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THE SECOND-ORDER OPTIMALITY CONDITIONS FOR VARIABLE PROGRAMMING*

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Abstract

We study in this paper the continuity of the objective function for variable programming. In particular, we study the second-order optimality conditions for unconstrained and constrained variable programming. Some new second-order sufficient and necessary conditions are obtained.

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Key words: Second-order optimality condition, Unconstrained, Constrained, Variable programming.

1. Introduction

Consider the unconstrained variable programming problem (VPI)

$$\min_{x\in R^n} \max_{i\in I(x)} f_i(x), \tag{1.1}$$

where

$$I(x) = \{j \in K | q_j(x) = q(x)\}, \tag{1.2a}$$

$$q(x) = \max_{l\in K}\{q_l(x)\}, \quad K = \{1, 2, \cdots, k\}. \tag{1.2b}$$

We also consider the constrained variable programming problem (VPII)

$$\min_{x} \max_{i\in I(x)} f_i(x) \tag{1.3}$$

$$s.t \quad c_j(x) \le 0, \qquad j = 1, 2, \cdots, p, \tag{1.4}$$

where

$$I(x) = \{j \in K | q_j(x) = \max_{l\in K} q_l(x)\}, \quad K = \{1, 2, \cdots, k\}. \tag{1.5}$$

In [8], Wang and Xu gave some theoretical results for the optimality conditions. In [3,4], Jiao et al. presented some useful theories and algorithms for (1.1)-(1.2) and (1.3)-(1.5). However, these theoretical results are only first-order optimality conditions. In this paper, we focus on the second-order optimality conditions for unconstrained and constrained variable programming.

Let

$$\varphi(x) = \max_{i\in I(x)} f_i(x). \tag{1.6}$$

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Fixing x, let us consider the set of indices $R(x)$ defined by

$$R(x) = \{i | i \in I(x), f_i(x) = \varphi(x)\}. \tag{1.7}$$

Lemma 1.1 ([3]) *For $x_0 \in R^n$, suppose that the functions $q_i(x), i \in K$, are continuous at point x_0, then there exists a real number $\delta > 0$ such that for all $x \in S(x_0, \delta) := \{x | ||x - x_0|| < \delta\}$),*

$$I(x) \subseteq I(x_0). \tag{1.8}$$

Lemma 1.2 ([3]) *For $x_0 \in R^n$, let the functions $f_i(x), q_i(x), i \in K$, be continuous at point x_0. If there exists a real number $\delta > 0$ such that for all $x \in S(x_0, \delta)$,*

$$I(x) \cap R(x_0) \neq \emptyset, \tag{1.9}$$

then

$$\varphi(x) = \max_{i \in I(x)} f_i(x) = \max_{i \in I(x) \cap R(x_0)} f_i(x). \tag{1.10}$$

Theorem 1.1. *For $x_0 \in R^n$, suppose that the functions $f_i(x), q_i(x), i \in K$, are continuous at point x_0. Then $\varphi(x)$ is continuous at point x_0 if and only if there exists a real number $\delta > 0$ such that for all $x \in S(x_0, \delta)$,*

$$I(x) \cap R(x_0) \neq \emptyset. \tag{1.11}$$

Proof. If $I(x) \cap R(x_0) \neq \emptyset$, we obtain from Lemma 1.2 that

$$\lim_{x \to x_0} \varphi(x) = \lim_{x \to x_0} \max_{i \in I(x) \cap R(x_0)} f_i(x) = \max_{i \in R(x_0)} f_i(x_0) = \varphi(x_0).$$

Hence, $\varphi(x)$ is continuous at point x_0. On the other hand, suppose that $\varphi(x)$ is continuous at point x_0. If there exists a sequence $x_i \to x_0$ such that $I(x_i) \cap R(x_0) = \emptyset$, then for $\forall \epsilon$ satisfying $0 < \epsilon \leq \frac{1}{2}(\varphi(x_0) - f_{j_0}(x_0))$, where

$$j_0 \in \Big\{ j | f_j(x_0) = \max_{j \in \{\lim\limits_{x_i \to x_0} I(x_i)\}} \{f_j(x)\} \Big\},$$

there exists an integer N_0 such that for $i > N_0$,

$$\varphi(x_i) = \max_{j \in I(x_i)} \{f_j(x_i)\} \leq f_{j_0}(x_0) + \epsilon.$$

