|_{2,\hat{K}} \|\hat{v}_{h\xi}^{[1]}\|_{0,\hat{K}}. \tag{4.14}$$

Hence

$$\begin{aligned}\int_K (p - Ip) v_{hx}^{[1]} dxdy &\le Ch^2 |p|_{2,K} \|v_{hx}^{[1]}\|_{0,K} + \frac{4h_{K2}^2}{3} \int_K p_y v_{hxy}^{[1]} dxdy \\ &\le Ch^2 |p|_{2,K} |v_h^{[1]}|_{1,K} + \frac{4h_{K2}^2}{3} \left(\int_{l_2} - \int_{l_4} \right) p_y v_{hy}^{[1]} dy.\end{aligned}$$

The above line integrals will be canceled through the summation of $K \in J_h$ since $p_y v_{hy}^{[1]}$ is continuous across l_i $(i = 2, 4)$ of each element of J_h, and $v_{hy}^{[1]}$ vanishes on the boundaries of $\partial\Omega$ which are parallel to the y-direction. As a result, we obtain

$$\int_\Omega (p - Ip) v_{hx}^{[1]} dxdy \le Ch^2 |p|_{2,\Omega} |v_h^{[1]}|_{1,\Omega}. \tag{4.15}$$

In a similar way,

$$\int_\Omega (p - Ip) v_{hy}^{[2]} dxdy \le Ch^2 |p|_{2,\Omega} |v_h^{[2]}|_{1,\Omega}. \tag{4.16}$$

Then the proof is completed by (4.15) and (4.16). □

Lemma 4.3. *There holds*

$$b(\vec{u} - \vec{\Pi}_h \vec{u}, q_h) = 0, \quad \forall q_h \in P_h. \tag{4.17}$$

Proof. Because $q_h|_K$ is a constant, by the definition of the interpolation, we have

$$\int_K \nabla\cdot(\vec{u}-\vec{\Pi}_K\vec{u})dxdy=\int_{\partial K}(\vec{u}-\vec{\Pi}\vec{u})\cdot\vec{n}ds=0.$$

Then (4.17) follows. □

From Lemmas 4.1-4.3, we can prove the following main result of this section.

Theorem 4.1. *Let* $(\vec{u},p)$ *be the solution of* (3.1) *with* $(\vec{u},p)\in[H^3(\Omega)\cap H_0^1(\Omega)]^2\times H^2(\Omega)$ *under anisotropic meshes. Then there holds*

$$|\vec{u}_h-\vec{\Pi}_h\vec{u}|_{1,\Omega}+\|p_h-I_hp\|_{0,\Omega}\le Ch^2(|\vec{u}|_{3,\Omega}+|p|_{2,\Omega}). \tag{4.18}$$

Proof. Since

$$a(\vec{u}-\vec{u}_h,\vec{v}_h)+b(\vec{v}_h,p-p_h)=b(\vec{u}-\vec{u}_h,q_h)=0,$$

by [5, 26], $\forall\ (\vec{v}_h,q_h)\in\vec{V}_h\times P_h$, there exists a constant C dependent of γ^*, such that

$$\begin{aligned}&C(|\vec{u}_h-\vec{\Pi}_h\vec{u}|_{1,\Omega}+\|p_h-I_hp\|_{0,\Omega})\\ \le&\sup_{(\vec{0},0)\neq(\vec{v}_h,q_h)\in\vec{V}_h\times P_h}\frac{a(\vec{u}_h-\vec{\Pi}_h\vec{u},\vec{v}_h)+b(\vec{v}_h,p_h-I_hp)-b(\vec{u}_h-\vec{\Pi}_h\vec{u},q_h)}{|\vec{v}_h|_{1,\Omega}+\|q_h\|_{0,\Omega}}\\ =&\sup_{(\vec{0},0)\neq(\vec{v}_h,q_h)\in\vec{V}_h\times P_h}\frac{a(\vec{u}-\vec{\Pi}_h\vec{u},\vec{v}_h)+b(\vec{v}_h,p-I_hp)-b(\vec{u}-\vec{\Pi}_h\vec{u},q_h)}{|\vec{v}_h|_{1,\Omega}+\|q_h\|_{0,\Omega}}.\end{aligned} \tag{4.19}$$

