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CONVERGENCE ANALYSIS ON ITERATIVE METHODS FOR SEMIDEFINITE SYSTEMS*

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Abstract

The convergence analysis on the general iterative methods for the symmetric and positive semidefinite problems is presented in this paper. First, formulated are refined necessary and sufficient conditions for the energy norm convergence for iterative methods. Some illustrative examples for the conditions are also provided. The sharp convergence rate identity for the Gauss-Seidel method for the semidefinite system is obtained relying only on the pure matrix manipulations which guides us to obtain the convergence rate identity for the general successive subspace correction methods. The convergence rate identity for the successive subspace correction methods is obtained under the new conditions that the local correction schemes possess the local energy norm convergence. A convergence rate estimate is then derived in terms of the exact subspace solvers and the parameters that appear in the conditions. The uniform convergence of multigrid method for a model problem is proved by the convergence rate identity. The work can be regraded as unified and simplified analysis on the convergence of iteration methods for semidefinite problems [8,9].

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1. Introduction

We consider the iterative methods for the following linear problem,

$$Au = b, \tag{1.1}$$

where A is a symmetric and positive semidefinite operator from V to V, V is a finite dimensional Hilbert space with the inner product $(\cdot,\cdot)$ and $b \in V$ is a vector in the range of A. Such semidefinite problems arise in many areas of applied mathematics. The finite element and/or

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finite difference discretizations of the Poisson equation with Neumann boundary conditions [2] and the linear elasticity equation with pure traction boundary conditions lead to such problems. Other more sophisticated examples can be found at the linear systems obtained from generalized finite element methods [15, 16], and the time dependent Navier-Stokes systems [3].

For such problems, in general, it is difficult to apply the direct methods in a straightforward manner (not to mention that direct methods are very expensive for large linear systems [23]). Iterative methods are desirable for large semidefinite systems and our focus in this paper will be made, in particular, on the convergence analysis of the classic iterative methods and the general subspace correction methods for semidefinte (singular) problems given in (1.1).

The studies of the classic iterative methods for singular systems and their convergence can be traced back to Keller, [7] and there have been many investigations by many researchers since then, see [1, 4, 8, 11] and also many references cited therein. The classical iterative methods discussed in those works are mainly based on a matrix splitting: $A = M - N$ and from the setting for the iterates $\{u^\ell\}_{\ell=0}$:

$$Mu^\ell = Nu^{\ell-1} + b \tag{1.2}$$

or equivalently,

$$M\big(u^\ell - u^{\ell-1}\big) = \big(b - Au^{\ell-1}\big). \tag{1.3}$$

All the convergence results require that the iterator M be an invertible matrix, except that in [8]. Furthermore, the setting in [8] requires $\mathcal{N}(M^t) \subset \mathcal{N}(A)$ which is necessary for the solvability of (1.2) for x^ℓ.

In this paper, we study iterative methods for (1.1) in the following form:

$$u^\ell = u^{\ell-1} + R\big(b - Au^{\ell-1}\big), \tag{1.4}$$

where R is a linear operator from V to V and it may be singular. We then present more refined necessary and sufficient conditions for the energy norm convergence of the iterative method (1.4). One advantage of such view is that no assumption on the null space is necessary.

The rest of paper will be devoted to establish a convergence rate identity for the general successive subspace correction method. The techniques of subspace correction methods are based on a "divide and conquer" strategy. Classic iterative methods as Gauss-Seidel method, and many multigrid and domain decomposition methods fall into this category of methods. Recently, authors provided a sharpest possible convergence estimate for the general subspace correction method for singular systems in a general Hilbert space setting, [9]. The current works are aimed to better understand the basic idea of obtaining the convergence rate estimate in a transparent manner restricting the problem in finite dimensional spaces.

The sharp convergence rate identity for the Gauss-Seidel method for the semidefinite system, is obtained relying only on the pure matrix manipulations. The idea will guide us to obtain the convergence rate identity for the general successive subspace correction methods. For the successive subspace correction methods, we assume that the local correction schemes possess the local energy norm convergence. We then derive a new version of the convergence rate identity [9] under minimal assumptions. We also get the convergence rate estimate in terms of the exact subspace solvers and the parameters that appear in the conditions. In Section 4, we give an example from an electrochemical model to illustrate how to apply our identity in designing an optimal multigrid method for such a singular system, and prove the uniform convergence for the multigrid method. As the results in this paper, we will be able to give the convergence criteria that are more refined and concise that that in [8, 9] and whose analysis becomes more transparent.