Thus,

$$\begin{aligned} |\varphi(x_i) - \varphi(x_0)| &\geq |f_{j_0}(x_0) + \epsilon - \varphi(x_0)| \\ &\geq |f_{j_0}(x_0) - \varphi(x_0)| - \epsilon \\ &\geq \frac{1}{2}(\varphi(x_0) - f_{j_0}(x_0)), \end{aligned}$$

which is a contradiction with the assumption that $\varphi(x)$ is continuous at point x_0. Hence, the theorem is proved. □

For $x_0 \in R^n$, and $\forall h \in R^n$, $R'(x_0, h)$ is defined by

$$R'(x_0, h) = \lim_{\alpha \to 0^+} I(x_0 + \alpha h) \cap R(x_0), \tag{1.12}$$

$$R'(x_0) = \cup_{||h||=1} R'(x_0, h). \tag{1.13}$$

Furthermore, let

$$L(x) = \Big\{ z = \sum_{i \in R'(x_0)} \mu_i \bigtriangledown f_i(x_0) | \mu_i \geq 0, i \in R'(x_0), \sum_{i \in R'(x_0)} \mu_i = 1 \Big\}. \tag{1.14}$$

2. Optimality Conditions for Unconstrained Variable Programming

Suppose $K = \{1, 2, \cdots, k\}$, $R'(x_0) = \{i_1, i_2, \cdots, i_l\}$. Let

$$\Gamma_K = \Big\{\mu | \mu = (\mu_i)_{k\times 1}, i \in K, \mu_i \geq 0, \sum_{i\in K} \mu_i = 1\Big\}, \tag{2.1}$$

$$\Gamma_{R'(x_0)} = \Big\{\mu | \mu = (\mu_i)_{l\times 1}, \mu_i \geq 0, i \in R'(x_0), \sum_{i\in R'(x_0)} \mu_i = 1\Big\}. \tag{2.2}$$

The variable programming (1.1)-(1.2) is called regular at point x_0 if $R'(x_0) = R(x_0)$.

Theorem 2.1. *Let the functions $f_i(x), i \in K$ be continuously differentiable, and $q_i(x), i \in K$ be continuous. Assume the variable programming* (1.1)-(1.2) *is regular at point x^*. Then $0 \in L(x^*)$ if and only if there exists a multiplier vector $\mu \in \Gamma_K$ such that*

$$\sum_{i\in K} \mu_i \bigtriangledown f_i(x^*) = 0, \tag{2.3a}$$

$$\sum_{i\in K} \mu_i(\varphi(x^*) - f_i(x^*)) = 0, \tag{2.3b}$$

$$\sum_{i\in K} \mu_i(q(x^*) - q_i(x^*)) = 0. \tag{2.3c}$$

Proof. From (2.3a), (2.3b) and (2.3c), we have

$$\mu_i \geq 0, \quad i \in R(x^*),$$
$$\mu_i = 0, \quad i \notin R(x^*).$$

Hence, we obtain this theorem from the regularity of (1.1)-(1.2). □

Theorem 2.2. *Let the functions $f_i(x), i \in K$ be twice continuously differentiable, and $q_i(x), i \in K$ be continuous. Assume that x^* is a local minimizer of $\varphi(x)$. Let the critical cone $H(x^*)$ be defined by*

$$H(x^*) = \{h \in R^n, d\varphi(x^*, h) = 0\}, \tag{2.4}$$

and for any $h \in R^n$

$$R''(x^*, h) = \{j \in R'(x^*, h) | d\varphi(x^*, h) = \bigtriangledown f_j(x^*)^T h\}. \tag{2.5}$$