Then the proof follows from Lemmas 4.1-4.3. □

5. Anisotropic Global Superconvergence

In this section, we will use proper postprocessing methods to get anisotropic global superconvergence. For this purpose, we further assume that J_h is obtained from J_{2h} (where J_{2h} is an anisotropic partition of Ω which satisfies the hypothesis in Section 3) by dividing each element K of J_{2h} into four congruent rectangles K_1,K_2,K_3,K_4. Here we take the same notations as defined in previous sections.

We firstly define two postprocessing operators as follows,

$$\begin{cases}\vec{\Pi}_{2h}^2\vec{u}\in\mathcal{Q}_2(K)\backslash\{x^2y^2\}\times\mathcal{Q}_2(K)\backslash\{x^2y^2\},\\ \qquad\vec{\Pi}_{2h}^2\vec{u}(a_i)=\vec{u}(a_i),\ i=1,2,\cdots,8,\end{cases} \tag{5.1}$$

$$\begin{cases}I_{2h}^1p\in\mathcal{Q}_1(K),\\ \displaystyle\int_{K_i}I_{2h}^1pdxdy=\int_{K_i}pdxdy,\ i=1,2,3,4,\end{cases} \tag{5.2}$$

where $a_i,i=1,2,3,4,\ a_5=(x_K,y_K-h_{K2}),\ a_6=(x_K+h_{K1},y_K),\ a_7=(x_K,y_K+h_{K2}),a_8=(x_K-h_{K1},y_K)$ are vertices of K_1,K_2,K_3,K_4.

Then the following properties can be easily verified:

$$\vec{\Pi}_{2h}^2\vec{\Pi}_h=\vec{\Pi}_{2h}^2,\quad I_{2h}^1I_h=I_{2h}^1. \tag{5.3}$$

The following lemma shows that the postprocessing operator $\vec{\Pi}_{2h}^2$ satisfies the anisotropic interpolation properties.

Lemma 5.1. *Denote* $\hat{\vec{\Pi}}_2 = \vec{\Pi}_{2h}^2 \circ F_K$. $\forall \hat{v} \in H^3(\hat{K})$, $|\alpha| = 1$, *there holds*

$$\|\hat{D}^\alpha(\hat{\vec{v}} - \hat{\vec{\Pi}}_2\hat{\vec{v}})\|_{0,\hat{K}} \le C|\hat{D}^\alpha\hat{\vec{v}}|_{2,\hat{K}}. \tag{5.4}$$

Proof. By (5.1), the interpolation of function $\hat{v}$ on $\hat{K}$ can be written as

$$\hat{\vec{\Pi}}_2\hat{\vec{v}} = \sum_{i=1}^{8} p_i\hat{\vec{v}}(a_i), \tag{5.5}$$

where $\hat{v}_i = \hat{v}(\hat{a}_i), i = 1, 2, \cdots, 8$,

$$\begin{aligned}
&p_i = -\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)(1-\xi_i\xi-\eta_i\eta), \quad i = 1,2,3,4,\\
&p_{i+4} = \frac{1}{2}(1-\xi^2)(1+\eta_i\eta), \quad i = 1,3, \quad p_{i+4} = \frac{1}{2}(1-\eta^2)(1+\xi_i\xi), \quad i = 2,4,\\
&(\xi_1,\xi_2,\xi_3,\xi_4) = (-1,1,1,-1), \quad (\eta_1,\eta_2,\eta_3,\eta_4) = (-1,1,1,-1).
\end{aligned}$$