In this paper, for the convenience of exposition, we introduce some standard notation. For any subspace W of V, i.e., $W \subset V$, $W^\perp$ denotes the orthogonal complement of W with respect to the inner product, $(\cdot,\cdot)$; for two subspaces N and W of V with $N \subset W$, W/N denotes the quotient space of W; for a given matrix M, the range of M and the null space of M are denoted with $\mathcal{R}(M)$ and $\mathcal{N}(M)$ respectively. For a given real matrix G (or an operator $B : V \mapsto V$), the real transpose of G (or the adjoint operator of B) shall be denoted by G^t (or B^t).

The structure of the paper is as follows. In Section 2, we derive the necessary and sufficient conditions for energy norm convergence of the scheme (1.4). In Section 3, we introduce the successive subspace correction methods for the system (1.1) and present the convergence rate identity and estimate. The proof of the technique lemma (Lemma 3.3) is given in Section 5. An example from an electrochemical model is given in Section 4.

2. Necessary and Sufficient Conditions for the Energy Norm Convergence of the Stationary Linear Iterative Method

A general stationary iterative method to solve the system of equation (1.1) is given by (1.4). Note that since $\mathcal{N}(A) \neq \{0\}$, the solution to (1.1) is not unique, but is unique in the quotient space $V/\mathcal{N}(A)$.

Denote $T = RA$, an obvious sufficient condition for convergence of (1.4) is given by

$$|I - T|_A < 1. \tag{2.1}$$

More precisely, let $\{u^\ell\}_{\ell=0}^\infty$ be the sequence of iterates generated by (1.4) with any given initial guess u^0. If (2.1) is satisfied, then the sequence $\{u^\ell\}_{\ell=0}^\infty$ is convergent, and the limit is a solution to (1.1). In case, the condition (2.1) is satisfied, we shall say that (1.4) is *energy norm convergent*. The goal of this section is to find necessary and sufficient conditions of the iterative method (1.4) for the energy norm convergence (2.1).

Note that,

$$\begin{aligned} |u|_A^2 - |(I-T)u|_A^2 &= \big((R + R^t - R^tAR)Au, u\big)_A \\ &= 2(Tu,u)_A - (Tu,Tu)_A, \quad \forall u \in V, \end{aligned} \tag{2.2}$$

Denote $\bar{R} = R + R^t - R^tAR$, and $\mathcal{Q} : V \mapsto \mathcal{R}(A)$ be the orthogonal projection under the inner product $(\cdot,\cdot)$. We get the following simple and important lemma.

Lemma 2.1. *The scheme* (1.4) *is energy norm convergent if and only if*

$$\mathcal{Q}\bar{R}\mathcal{Q} \text{ is symmetric and positive definite on } \mathcal{R}(A), \tag{2.3}$$

or equivalently there exists a positive constant α *such that*

$$2(Tu,u)_A - (Tu,Tu)_A \geq \alpha(u,u)_A \qquad \forall u \in V. \tag{2.4}$$

Proof. The conclusion follows directly by (2.2). □

The following theorem on the convergence of the iteration given by (1.4) is the main result in this section.

Theorem 2.1. *The assumption* (2.3) *or* (2.4) *is equivalent to the following two assumptions,*

(A1) $\exists \omega \in (0,2)$ *such that* $(Tu,Tu)_A \leq \omega (Tu,u)_A \qquad \forall u \in V$,

(A2) $\exists \beta > 0$ *such that* $(Tu,Tu)_A \geq \beta (u,u)_A \qquad \forall u \in V$.

The iterative scheme (1.4) *is convergent if both* (A1) *and* (A2) (*equivalently* (2.3) *or* (2.4)) *are satisfied. Furthermore* (A1) *and* (A2) (*equivalently* (2.3) *or* (2.4)) *are also necessary conditions for the energy norm convergence.*

Proof. We only need to prove that (A1) and (A2) are equivalent to (2.4). Assume (A1) and (A2) hold, we have

$$\begin{aligned} 2(Tu,u)_A - (Tu,Tu) &\geq (\frac{2}{\omega} - 1)(Tu,Tu)_A \\ &\geq \frac{\beta(2-\omega)}{\omega}(u,u)_A, \quad \forall u \in V. \end{aligned}$$

Then (2.4) holds with $\alpha = \beta(2-\omega)/\omega$.

Now we assume (2.4). Since T is a linear operator, we have

$$(Tu,Tu)_A \leq \|T\|^2 (u,u)_A.$$

Furthermore,

$$\begin{aligned} &2(Tu,u)_A - (Tu,Tu)_A \geq \alpha (u,u)_A \geq \frac{\alpha}{\|T\|^2}(Tu,Tu)_A, \quad \forall u \in V, \\ &(Tu,Tu)_A \leq 2/\left(1 + \frac{\alpha}{\|T\|^2}\right)(Tu,u)_A, \quad \forall u \in V, \end{aligned}$$

which gives (A1) with $\omega = 2/\left(1 + \frac{\alpha}{\|T\|^2}\right)$.