Then

$$\max_{j\in R''(x^*,h)} h^T \bigtriangledown^2 f_j(x^*)h \geq 0, \qquad \forall h \in H(x^*). \tag{2.6}$$

Proof. Since x^* is a local minimizer of $\varphi(x)$, it is a stationary point for the problem (VPI). Furthermore, for any $h \in R^n$, there exists a $t > 0$, such that

$$\begin{aligned} 0 &\leq \varphi(x^* + th) - \varphi(x^*) \\ &= \max_{i\in R'(x^*,h)} \{f_i(x^* + th) - f_i(x^*)\} \\ &= \max_{i\in R'(x^*,h)} \{t \bigtriangledown f_i(x^*)^T h + \frac{t^2}{2} h^T \bigtriangledown^2 f_i(x^* + s_i th)h\}, \end{aligned}$$

where $s_i = s_i(t, h) \in [0, 1]$. Now suppose $h \in H(x^*)$. Then $i \in R''(x^*, h)$,

$$\bigtriangledown f_i(x^*)^T h = 0.$$

The above two results yield (2.6), which completes the proof of this theorem. □

Theorem 2.3. *Let the functions $f_i(x), i \in K$ be twice continuously differentiable, and $q_i(x), i \in K$ be continuous. Assume that x^* is a stationary point of the problem (VPI). If there exist a $\mu^* \in \Gamma_{R'(x^*)}$ and an $\epsilon > 0$ such that for all $h \in H(x^*)$*

$$h^T \left(\sum_{i \in R'(x^*,h)} \mu_i^* \bigtriangledown^2 f_i(x^*) \right) h \geq \epsilon \|h\|^2. \tag{2.7}$$

Then x^* is a strict local minimizer of the problem (VPI).

Proof. Observe that

$$\begin{aligned} \varphi(x^* + th) - \varphi(x^*) &= \max_{i \in I(x^*+th) \cap R(x^*)} f_i(x^* + th) - \max_{i \in R(x^*)} f_i(x^*) \\ &= \max_{i \in R'(x^*,h)} \{ f_i(x^* + th) - f_i(x^*) \} \\ &= \max_{i \in R'(x^*,h)} \{ t \bigtriangledown f_i(x^* + s_i th)^T h \}, \quad 0 \leq s_i \leq 1. \end{aligned}$$

If $h \notin H(x^*)$, then $d\varphi(x^*, h) > 0$. Hence, there exists a $\delta > 0$, such that $x \in S(x^*, \delta)$,

$$\max_{i \in R'(x^*,h)} \bigtriangledown f_i(x)^T h > 0. \tag{2.8}$$

Let $0 < t < \frac{\delta}{\|h\|}$ and $x^* + th \in S(x^*, \delta)$. We have

$$\varphi(x^* + th) - \varphi(x^*) > 0.$$

We know from (2.7) that there exists a $\delta > 0$, such that for $x \in S(x^*, \delta)$,

$$h^T \left(\sum_{i \in R'(x^*,h)} \mu_i^* \bigtriangledown^2 f_i(x) \right) h > \frac{\epsilon}{2} ||h||^2.$$

On the other hand, if $h \in H(x^*)$, then

$$\begin{aligned} &\varphi(x^* + th) - \varphi(x^*) \\ &= \max_{i \in R'(x^*,h)} \{ f_i(x^* + th) - f_i(x^*) \} \\ &\geq \sum_{i \in R'(x^*,h)} \mu_i^* [f_i(x^* + th) - f_i(x^*)] \\ &= \sum_{i \in R'(x^*,h)} \int_0^1 \mu_i^* h^T \bigtriangledown^2 f_i(x^* + th)(1-t) h dt + \sum_{i \in R'(x^*,h)} t \mu_i^* \bigtriangledown f_i(x^*)^T h \\ &= \int_0^1 h^T \left(\sum_{i \in R'(x^*,h)} \mu_i^* \bigtriangledown^2 f_i(x^* + th)(1-t) \right) h dt \\ &\geq \frac{\epsilon}{2} \|h\|^2 \int_0^1 (1-t) dt = \frac{\epsilon}{4} \|h\|^2 > 0. \end{aligned}$$