Consider $\alpha = (1,0)$, we have

$$\hat{D}^\alpha\hat{\vec{\Pi}}_2\hat{\vec{v}} = \frac{(1-\eta)\eta}{4}\beta_{11} + \frac{(1+\eta)\eta}{4}\beta_{12} + \frac{1-\eta^2}{2}\beta_{13} + (1-\eta)\xi\beta_{14} + (1+\eta)\xi\beta_{15},$$

where

$$\begin{aligned}
&\beta_{11} = \hat{\vec{v}}(a_1) - \hat{\vec{v}}(a_2), \quad \beta_{12} = \hat{\vec{v}}(a_3) - \hat{\vec{v}}(a_4), \quad \beta_{13} = \hat{\vec{v}}(a_8) - \hat{\vec{v}}(a_6),\\
&\beta_{14} = -\hat{\vec{v}}(a_5) + \frac{\hat{\vec{v}}(a_1) + \hat{\vec{v}}(a_2)}{2}, \quad \beta_{15} = -\hat{\vec{v}}(a_7) + \frac{\hat{\vec{v}}(a_3) + \hat{\vec{v}}(a_4)}{2}.
\end{aligned}$$

Obviously, $\{\frac{(1-\eta)\eta}{4}, \frac{(1+\eta)\eta}{4}, \frac{1-\eta^2}{2}, (1-\eta)\xi, (1+\eta)\xi\}^2$ is a basis of $\hat{D}^\alpha([\mathcal{Q}_2(\hat{K})\backslash\{\xi^2\eta^2\}]^2)$, and

$$\begin{aligned}
\beta_{11} &= -\int_{-1}^{1}\frac{\partial\hat{\vec{v}}(\xi,-1)}{\partial\xi}d\xi \triangleq E_{11}(\hat{D}^\alpha\hat{\vec{v}}),\\
\beta_{12} &= \int_{-1}^{1}\frac{\partial\hat{\vec{v}}(\xi,1)}{\partial\xi}d\xi \triangleq E_{12}(\hat{D}^\alpha\hat{\vec{v}}),\\
\beta_{13} &= -\int_{-1}^{1}\frac{\partial\hat{\vec{v}}(\xi,0)}{\partial\xi}d\xi \triangleq E_{13}(\hat{D}^\alpha\hat{\vec{v}}),\\
\beta_{14} &= \frac{1}{2}(\int_{0}^{1}\frac{\partial\hat{\vec{v}}(\xi,-1)}{\partial\xi}d\xi - \int_{-1}^{0}\frac{\partial\hat{\vec{v}}(\xi,-1)}{\partial\xi}d\xi) \triangleq E_{14}(\hat{D}^\alpha\hat{\vec{v}}),\\
\beta_{15} &= \frac{1}{2}(\int_{0}^{1}\frac{\partial\hat{\vec{v}}(\xi,1)}{\partial\xi}d\xi - \int_{-1}^{0}\frac{\partial\hat{\vec{v}}(\xi,1)}{\partial\xi}d\xi) \triangleq E_{15}(\hat{D}^\alpha\hat{\vec{v}}).
\end{aligned}$$

By virtue of Cauchy-Schwarz inequality and the trace theorem, we can show that

$$|E_{1i}(\hat{\vec{w}})| \le C\|\hat{\vec{w}}\|_{1,\hat{K}} \le C\|\hat{\vec{w}}\|_{2,\hat{K}}, \quad i = 1,2,3,4,5,$$

i.e., $E_{1i}, i = 1,2,3,4,5$, are bounded linear functionals on $[H^2(\hat{K})]^2$.

Then we can employ the basic anisotropic interpolation theorem ([12,13]) and get that

$$\|\hat{D}^\alpha(\hat{\vec{v}} - \hat{\vec{\Pi}}_2\hat{\vec{v}})\|_{0,\hat{K}} \le C|\hat{D}^\alpha\hat{\vec{v}}|_{2,\hat{K}}.$$

Similarly, (5.4) holds for $\alpha = (0,1)$. The proof is complete. □

Below we will show the stability of the two operators $\vec{\Pi}_{2h}^2$ and I_{2h}^1.