By (2.4),

$$\begin{aligned} &(Tu,Tu)_A^{1/2}(u,u)_A^{1/2} \geq (Tu,u)_A \geq \frac{\alpha}{2}(u,u)_A, \quad \forall u \in V, \\ &(Tu,Tu)_A \geq \frac{\alpha^2}{4}(u,u)_A, \quad \forall u \in V, \end{aligned}$$

which gives (A2) with $\beta = \alpha^2/4$. □

We would like to remark that (A1) alone is not sufficient for convergence. For example, whenever $\mathcal{R}(T) = \mathcal{N}(A)$, (A1) is true; but there is no energy norm convergence in this case (see Example 1).

For an operator $B : V \mapsto V$, denote the Moore-Penrose generalized inverse [6] by $B^\dagger$. If B satisfies $\mathcal{N}(B) = \mathcal{N}(A), \quad \mathcal{R}(B) = \mathcal{R}(A)$, then $B^\dagger : V \mapsto V$ is a simple zero extension of $B^{-1} : \mathcal{R}(A) \mapsto \mathcal{R}(A)$, namely

$$\begin{aligned} B^\dagger c &= 0, \qquad \forall c \in \mathcal{N}(A), \\ B^\dagger v &= B^{-1} v, \quad \forall v \in \mathcal{R}(A). \end{aligned}$$

Now we try to estimate the convergence rate of the iteration (1.4).

Lemma 2.2. *Under the assumptions* (A1) *and* (A2)*, the energy norm convergence rate* $|I-T|_A$ *of the iterative method* (1.4) *can be estimated by*

$$|I-T|_A \leq \sqrt{1 - \frac{\beta(2-\omega)}{\omega}},$$

and $|I-T|_A^2 = 1 - 1/K$ *with*

$$K = \sup_{v\in\mathcal{R}(A),|v|_A=1} \inf_{c\in\mathcal{N}(A)} (v+c, v+c)_{(\mathcal{Q}\bar{R}\mathcal{Q})^\dagger}. \tag{2.5}$$

Proof. Note that

$$|(I-T)u|_A^2 = |u|_A^2 - \big((R+R^t-R^tAR)Au,u\big)_A,$$

and

$$\begin{aligned}\big((R+R^t-R^tAR)Au,u\big)_A &= 2(RAu,u)_A - (RAu,RAu)_A\\ \geq \left(\frac{2}{\omega}-1\right)(RAu,RAu)_A &\geq \frac{\beta(2-\omega)}{\omega}|u|_A^2,\end{aligned}$$

from which, we obtain

$$|I-T|_A^2 \leq 1 - \frac{\beta(2-\omega)}{\omega}, \tag{2.6}$$

$$\begin{aligned}|I-T|_A^2 &= \sup_{v\in\mathcal{R}(A)} \frac{(I-RA)v, \big((I-RA)v\big)_A}{(v,v)_A}\\ &= 1 - \inf_{v\in\mathcal{R}(A)} \frac{(\bar{R}Av,v)_A}{(v,v)_A}\\ &= 1 - \inf_{v\in\mathcal{R}(A)} \frac{(\mathcal{Q}\bar{R}\mathcal{Q}v,v)}{(A^\dagger v,v)}.\end{aligned}$$

Noting that $\mathcal{Q}\bar{R}\mathcal{Q}$ and $A^\dagger$ are symmetric and positive definite on $\mathcal{R}(A)$, then we get that $|I-T|_A^2 = 1-1/K$ with

$$\begin{aligned}K &= \sup_{v\in\mathcal{R}(A)} \frac{(A^\dagger v,v)}{(\mathcal{Q}\bar{R}\mathcal{Q}v,v)} = \sup_{v\in\mathcal{R}(A)} \frac{\big((\mathcal{Q}\bar{R}\mathcal{Q})^\dagger v,v\big)}{(v,v)_A}\\ &= \sup_{v\in\mathcal{R}(A),|v|_A=1} \inf_{c\in\mathcal{N}(A)} (v+c,v+c)_{(\mathcal{Q}\bar{R}\mathcal{Q})^\dagger}.\end{aligned}$$