This completes the proof of the theorem. □

Theorem 2.4. *Let the functions $f_i(x), i \in K$ be twice continuously differentiable, and $q_i(x), i \in K$ be continuous. Assume that x^* is a stationary point of the problem (VPI). If there exist $\mu^* \in \Gamma_{R'(x^*)}$ and positive constants α, δ, ϵ, such that for all $x \in S(x^*, \delta)$ and for all $h \in H_\alpha(x^*)$,*

$$h^T \left(\sum_{i \in R'(x^*,h)} \mu_i^* \bigtriangledown^2 f_i(x) \right) h \geq 0, \tag{2.9}$$

and for all $h \in \widehat{H}_\alpha(x^*)$

$$\max_{i \in R''(x^*,h)} h^T \bigtriangledown^2 f_i(x^*) h \geq \epsilon \|h\|^2, \tag{2.10}$$

where

$$\widehat{H}_\alpha(x^*) = \{h \in H_\alpha(x^*) | h^T (\sum_{i \in R'(x^*,h)} \mu_i^* \bigtriangledown^2 f_i(x)) h = 0\}, \tag{2.11}$$

$$H_\alpha(x^*) = \{h \in R^n | d\varphi(x^*, h) \leq \alpha \|h\|\}. \tag{2.12}$$

Then x^* is a strict local minimizer of the problem (VPI).

Proof. Observe that

$$\varphi(x^* + th) - \varphi(x^*) = \max_{i \in R'(x^*,h)} \{f_i(x^* + th) - f_i(x^*)\}.$$

If $h \notin H_\alpha(x^*), d\varphi(x^*, h) > \alpha \|h\| > 0$.

Arguing in the same way as in the proof of Theorem 2.3, we obtain

$$\varphi(x^* + th) - \varphi(x^*) > 0.$$

Assume that $h \in H_\alpha(x^*)$, but $h \notin \widehat{H}_\alpha(x^*)$. In this case, due to (2.9) and the fact that the functions $f_i, i \in K$ are twice continuously differentiable, there exists a $\sigma = \sigma(h)$ such that for all $t \in [0, \sigma]$

$$h^T \Big(\sum_{i \in R'(x^*,h)} \mu_i^* \bigtriangledown^2 f_i(x^* + th) \Big) h > 0.$$

Hence, it follows that

$$\begin{aligned} &\varphi(x^* + th) - \varphi(x^*) \\ \geq &\sum_{i \in R'(x^*,h)} \mu_i^* (f_i(x^* + th) - f_i(x^*)) \\ = &\sum_{i \in R'(x^*,h)} \frac{t^2}{2} \mu_i^* h^T \bigtriangledown^2 f_i(x^* + s_i th) h + \sum_{i \in R'(x^*,h)} t \mu_i^* \bigtriangledown f_i(x^*)^T h. \end{aligned} \tag{2.13}$$

In (2.13), the first term is greater than zero because $h \in H_\alpha(x^*)$, but $h \notin \widehat{H}_\alpha(x^*)$. The second term is equal to zero, because x^* is a stationary point of the problem (VPI). Thus, we obtain

$$\varphi(x^* + th) - \varphi(x^*) > 0.$$

If $h \in \widehat{H}_\alpha(x^*)$, let $i \in \arg\max_{j\in R''(x^*,h)} h^T \bigtriangledown^2 f_j(x^*)h$. Then

$$\begin{aligned}
&\varphi(x^*+th)-\varphi(x^*)\\
&\geq f_i(x^*+th)-f_i(x^*)\\
&= t\bigtriangledown f_i(x^*)^T h+\int_0^1 (1-t)h^T \bigtriangledown^2 f_i(x^*+th)hdt\\
&\geq \int_0^1 (1-t)h^T \bigtriangledown^2 f_i(x^*+th)hdt\\
&\geq \frac{\epsilon}{2}\|h\|^2\int_0^1 (1-t)dt=\frac{\epsilon}{4}\|h\|^2>0,
\end{aligned}$$

where the second inequality holds because $\bigtriangledown f_i(x^*)^T h \geq 0, i \in R''(x^*,h)$, and the third inequality holds because of (2.10). Thus, the proof of the theorem is completed. □