Lemma 5.2. *$\forall \vec{v}_h = (v_h^{[1]}, v_h^{[2]}) \in \vec{V}_h,\ q_h \in P_h$, there hold*

$$|\vec{\Pi}_{2h}^2 \vec{v}_h|_{1,\Omega} \le C|\vec{v}_h|_{1,\Omega}, \quad \|I_{2h}^1 q_h\|_{0,\Omega} \le C\|q_h\|_{0,\Omega}. \tag{5.6}$$

Proof. Define an operator $T_3 : [H^1(\hat{K})]^2 \to \hat{D}^{(1,0)}([\mathcal{Q}_2(\hat{K})\backslash\{\xi^2\eta^2\}]^2)$ as follows:

$$\begin{aligned} T_3(\vec{w}) = {} & \frac{(1-\eta)\eta}{4} F_{11}(\vec{w}) + \frac{(1+\eta)\eta}{4} F_{12}(\vec{w}) \\ & + \frac{1-\eta^2}{2} F_{13}(\vec{w}) + (1-\eta)\xi F_{14}(\vec{w}) + (1+\eta)\xi F_{15}(\vec{w}). \end{aligned} \tag{5.7}$$

Thanks to the equivalent norms over the finite dimensional space, we have

$$\|\frac{\partial \vec{\hat{\Pi}}_2 \vec{\hat{v}}_h}{\partial \xi}\|_{0,\hat{K}} = \|T_3(\frac{\partial \vec{\hat{v}}_h}{\partial \xi})\|_{0,\hat{K}} \le C\|\frac{\partial \vec{\hat{v}}_h}{\partial \xi}\|_{1,\hat{K}} \le C\|\frac{\partial \vec{\hat{v}}_h}{\partial \xi}\|_{0,\hat{K}}, \tag{5.8}$$

so by the scaling argument,

$$\|\frac{\partial \vec{\Pi}_{2h}^2 \vec{v}_h}{\partial x}\|_{0,K} \le C\|\frac{\partial \vec{v}_h}{\partial x}\|_{0,K}. \tag{5.9}$$

Similarly, one can show that

$$\|\frac{\partial \vec{\Pi}_{2h}^2 \vec{v}_h}{\partial y}\|_{0,K} \le C\|\frac{\partial \vec{v}_h}{\partial y}\|_{0,K}. \tag{5.10}$$

Then the first term of (5.6) follows from (5.9) and (5.10).

Now we come to prove the second term of (5.6). Denote $\hat{I}_1 = I_{2h}^1 \circ F_K$. Noting the equivalent norms over the finite dimensional space, we have

$$\begin{aligned} \|I_{2h}^1 q_h\|_{0,\Omega}^2 &= \sum_K \|I_{2h}^1 q_h\|_{0,K}^2 = \sum_K (h_{K1}h_{K2})\|\hat{I}_1 \hat{q}_h\|_{0,\hat{K}}^2 \\ &\le \sum_K C(h_{K1}h_{K2})\|\hat{q}_h\|_{0,\infty,\hat{K}}^2 \le \sum_K C(h_{K1}h_{K2})\|\hat{q}_h\|_{0,\hat{K}}^2 = C\|q_h\|_{0,\Omega}^2. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 5.3. *For the two postprocessing operators $\vec{\Pi}_{2h}^2$ and I_{2h}^1, there hold the following interpolation estimates,*

$$|\vec{u} - \vec{\Pi}_{2h}^2 \vec{u}|_{1,\Omega} \le Ch^2|\vec{u}|_{3,\Omega}, \quad \|p - I_{2h}^1 p\|_{0,\Omega} \le Ch^2|p|_{2,\Omega}. \tag{5.11}$$