□

Let us consider the Gauss-Seidel method and obtain the convergence rate identity by (2.5). We note that the Gauss-Seidel method can also be viewed as a simple successive subspace correction method, the result shall guide us how to study the general case of successive subspace corrections. The Gauss-Seidel method is given by

$$u^\ell = u^{\ell-1} + B\left(b - Au^{\ell-1}\right), \tag{2.7}$$

where $B = (D-L)^{-1}$ with $A = D-L-L^t$ where D and $-L$ are the diagonal matrix and strict lower triangular matrix of A respectively. We shall now see that

Lemma 2.3. *The energy norm convergence rate* $|I-BA|_A$ *for* (2.7) *is given by*

$$|I-BA|_A^2 = \rho^2 = 1 - \frac{1}{K}, \tag{2.8}$$

with

$$K = \sup_{v\in\mathcal{R}(A)} \inf_{c\in\mathcal{N}(A)} \frac{\big(S(v+c),(v+c)\big)}{(v,v)_A}, \tag{2.9}$$

where $S = (D-L)D^{-1}(D-L^t)$.

Proof. Using $\bar{B} = B + B^t - B^tAB$, a simple manipulation lead to

$$\bar{B} = S^{-1}.$$

By (2.5), we have

$$K = \sup_{v\in\mathcal{R}(A),|v|_A=1} \inf_{c\in\mathcal{N}(A)} (v+c, v+c)_{(\mathcal{Q}S^{-1}\mathcal{Q})^\dagger} = \sup_{v\in\mathcal{R}(A)} \frac{\big((\mathcal{Q}S^{-1}\mathcal{Q})^\dagger v, v\big)}{(v,v)_A}.$$

We now introduce $w = (\mathcal{Q}S^{-1}\mathcal{Q})^\dagger v$, then $w \in \mathcal{R}(A)$ and $w = S\big(v + c(v)\big)$ where $c(v) = S^{-1}w - v \in \mathcal{N}(A)$ is uniquely determined by $v \in \mathcal{R}(A)$. Then

$$K = \sup_{v\in\mathcal{R}(A)} \frac{\big(S(v+c(v)), v+c(v)\big)}{(v,v)_A}.$$

It is left to prove that

$$\big(S(v+c(v)), v+c(v)\big) = \inf_{c\in\mathcal{N}(A)} (S(v+c), v+c),$$

for any $v \in \mathcal{R}(A)$. Assume that

$$\xi = \arg\inf_{c\in\mathcal{N}(A)} \big(S(w+c), (w+c)\big). \tag{2.10}$$

Then ξ satisfies

$$\big(S(w+\xi), c\big) = 0, \quad \forall c \in \mathcal{N}(A). \tag{2.11}$$

S is symmetric and positive definite, then (2.11) admits a unique $\xi \in \mathcal{N}(A)$ for a given $v \in \mathcal{R}(A)$. Obviously, $c(v) \in \mathcal{A}$ also satisfies (2.11) which implies that $\xi = c(v)$. We then complete the proof. □

We now consider the iteration method based on matrix splitting. Assume that

$$A = M - N,$$

the corresponding iteration method can be defined in the form of (1.4) with $R = M^\dagger$, where $M^\dagger$ is the Moore-Penrose generalized inverse [14] of M, namely,

$$u^\ell = u^{\ell-1} + M^\dagger(b - Au^{\ell-1}).$$

One main convergence result for (1.4) by Keller [7] is summarized as follows. The iterative scheme (1.4) is energy norm convergent if the splitting $A = M - N$ is P-regular, in the sense that M is invertible, and $M^T + M - A$ is positive definite.

For some special singular systems, considered by e.g., Marek and Szyld [12], the convergence has been studied via the theory of nonnegative matrices, for which the *weak-regularity* condition, proposed in Ortega and Rheinboldt, [14] is often applied as a sufficient condition. A version of the weak regularity condition, (see e.g., Berman and Plemmons [1]) is as follows: A splitting $A = M - N$ is called weakly-regular if M is invertible, and in addition, both M^{-1} and $M^{-1}N$ are nonnegative matrices. An example given in [8] have showed that neither P-regularity, nor weak regularity of the matrix splitting is necessary for the convergence.

For the energy norm convergence of (1.4) with $R = M^\dagger$, the result in Theorem 2.1 is new. Example 2 shows that, in the framework here, R may be singular and the assumption $\mathcal{N}(M^t) \subset \mathcal{N}(A)$ imposed in [8] is not necessary.