Theorem 2.5. *Let the functions $f_i(x), i \in K$ be twice continuously differentiable, and $q_i(x), i \in K$ be continuous. Assume that x^* is a stationary point of the problem (VPI). If the functions $f_i(x), i \in K$ are convex, then x^* is a local minimizer of the problem (VPI).*

Proof. For the stationary point x^* and a neighborhood $S(x^*,\delta)$ of the x^*, let $h=\frac{x-x^*}{||x-x^*||}, t=||x-x^*||$. By the same arguments as in the proof of Theorem 2.3, we have

$$\begin{aligned}
&\varphi(x^*+th)-\varphi(x^*)\\
&\geq \sum_{i\in R'(x^*,h)} \mu_i^*[f_i(x^*+th)-f_i(x^*)]\\
&= \sum_{i\in R'(x^*,h)} \mu_i^*\Big[t\bigtriangledown f_i(x^*)^T h+\frac{t^2}{2}h^T\bigtriangledown^2 f_i(x^*+s_ith)h\Big]\\
&= \sum_{i\in R'(x^*,h)} \Big[\frac{t^2}{2}h^T\mu_i^*\bigtriangledown^2 f_i(x^*+s_ith)h\Big],
\end{aligned}$$

where $s_i = s_i(t,h) \in [0,1]$. Since x^* is a stationary point and the functions $f_i, i \in K$, are convex, $h^T \bigtriangledown^2 f_i(x^*+s_ith)h \geq 0$. Hence,

$$\varphi(x)-\varphi(x^*)\geq 0, \quad \forall x \in S(x^*,\delta).$$

Consequently, x^* is a local minimizer of the problem (VPI). □

Example 2.1. Consider

$$\begin{aligned}
&f_1(x_1,x_2)=x_1^2+x_2^3, \quad f_2(x_1,x_2)=x_2^2+x_2+x_1^3,\\
&f_3(x_1,x_2)=x_2^3-x_2-x_1^3, \quad f_4(x_1,x_2)=x_2^2+x_1^3.
\end{aligned}$$

Let

$$I(x)=\begin{cases}\{1,2,3\}, & x_1+x_2<1,\\ \{1,2,3,4\}, & x_1+x_2=1,\\ \{3,4\}, & x_1+x_2>1.\end{cases}$$

It can be verified that $x^* = (0,0)$ and $\mu_1^* = 0,\ \mu_2^* = \mu_3^* = 0.5$. Moreover,

$$\nabla f_1(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \nabla f_2(0,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \nabla f_3(0,0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad R'(0) = \{1,2,3\},$$
$$\nabla^2 f_1(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \nabla^2 f_2(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad \nabla^2 f_3(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It can be shown that the necessary condition (2.6) is satisfied. Furthermore,

$$H_\alpha(0,0) = \{h = (h_1,h_2)^T \,|\, \max\{h_2,-h_2\} \le \alpha\sqrt{h_1^2+h_2^2}\}$$

$\forall h \in H_\alpha(0,0)$,

$$h^T \left(\sum_{i \in R'(x^*,h)} \mu_i^* \nabla^2 f_i(x) \right) h = (2+6x_2)h_2^2 \ge 0.$$

Hence, (2.9) is satisfied. Now, $\widehat{H}_\alpha(0,0) = \{h = (h_1,0)\}$, and for $h \in \widehat{H}_\alpha(0,0),\ R''(0,h) = \{1\}$,

$$\max_{i \in R''(0,h)} h^T \nabla^2 f_i(0,0) h = 2h_1^2,$$

which shows that the inequality (2.10) is also satisfied. Therefore, x^* is a strict local minimizer. In fact, the set $I(x)$ is very important. If $I(x) = \{1,2\}, x_1 + x_2 \le 1$, then $x^* = (0,0)$ is not a local minimizer.