Proof. By Lemma 5.2, we have

$$\begin{aligned} |\vec{u} - \vec{\Pi}_{2h}^2 \vec{u}|_{1,\Omega}^2 &= \sum_K \sum_{|\alpha|=1} \|D^\alpha(\vec{u} - \vec{\Pi}_{2h}^2 \vec{u})\|_{0,K}^2 \\ &= \sum_K \sum_{|\alpha|=1} (h_{K1}h_{K2})h_K^{-2\alpha}\|\hat{D}^\alpha(\vec{\hat{u}} - \vec{\hat{\Pi}}_2 \vec{\hat{u}})\|_{0,\hat{K}}^2 \\ &\le \sum_K \sum_{|\alpha|=1} C(h_{K1}h_{K2})h_K^{-2\alpha}|\hat{D}^\alpha \vec{\hat{u}}|_{2,\hat{K}}^2 \\ &= \sum_K \sum_{|\alpha|=1} \sum_{|\beta|=2} C(h_{K1}h_{K2})h_K^{-2\alpha}\|\hat{D}^{\alpha+\beta} \vec{\hat{u}}\|_{0,\hat{K}}^2 \\ &= \sum_K \sum_{|\alpha|=1} \sum_{|\beta|=2} Ch_K^{2\beta}\|D^{\alpha+\beta}\vec{u}\|_{0,K}^2 \le Ch^4|\vec{u}|_{3,\Omega}^2. \end{aligned}$$

This proves the first estimate of (5.11). By employing the Bramble-Hilbert lemma, we have

$$\begin{aligned}\|p - I_{2h}^1 p\|_{0,\Omega}^2 &= \sum_K \|p - I_{2h}^1 p\|_{0,K}^2 = \sum_K (h_{K1}h_{K2})\|\hat{p} - \hat{I}_1\hat{p}\|_{0,\hat{K}}^2 \\ &\le C\sum_K (h_{K1}h_{K2})|\hat{p}|_{2,\hat{K}}^2 = C\sum_K \sum_{|\beta|=2} (h_{K1}h_{K2})\|\hat{D}^\beta \hat{p}\|_{0,\hat{K}}^2 \\ &= C\sum_K \sum_{|\beta|=2} h_K^{2\beta}\|D^\beta p\|_{0,K}^2 \le Ch^4|p|_{2,\Omega}^2.\end{aligned}$$

Thus we complete the proof of Lemma 5.3. □

The following theorem is the main result of this section.

Theorem 5.1. *Under the assumption of Theorem* 4.1*, we have the following global superconvergence for the gradient of the velocity and the pressure:*

$$|\vec{u} - \vec{\Pi}_{2h}^2 \vec{u}_h|_{1,\Omega} + \|p - I_{2h}^1 p_h\|_{0,\Omega} \le Ch^2(|\vec{u}|_{3,\Omega} + |p|_{2,\Omega}). \tag{5.12}$$

Proof. By (5.3), Theorem 4.1, Lemmas 5.2 and 5.3, we have

$$\begin{aligned}&|\vec{u} - \vec{\Pi}_{2h}^2 \vec{u}_h|_{1,\Omega} + \|p - I_{2h}^1 p_h\|_{0,\Omega} \\ \le\ & |\vec{u} - \vec{\Pi}_{2h}^2 \vec{u}|_{1,\Omega} + |\vec{\Pi}_{2h}^2(\vec{\Pi}_h\vec{u} - \vec{u}_h)|_{1,\Omega} + \|p - I_{2h}^1 p\|_{0,\Omega} + \|I_{2h}^1(I_h p - p_h)\|_{0,\Omega} \\ \le\ & |\vec{u} - \vec{\Pi}_{2h}^2 \vec{u}|_{1,\Omega} + \|p - I_{2h}^1 p\|_{0,\Omega} + C(|\vec{u}_h - \vec{\Pi}_h\vec{u}|_{1,\Omega} + \|p_h - I_h p\|_{0,\Omega}) \\ \le\ & Ch^2(|\vec{u}|_{3,\Omega} + |p|_{2,\Omega}).\end{aligned}$$