Example 1: (A2) is necessary for the convergence

Let

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}. \tag{2.12}$$

Introduce an iteration scheme (1.4), where R is given by

$$R = \begin{pmatrix} 1 & 2/3 \\ -1/3 & 0 \end{pmatrix}.$$

Simple algebraic computation yields

$$\mathcal{R}(T) = \mathcal{N}(A).$$

It is then straightforward to see that

$$|(I-T)u|_A^2 = |u|_A^2, \quad \forall u \in V.$$

This means that the iteration is not convergent.

Example 2: $R = M^\dagger$ may be singular and $\mathcal{N}(M^t) \subset \mathcal{N}(A)$ in [8] is unnecessary

Assume

$$A = \begin{pmatrix} 1 & -\frac{2}{3}\sqrt{2} & 0 \\ -\frac{2}{3}\sqrt{2} & 2 & -\frac{1}{3}\sqrt{10} \\ 0 & -\frac{1}{3}\sqrt{10} & 1 \end{pmatrix}.$$

We have the following matrix splitting,

$$A = M - N = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \frac{2}{3}\sqrt{2}-1 & 0 \\ \frac{2}{3}\sqrt{2}-1 & 0 & \frac{1}{3}\sqrt{10}-1 \\ 0 & \frac{1}{3}\sqrt{10}-1 & 0 \end{pmatrix}.$$

Consider the iteration scheme (1.4), where $R = M^\dagger$ is the Moore-Penrose generalized inverse of M. Namely, R is given by

$$R = \begin{pmatrix} 5/9 & -1/9 & -4/9 \\ -1/9 & 9/2 & -1/9 \\ -4/9 & -1/9 & 5/9 \end{pmatrix}.$$

Then

$$I - T = \begin{pmatrix} \frac{4}{9} - \frac{2}{27}\sqrt{2} & \frac{10}{27}\sqrt{2} + \frac{2}{9} - \frac{4}{27}\sqrt{10} & \frac{4}{9} - \frac{1}{27}\sqrt{10} \\ \frac{1}{9} + \frac{4}{27}\sqrt{2} & \frac{5}{9} - \frac{2}{27}\sqrt{2} - \frac{1}{27}\sqrt{10} & \frac{1}{9} + \frac{2}{27}\sqrt{10} \\ \frac{4}{9} - \frac{2}{27}\sqrt{2} & -\frac{8}{27}\sqrt{2} + \frac{2}{9} + \frac{5}{27}\sqrt{10} & \frac{4}{9} - \frac{1}{27}\sqrt{10} \end{pmatrix}.$$

By direct computation, it can be verified that

$$|I - T|_A < 1,$$

which implies that the iteration is convergent. From

$$\mathcal{N}(M^t) = \operatorname{span}\{(1,1,1)^t\}, \quad \mathcal{N}(A) = \left\{ \left(1, \frac{3}{4}\sqrt{2}, \frac{1}{2}\sqrt{5}\right)^t \right\},$$

we get that $\mathcal{N}(M^t) \not\subset \mathcal{N}(A)$.

3. Convergence Rate Estimate for Subspace Correction Method

In this section, we first introduce the (successive) subspace correction methods for the semi-definite system (1.1). Then, we present the convergence rate identity and estimate for general subspace correction methods under the minimal assumptions on subspace solvers.

First of all, we assume that V is decomposed into a finite number of subspaces $\{V_i\}_{i=1}^J$ such that

$$V = \sum_{i=1}^{J} V_i. \tag{3.1}$$

Following [20], the general method of successive subspace corrections is given as follows.

Algorithm 3.1. *If u^{l-1} is known, the next iterate u^l is decided by the following procedure. Define $u_0^{l-1} = u^{l-1}$*

$$u_i^{l-1} = u_{i-1}^{l-1} + R_i Q_i \big(b - A u_{i-1}^{l-1}\big), \quad i = 1, \cdots, J. \tag{3.2}$$

Set $u^l = u_J^{l-1}$
where R_i is an operator from V_i to V_i, and the orthogonal projection with respect to $(\cdot,\cdot)$, $Q_i : V \mapsto V_i$ is defined as

$$(Q_i v, w_i) = (v, w_i), \quad \forall v \in V, w \in V_i.$$

3.1. Notation and assumptions

In what follows, we will use the following notation,

$$\mathcal{N} = \mathcal{N}(A), \quad \mathcal{N}_i = \mathcal{N} \cap V_i,$$

$\mathcal{N}_i^{\perp} \subset V_i$ are the orthogonal complement of $\mathcal{N}_i$ with respect to the inner product $(\cdot,\cdot)$. Define $A_i : V_i \mapsto V_i$ such that

$$(v_i, w_i)_{A_i} = (v_i, w_i)_A, \quad \forall v_i, w_i \in V_i,$$

the restriction of A on V_i. Then R_i is in some sense the approximation of A_i^{-1} if A_i is invertible,

$$\begin{aligned} &A_i = Q_i A \quad \text{on } V_i, \\ &\mathcal{N}(A_i) = \mathcal{N}_i, \quad \mathcal{R}(A_i) = \mathcal{N}_i^{\perp}. \end{aligned}$$