Remark 2.1. Theorem 17 in [3] demonstrated that x^* is a local minimizer of the problem (VPI) if $\varphi(x)$ is convex. It is also shown in [8] that $\varphi(x)$ is not convex even though the functions $f_i(x), i \in K$ are convex. Hence, Theorem 2.5 is sharper than Theorem 17 in [3]. But the minimizer x^* in Theorem 2.5 is a local minimizer.

Example 2.2. Consider $f_1(x) = x^2$ and $f_2(x) = (x-2)^2 - 2$, and

$$q_1(x) = \begin{cases} (x-0.5)^2 & x < 0.5, \\ 0 & x \ge 0.5, \end{cases} \qquad q_2(x) = \begin{cases} 0 & x < 0.5, \\ (x-0.5)^2 & x \ge 0.5. \end{cases}$$

Then

$$\varphi(x) = \begin{cases} x^2 & x < 0.5, \\ (x-2)^2 - 2 & x \ge 0.5. \end{cases}$$

Obviously, $x = 0$ is a local minimizer of $\varphi(x)$ even though $f_1(x), f_2(x), q_1(x)$ and $q_2(x)$ are convex in R.

3. Optimality Conditions for Constrained Variable Programming

Let x^* be a local minimizer of the problem (VPII). Then we define:

$$g_i(x) = f_i(x) - \varphi(x^*), \quad i \in K. \tag{3.1}$$

Next, as before, we define

$$\min \max_{i \in I(x), j \in P} \{g_i(x), c_j(x)\}, \tag{3.2}$$

where

$$P = \{1, 2, \cdots, p\},$$

$$\psi(x) = \max_{i \in I(x), j \in P} \{g_i(x), c_j(x)\}, \tag{3.3}$$

$$J(x) = \{j | c_j(x) = \psi(x), j = 1, 2, \cdots, p\}, \tag{3.4}$$

$$\hat{L}(x_0) = \Big\{ z = \sum_{i \in R'(x_0)} \mu_i \bigtriangledown f_i(x_0) + \sum_{j \in J(x_0)} \lambda_j \bigtriangledown c_j(x_0) | \mu_i, \\ \lambda_j \geq 0, i \in R'(x_0), j \in J(x_0), \sum_{i \in R'(x_0)} \mu_i + \sum_{j \in J(x_0)} \lambda_j = 1 \Big\}. \tag{3.5}$$

Theorem 3.1. *If x^* is a local minimizer of the problem (VPII), then it is a local minimizer of* (3.2).

Proof. Observe that

$$\psi(x^* + th) - \psi(x^*) = \psi(x^* + th) \\ = \max_{i \in R'(x^*, h), j \in P} \{g_i(x^* + th), c_j(x^* + th)\} \geq 0.$$

This completes the proof of the theorem. □

Theorem 3.2. *x^* is a strict local minimizer of the problem (VPII) if and only if it is a strict local minimizer of* (3.2).

Proof. If x^* is a strict local minimizer of (3.2), we have $\psi(x^*) = 0$. Moreover, there exists a neighborhood $S(x^*, \delta)$ such that for $x \neq x^*, x \in S(x^*, \delta)$,

$$\psi(x) > \psi(x^*) = 0.$$

Hence, if x is feasible, it follows that

$$\varphi(x) > \varphi(x^*).$$

Consequently, x^* is a strict local minimizer of the problem (VPII). On the other hand, suppose x^* is a strict local minimizer of the problem (VPII). Then there exists a neighborhood $S(x^*, \delta)$ such that for $x \neq x^*$, $x \in S(x^*, \delta)$, either x is infeasible, i.e., $c_j(x) > 0$ for some $j \in \{1, 2, \cdots, p\}$, or x is feasible but $\varphi(x) > \varphi(x^*)$. In both cases, we have