Then the anisotropic global superconvergence is obtained. □

Remark 5.1. Compared with [18, 19], our postprocessing method will save more computational cost as we only use three quarters of the nodes of all the elements in J_h while those in [18, 19] use all the nodes. Most importantly, the meshes in this paper need not satisfy the conventional shape regularity assumption and inverse assumption.

6. Natural Superconvergence at Central Points

As one can see from the previous section, the meshes to obtain the global superconvergence is somehow heuristic (or in an adhoc manner). In fact, there exists some potential natural superconvergence with less restrictions on the meshes than those in Section 5. We assume that the meshes considered in this section are the same as those in Section 3. We will show that the gradient of the velocity and the pressure are superconvergent at the central points of all the rectangular elements.

Theorem 6.1. *Let* $(\vec{u}, p)$ *be the solution of* (3.1) *with* $(\vec{u}, p) \in [H^3(\Omega)\cap H_0^1(\Omega)]^2 \times H^2(\Omega)$*. Then we have the following superconvergence results at the central points,*

$$\Big(\sum_K \sum_{|\alpha|=1} |D^\alpha(\vec{u} - \vec{u}_h)(x_K, y_K)|^2 h_{K1}h_{K2}\Big)^{\frac{1}{2}} \le Ch^2(|\vec{u}|_{3,\Omega} + |p|_{2,\Omega}), \tag{6.1}$$

$$\Big(\sum_K |(p - p_h)(x_K, y_K)|^2 h_{K1}h_{K2}\Big)^{\frac{1}{2}} \le Ch^2(|\vec{u}|_{3,\Omega} + |p|_{2,\Omega}). \tag{6.2}$$

Proof. The triangle inequality gives

$$
\begin{aligned}
&|D^\alpha(\vec{u}-\vec{u}_h)(x_K,y_K)|^2 \\
\le\,& 2(|D^\alpha(\vec{u}-\vec{\Pi}_h\vec{u})(x_K,y_K)|^2+|D^\alpha(\vec{\Pi}_h\vec{u}-\vec{u}_h)(x_K,y_K)|^2).
\end{aligned}
\tag{6.3}
$$

Thanks to the equivalent norm over the finite dimensional space, we have

$$
\begin{aligned}
&|D^\alpha(\vec{\Pi}_h\vec{u}-\vec{u}_h)(x_K,y_K)| \\
=&h_K^{-\alpha}|\hat{D}^\alpha(\hat{\vec{\Pi}}\hat{\vec{u}}-\hat{\vec{u}}_h)(0,0)| \le Ch_K^\alpha\|\hat{D}^\alpha(\hat{\vec{\Pi}}\hat{\vec{u}}-\hat{\vec{u}}_h)\|_{0,\infty,\hat{K}} \\
\le& Ch_K^\alpha\|\hat{D}^\alpha(\hat{\vec{\Pi}}\hat{\vec{u}}-\hat{\vec{u}}_h)\|_{0,\hat{K}}\le C(h_{K1}h_{K2})^{-\frac{1}{2}}|\vec{\Pi}_h\vec{u}-\vec{u}_h|_{1,K}.
\end{aligned}
\tag{6.4}
$$

Now, let us consider the first term at the right hand of (6.3). Firstly, we focus on $\alpha=(1,0)$. By the scaling technique,

$$
\begin{aligned}
&|D^{(1,0)}(u^{[1]}-\Pi_h^{[1]}u^{[1]})(x_K,y_K)| \\
=&h_{K1}^{-1}|\hat{D}^{(1,0)}(\hat{u}^{[1]}-\hat{\Pi}^{[1]}\hat{u}^{[1]})(0,0)|=h_{K1}^{-1}|l(\hat{D}^{(1,0)}\hat{u}^{[1]})|,
\end{aligned}
\tag{6.5}
$$

where in the last step we have used (2.6) and (2.7). Here $l(\hat{w})=\hat{w}(0,0)-\frac{1}{4}E_1(\hat{w})-\frac{1}{4}E_2(\hat{w})-\frac{3}{8}E_3(\hat{w})$.