Denote

$$T_i = R_i Q_i A,$$

which is also called the (local) subspace solver on V_i. For energy norm convergence of Algorithm 3.1, some additional assumptions are necessary. These assumptions are motivated by the simple observation that the local energy norm convergence is guaranteed, namely,

$$|I - T_i|_{A_i} = \sup_{v_i \in V_i} \frac{|(I - T_i)v_i|_{A_i}}{|v_i|_{A_i}} \le \delta_i < 1, \tag{3.3}$$

where $\delta_i \in [0, 1)$.

The necessary and sufficient conditions for the local energy norm convergence would be the local version assumption of (A1) and (A2). To state these assumptions, we denote $\mathcal{Q}_i : V_i \mapsto \mathcal{R}(A_i) = \mathcal{N}_i^{\perp}$ be the orthogonal projection from V_i to $\mathcal{N}_i^{\perp}$ with respect to $(\cdot,\cdot)$, and

$$\bar{R}_i = R_i + R_i^t - R_i^t Q_i A R_i = R_i + R_i^t - R_i^t A_i R_i. \tag{3.4}$$

The assumption (3.3) is equivalent to

$$\mathcal{Q}_i \bar{R}_i \mathcal{Q}_i \ \text{ is symmetric and positive definite on } \ \mathcal{N}_i^{\perp},$$

or there exits a positive constant α_i such that

$$2(T_i v_i, v_i)_A - (T_i v_i, T_i v_i)_A \geq \alpha_i (v_i, v_i)_A.$$

Furthermore, (3.3) is equivalent to the following two assumptions:

(H1) $\exists \omega_i \in (0,2)$ such that $(T_i v_i, T_i v_i)_A \leq \omega_i (T_i v_i, v_i)_A, \quad \forall v_i \in V_i,$

(H2) $\exists \beta_i > 0$ such that $(T_i v_i, T_i v_i)_A \geq \beta_i (v_i, v_i)_A, \quad \forall v_i \in V_i.$

Lemma 3.1. *Assume that* (H1) *and* (H2) *hold. Then for all* $1 \leq i \leq J$, *there exists* $\delta_i \in [0,1)$ *such that*

$$|(I - T_i) v_i|_A \leq \delta_i |v_i|_A, \quad \forall v_i \in V_i.$$

where $\delta_i^2 = 1 - \beta_i(2 - \omega_i)/\omega_i$.

Proof. The proof is the same as that of Lemma 2.2. □

We remark that when A is symmetric and positive definite, the assumption (H1) and (H2) (For this case, (H2) is equivalent to $T_i : V_i \mapsto V_i$ is isomorphic) are also sufficient conditions for the energy norm convergence of the subspace correction method [20,21]. The similar assumption to (H2) has been added to guarantee the local parameter independent convergence in [10].

3.2. Convergence rate identity and estimate

Let $u \in V$ be a solution to (1.1) and $\{u^l : l = 0, \cdots\}$ be the iterates generated by Algorithm 3.1 respectively. The following relation is standard,

$$u - u^l = E_J\big(u - u^{l-1}\big) = \cdots = E_J^l\big(u - u^0\big),$$

where

$$E_J = (I - T_J) \cdots (I - T_1).$$

The convergence rate can be estimated by the energy norm of E_J,

$$|E_J|_A = \sup_{v \in \mathcal{R}(A)} \frac{|E_J v|_A}{|v|_A}. \tag{3.5}$$

We first present a new version of the convergence rate identity when A is symmetric and positive definite [8, 21].

Lemma 3.2. *If A is symmetric and positive definite, under the assumptions* (H1) *and* (H2)*, the convergence rate of Algorithm* 3.1 *is given by $\|E_J\|_A$ where*

$$\|E_J\|_A^2 = 1 - \frac{1}{K},$$

with

$$K = \sup_{\|v\|_A=1} \sum \inf_{v_k=v} \sum_{k=1}^{J} \left(v_k + T_k^* w_k, v_k + T_k^* w_k\right)_{\bar{R}_k^{-1}}, \tag{3.6}$$

where $w_k = \sum_{i=k+1}^{J} v_i$ and $T_k^* = R_k^t Q_k A$.