$$\psi(x) > \psi(x^*).$$

Hence, x^* is a strict local minimizer of (3.2). □

Theorem 3.3. *Let the functions $f_i(x), c_j(x), i \in K, j \in P$ be continuously differentiable, and $q_i(x), i \in K$ be continuous. Suppose that the variable programming* (1.3)-(1.4) *is regular. Then, $0 \in \hat{L}(x^*)$ holds if and only if there exist $\mu^* \in \Gamma_K$, $\lambda^* \in \Gamma_P$, such that*

$$\sum_{i \in K} \mu_i^* \bigtriangledown f_i(x^*) + \sum_{j=1}^{p} \lambda_j^* \bigtriangledown c_j(x^*) = 0, \tag{3.6}$$

$$\sum_{i \in K} \mu_i^* (f_i(x^*) - \varphi(x^*)) + \sum_{j=1}^{p} \lambda_j^* c_j(x^*) = 0. \tag{3.7}$$

Proof. The assertions follow by using the same arguments as in the proof of Theorem 2.1. □

Theorem 3.4. *Let the functions $f_i(x), c_j(x), i \in K, j \in P$ be twice continuously differentiable, and $q_i(x), i \in K$ is continuous. Suppose x^* is a local minimizer of the problem (VPII). Let the critical cone for the programming (VPII) at x^* be defined by*

$$\begin{aligned} H(x^*) &= \{h \in R^n | d\psi(x^*, h) = 0\} \\ &= \Big\{h \in R^n | \max_{\substack{i \in R'(x^*, h) \\ j \in J(x^*)}} \{\triangledown f_i(x^*)^T h, \triangledown c_j(x^*)^T h\} = 0\Big\}, \end{aligned} \tag{3.8}$$

and for all $h \in R^n$, let

$$u(x^*, h) = \{i \in R'(x^*, h) | d\psi(x^*, h) = \triangledown f_i(x^*)^T h\}, \tag{3.9}$$

$$v(x^*, h) = \{j \in J(x^*) | d\psi(x^*, h) = \triangledown c_j(x^*)^T h\}. \tag{3.10}$$

Then for all $h \in H(x^*)$

$$\max_{i \in u(x^*, h), j \in v(x^*, h)} \{h^T \triangledown^2 f_i(x^*) h, h^T \triangledown^2 c_j(x^*) h\} \geq 0. \tag{3.11}$$

Proof. The proof is similar to that used in the proof of Theorem 2.2. □

Theorem 3.5. *Let the functions $f_i(x)$, $i \in K$, $c_j(x)$, $j \in P$ be twice continuously differentiable, and $q_i(x)$, $i \in K$ be continuous. Suppose that x^* is a stationary point of the problem (VPII), and μ^*, λ^* are the corresponding Lagrange multipliers. If there exists an $\varepsilon > 0$ such that for all $h \in H(x^*)$,*

$$h^T \left(\sum_{i \in R'(x^*, h)} \mu_i^* \triangledown^2 f_i(x^*) + \sum_{j \in J(x^*)} \lambda_j^* \triangledown^2 c_j(x^*) \right) h \geq \varepsilon ||h||^2. \tag{3.12}$$

Then x^* is a strict local minimizer of the problem (VPII).

Proof. By the same arguments as in the proof of Theorem 2.3, we deduce that x^* is a strict local minimizer of $\psi(x)$. Thus, we conclude this theorem in view of Theorem 3.2. □