From (2.6) and (2.7) it can be easily checked that for all $\hat{w}\in P_1(\hat{K})$,

$$
l(\hat{w})=0. \tag{6.6}
$$

Since $H^2(\hat{K})\hookrightarrow L^\infty(\hat{K})$, by the Bramble-Hilbert lemma we have

$$
|l(\hat{w})|\le C|\hat{w}|_{2,\hat{K}},\quad \forall \hat{w}\in H^2(\hat{K}). \tag{6.7}
$$

Consequently,

$$
\begin{aligned}
&|D^{(1,0)}(u^{[1]}-\Pi_h^{[1]}u^{[1]})(x_K,y_K)| = h_{K1}^{-1}|l(\hat{D}^{(1,0)}\hat{u}^{[1]})| \\
\le\,& Ch_{K1}^{-1}|\hat{D}^{(1,0)}\hat{u}^{[1]}|_{2,\hat{K}}\le C(h_{K1}h_{K2})^{-\frac{1}{2}}h_K^2|u^{[1]}|_{3,K}.
\end{aligned}
\tag{6.8}
$$

By the same arguments, we can prove that

$$
|D^{(1,0)}(u^{[2]}-\Pi_h^{[2]}u^{[2]})(x_K,y_K)| \le C(h_{K1}h_{K2})^{-\frac{1}{2}}h_K^2|u^{[2]}|_{3,K}, \tag{6.9}
$$

$$
|D^{(0,1)}(u^{[1]}-\Pi_h^{[1]}u^{[1]})(x_K,y_K)| \le C(h_{K1}h_{K2})^{-\frac{1}{2}}h_K^2|u^{[1]}|_{3,K}, \tag{6.10}
$$

$$
|D^{(0,1)}(u^{[2]}-\Pi_h^{[2]}u^{[2]})(x_K,y_K)| \le C(h_{K1}h_{K2})^{-\frac{1}{2}}h_K^2|u^{[2]}|_{3,K}. \tag{6.11}
$$

Therefore, a combination of (6.8)-(6.11), (6.4) and Theorem 4.1 yields (6.1).

Following the lines of the above arguments, we can prove (6.2), which completes the proof of the theorem. □

It is useful that both the gradient of the velocity and the pressure are superconvergent at the central points of all the rectangular elements. From the viewpoint of practical computations, we are more interested in the accuracy of the solution at the refined meshes. In fact, the central points of elements are very dense in these domains, so the accuracy of the finite element solution may be improved considerably. In our future work, we will do some numerical experiments to demonstrate our competing scheme for these problems.

We close this section by noting that the results in this section also hold for any quasi-uniform rectangular meshes.

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References

[1] T. Apel and M. Dobrowolski, Anisotropic interpolation with applications to the finite element method, *Computing*, **47** (1992), 277-293.

[2] T. Apel and S. Nicaise, The inf-sup condition for low order elements on anisotropic meshes, *Calcolo*, **41**:2 (2004), 89-113.

[3] T. Apel, S. Nicaise and J. Schöberl, A nonconforming finite element method with anisotropic mesh grading for the Stokes problem in domains with edges, *IMA J. Numer. Anal.*, **21** (2001), 843-856.

[4] T. Apel, Anisotropic Finite Elements: Local Estimates and Applications, Stuttgart Teubner, 1999.