Now we try to extend Lemma 3.2 to semidefinite case. First we consider the case that for all $1 \le i \le J$,

$$\mathcal{R}(R_i) = \mathcal{N}_i^{\perp} \quad \text{and} \quad \mathcal{N}(R_i) = \mathcal{N}_i. \tag{3.7}$$

Because R_i may be singular, we introduce the Moore-Penrose generalized inverse [6] for this kind of singular operators. For an operator $B_i : V_i \mapsto V_i$ satisfying $\mathcal{N}(B_i) = \mathcal{N}_i, \quad \mathcal{R}(B_i) = \mathcal{N}_i^{\perp}$, the Moore-Penrose generalized inverse, $B_i^{\dagger} : V_i \mapsto V_i$ is a simple zero extension of $B_i^{-1} : \mathcal{N}_i^{\perp} \mapsto \mathcal{N}_i^{\perp}$, namely

$$\begin{aligned} B_i^{\dagger} c_i &= 0, && \forall c_i \in \mathcal{N}_i, \\ B_i^{\dagger} v_i &= B_i^{-1} v_i, && \forall v_i \in \mathcal{N}_i^{\perp}. \end{aligned}$$

Lemma 3.3. *Under the assumptions,* (3.7)*,* (H1) *and* (H2)*, the convergence rate of Algorithm* 3.1 *is given by $|E_J|_A$ where*

$$|E_J|_A^2 = 1 - \frac{1}{K},$$

with $K = \sup_{|v|_A=1, v\in\mathcal{R}(A)} K(v)$ and

$$K(v) = \inf_{c\in\mathcal{N}} \sum \inf_{v_k=v+c} \sum_{k=1}^{J} \left(v_k + T_k^* w_k, v_k + T_k^* w_k\right)_{\bar{R}_k^{\dagger}},$$

where $w_k = \sum_{i=k+1}^{J} v_i$ and $T_k^* = R_k^t Q_k A$.

Proof. The proof is given in Section 5. □

For the general case, we have the following theorem, which is the main result of this section.

Theorem 3.1. *Under the assumptions* (H1) *and* (H2)*, then the convergence rate of Algorithm* 3.1 *is given by $|E_J|_A$ where*

$$|E_J|_A^2 = 1 - \frac{1}{K},$$

with $K = \sup_{|v|_A=1, v\in\mathcal{R}(A)} K(v)$ and

$$K(v) = \inf_{c\in\mathcal{N}} \sum \inf_{v_k=v+c} \sum_{k=1}^{J} \left(v_k + T_k^* w_k, v_k + T_k^* w_k\right)_{(\mathcal{Q}_k \bar{R}_k \mathcal{Q}_k)^{\dagger}}, \tag{3.8}$$

where $w_k = \sum_{i=k+1}^{J} v_i$ and $T_k^* = R_k^t Q_k A$.

Proof. In Lemma 3.3, we assume that $\mathcal{N}(R_i) = \mathcal{N}_i$ and $\mathcal{R}(R_i) = \mathcal{N}_i^{\perp}$. For the general case, we define the auxiliary operators $\underline{R}_i$ as

$$\underline{R}_i = \mathcal{Q}_i R_i \mathcal{Q}_i.$$

By the definition of $\mathcal{Q}_i$, we get that

$$\mathcal{N}(\underline{R}_i) = \mathcal{N}_i \quad \text{and} \quad \mathcal{R}(\underline{R}_i) = \mathcal{N}_i^{\perp}.$$

Similarly, we can define the operators $\underline{T}_i$, $\underline{E}_J$, $\bar{R}_i$ and $\bar{T}_i$ corresponding to $\underline{R}_i$.

Since $\mathcal{R}(Q_i A) = \mathcal{N}_i^{\perp}$, we have

$$\underline{T}_i v - T_i v = (\mathcal{Q}_i - I) R_i Q_i A v \in \mathcal{N}_i, \quad \forall v \in V,$$

and furthermore

$$\sup_{v \in \mathcal{R}(A)} \frac{|E_J v|_A}{(v,v)_A} = \sup_{v \in \mathcal{R}(A)} \frac{|\underline{E}_J v|_A}{(v,v)_A}.$$

Applying Lemma 3.3 for $\underline{R}_i$ and $\underline{E}_J$, we finally get Theorem 3.1. □

By the definition of $\mathcal{Q}_k$, $\mathcal{Q}_k \bar{R}_k \mathcal{Q}_k : \mathcal{N}_k^{\perp} \mapsto \mathcal{N}_k^{\perp}$ is symmetric and positive definite under the assumptions (H1) and (H2), the generalized inverse of $\mathcal{Q}_k \bar{R}_k \mathcal{Q}_k$ is relatively easy to understand.