Theorem 3.6. *Let the functions $f_i(x)$, $c_j(x), i \in K, j \in P$ be twice continuously differentiable, and $q_i(x), i \in K$ be continuous. Assume that x^* is a stationary point of the problem (VPII). If there exist $\mu^* \in \Gamma_{R'(x^*)}$, $\lambda^* \in \Gamma_{J(x^*)}$, and positive constants α, δ, ϵ such that for all $x \in S(x^*, \delta)$ and for all $h \in H_\alpha(x^*)$*

$$h^T \left(\sum_{i \in R'(x^*, h)} \mu_i^* \triangledown^2 f_i(x) + \sum_{j \in J(x^*)} \lambda_j^* \triangledown^2 c_j(x) \right) h \geq 0, \tag{3.13}$$

where

$$H_\alpha(x^*) = \{h \in R^n | d\psi(x^*, h) \leq \alpha ||h||\}, \tag{3.14}$$

and for all $h \in \hat{H}_\alpha(x^*)$

$$\max_{\substack{i \in u(x^*, h) \\ j \in v(x^*, h)}} \{h^T \triangledown^2 f_i(x^*) h,\ h^T \triangledown^2 c_j(x^*) h\} \geq \epsilon \|h\|^2, \tag{3.15}$$

where

$$\hat{H}_\alpha(x^*) = \Big\{ h \in H_\alpha(x^*) \Big| h^T\Big(\sum_{i \in u(x^*,h)} \mu_i^* \bigtriangledown^2 f_i(x^*))h + h^T(\sum_{j \in v(x^*,h)} \lambda_j^* \bigtriangledown^2 c_j(x^*)\Big)h = 0 \Big\}. \tag{3.16}$$

Then x^* is a strict local minimizer of the problem (VPII).

Proof. By the same arguments as in the proof of Theorem 2.4, we deduce that x^* is a strict local minimizer of $\psi(x)$. Thus, we conclude this theorem in view of Theorem 3.2. □

Corollary 3.1. *Let the functions $f_i(x)$, $c_j(x), i \in K, j \in P$ be twice continuously differentiable, and $q_i(x), i \in K$ be continuous. Suppose that x^* is a stationary point of the problem (VPII). If the functions $f_i(x)$, $c_j(x), i \in R(x^*), j \in J(x^*)$ are strict convex, then x^* is a strict local minimizer of the problem (VPII).*

Proof. If $f_i(x)$, $c_j(x), i \in R(x^*), j \in J(x^*)$ are strict convex, then (3.13) and (3.15) are satisfied. Taking Theorems 3.2 and 3.6 into account, we complete the proof. □

Example 3.1. Consider $f_1(x_1, x_2) = x_1^2 + x_2,\ f_2(x_1, x_2) = x_2^2 + x_1,\ f_3(x_1, x_2) = x_1^3 - x_1 - x_2$, s.t. $-\frac{1}{2} \le x_1 + x_2 \le 1$. Let

$$I(x) = \begin{cases} \{1,2\}, & -\frac{1}{2} \le x_1 + x_2 < \frac{1}{2}, \\ \{1,2,3\}, & x_1 + x_2 = \frac{1}{2}, \\ \{3,1\}, & \frac{1}{2} < x_1 + x_2 \le 1. \end{cases}$$

It can be verified that $x^* = (-0.25, -0.25)$. Then

$$\nabla f_1(x_1^*, x_2^*) = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}, \quad \nabla f_2(x_1^*, x_2^*) = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix},$$

$$\nabla c_1(x_1^*, x_2^*) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \nabla c_2(x_1^*, x_2^*) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We obtain only one solution: $\mu_1^* = \mu_2^* = \frac{2}{5},\ \lambda_1^* = \frac{1}{5},\ \lambda_2^* = 0$. Moreover,

$$\nabla^2 f_1(x_1^*, x_2^*) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \nabla^2 f_2(x_1^*, x_2^*) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

$$\nabla^2 c_1(x_1^*, x_2^*) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \nabla^2 c_2(x_1^*, x_2^*) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It can be shown that $\forall h \in H(x_1^*, x_2^*)$,

$$h^T\Big(\sum_{i \in R'(x^*,h)} \mu_i^* \bigtriangledown^2 f_i(x^*)\Big)h + h^T\Big(\sum_{j \in J(x^*)} \lambda_j^* \bigtriangledown^2 c_j(x^*)\Big)h = 2(h_1^2 + h_2^2),$$

which implies that (3.12) is satisfied. Therefore, x^* is a local strict minimizer.

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