[5] I. Babuška, The finite element method with Lagrangian multipliers, *Numer. Math.*, **20** (1973), 179-192.

[6] R. Becker, An adaptive finite element method for the incompressible Navier-Stokes equations on time-dependent domains, PhD thesis, Ruprecht-Karls-Universitat Heidelberg, 1995.

[7] R. Becker and R. Rannacher, Finite element solution of the incompressible Navier-Stokes equations on anisotropically refined meshes, In Fast Solvers for Flow Problems, Vol 49 of Notes on Numerical Fluid Mechanics, pp. 52-62, Wiesbaden, 1995.

[8] C. Bernardi and G. Raugel, Analysis of some finite elements for the Stokes problem, *Math. Comput.*, **44** (1985), 71-79.

[9] S.C. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods, Springer-Verlag, New York, 1994.

[10] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, New York, 1991.

[11] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers, *RAIRO Anal. Numer.*, **8** (1974), 129-151.

[12] S.C. Chen, D.Y. Shi and Y.C. Zhao, Anisotropic interpolation and quasi-Wilson element for narrow quadrilateral meshes, *IMA J. Numer. Anal.*, **24** (2004),77-95.

[13] S.C. Chen, Y.C. Zhao and D.Y. Shi, Anisotropic interpolations with application to nonconforming elements, *Appl. Numer. Math.*, **49** (2004), 135-152.

[14] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[15] M. Crouzeix and P.A. Raviart, Conforming and non-conforming finite elements for solving the stationary Stokes equations. *RAIRO. Anal. Numer.*, **7** (1973), 33-76.

[16] V. Girault and P.A. Raviart, Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms, Vol 5 of Springer Series in Computational Mathematics, Springer, Berlin, 1986.

[17] Q. Lin, J. Li and A. Zhou, A rectangle test for the Stokes problem, In Proc. Systems Science and Systems Engineering, Great Hall (H.K.) Culture Publishing Co.(1991), 236-247.

[18] Q. Lin and J. Pan, Global superconvergence for rectangular elements in Stokes problem, In Proc. Systems Science and Systems Engineering, Great Hall (H.K.) Culture Publishing Co. (1991), 371-376.

[19] Q. Lin and N. Yan, The Construction and Analysis of High Efficient Elements, Hebei University Press, 1996 (in Chinese).

[20] J. Pan, Global superconvergence for the bilinear-constant scheme for the Stokes problem, *SIAM J. Numer. Anal.*, **34** (1997), 2424-2430.

[21] R. Rannacher and S. Turek, Simple nonconforming quadrilateral Stokes element, *Numer. Meth. Part. Diff. Eqn.*, **8** (1992), 97-111.

[22] A. Russo, Communication at Recent Adances in Numerical Methods for Partial Differential Equations, ICTP, Trieste. September (1996), 23-27.

[23] D. Schötzau and C. Schwab, Mixed *hp*-FEM on anisotropic meshes, *Math. Mod. Meth. Appl. S.*, **8** (1998), 787-820.

[24] D. Schötzau , C. Schwab and R. Stenberg, Mixed *hp*-FEM on anisotropic meshes *II*: Hanging nodes and tensor products of boundary layer meshes, *Numer. Math.*, **83** (1999), 667-697.

[25] R. Stenberg, Analysis of some mixed finite element methods for the Stokes problem: A unified

approach, *Math. Comput.*, **42** (1984), 9-23.

[26] J.C. Xu and L. Zikatanov, Some observations on Babuška and Brezzi theories, *Numer. Math.*, **94** (2003), 195-202.

[27] S.P. Mao and S.C. Chen, Convergence analysis of Morley element on anisotropic meshes, *J. Comput. Math.*, **24** (2006), 169-180.

[28] D.Y. Shi and Y.R. Zhang, A nonconforming anisotropic finite element approximation with moving grids for Stokes problem, *J. Comput. Math.*, **24** (2006), 561-578.