Now we define $P_i : V \mapsto \mathcal{N}_i^{\perp}$, such that

$$\big(P_i v, w_i\big)_A = (v, w_i)_A, \quad \forall v \in V, v_i \in \mathcal{N}_i^{\perp}.$$

The assumptions (H1) and (H2) are automatic for $T_i = P_i$ (corresponding to $R_i = A_i^{\dagger}$) with $\omega_i = \beta_i = 1$. If $T_i = P_i$ for all i , which is an important application, we have the following:

Corollary 3.1. *Assume $T_i = P_i$ for $i = 1, \cdots, J$. We can obtain that*

$$K = \sup_{|v|_A = 1, v \in \mathcal{R}(A)} \inf_{c \in \mathcal{N}} \sum \inf_{v_k = v + c} \sum_{k=1}^{J} \left(P_k \sum_{i=k}^{J} v_i, \sum_{i=k}^{J} v_i \right)_A.$$

By the assumptions (H1) and (H2), we try to estimate the related terms in the convergence rate identity (3.8).

Lemma 3.4. *Assume that* (H1) *and* (H2) *hold, then*

$$(v_i, v_i)_{(\mathcal{Q}_k \bar{R}_k \mathcal{Q}_k)^{\dagger}} \le \frac{\omega_i}{\beta_i(2-\omega_i)} (v_i, v_i)_A, \quad \forall v_i \in V_i, \tag{3.9}$$

$$\big(T_i^* v_i, T_i^* v_i\big)_A \le \omega_i^2 (v_i, v_i)_A, \quad \forall v_i \in V_i. \tag{3.10}$$

Proof. First we see that

$$\big(A_i v_i, A_i v_i\big)_{\bar{R}_i} = 2\big(T_i v_i, v_i\big) - \big(T_i v_i, T_i v_i\big) \ge \frac{\beta_i(2-\omega_i)}{\omega_i} (v_i, v_i)_A.$$

Since

$$(v_i, v_i)_A = (v_i, v_i)_{A_i} = \big(A_i v_i, A_i v_i\big)_{A_i^{\dagger}},$$

we get that

$$\big(A_i v_i, A_i v_i\big)_{\mathcal{Q}_i \bar{R}_i \mathcal{Q}_i} \ge \frac{\beta_i(2-\omega_i)}{\omega_i} \big(A_i v_i, A_i v_i\big)_{A_i^{\dagger}}.$$

$\mathcal{Q}_i\bar{R}_i\mathcal{Q}_i$ and $A_i^\dagger$ are symmetric and positive definite on $\mathcal{R}(A_i)=\mathcal{N}_i^\perp$. By the above inequality, we obtain

$$(v_i,v_i)_{(Q_i\bar{R}_i\mathcal{Q}_i)^\dagger}\le\frac{\omega_i}{\beta_i(2-\omega_i)}(v_i,v_i)_{A_i}=\frac{\omega_i}{\beta_i(2-\omega_i)}(v_i,v_i)_A,\quad\forall v_i\in\mathcal{N}_i^\perp.$$

Noting that $(\mathcal{Q}_i\bar{R}_i\mathcal{Q}_i)^\dagger c_i=0$, $Ac_i=0$ for any $c_i\in\mathcal{N}_i$, we get the first inequality (3.9).

By the definition of T_i^*, it is easy to see that

$$\big(T_iv,w\big)_A=\big(v,T_i^*w\big)_A\quad\forall v,w\in V.$$

By (H1) we get that for any $v_i\in\mathcal{N}_i^\perp$,

$$\big(T_iv_i,T_iv_i\big)_A\le\omega_i^2(v_i,v_i)_A,$$

then

$$(T_i^*v_i,T_i^*v_i)_A\le\omega_i^2(v_i,v_i)_A,\quad\forall v_i\in\mathcal{N}_i^\perp.$$

Since $T_i^*c_i=0$ and $Ac_i=0$ for any $c_i\in\mathcal{N}_i$, the above inequality is also valid for any $v_i\in V_i$, which gives (3.10). □

By Lemma 3.4 and Theorem 3.1, we get the following theorem on the convergence rate estimate.

Theorem 3.2. *Assume that* (H1) *and* (H2) *hold, we have that*

$$K\le\sup_{|v|_A=1,v\in\mathcal{R}(A)}\inf_{c\in\mathcal{N}}\inf_{\sum v_k=v+c}\sum_{k=1}^J\frac{2\omega_k}{\beta_k(2-\omega_k)}\left((v_k,v_k)_A+\omega_k^2\Big(P_k\sum_{i=k+1}^J v_i,\sum_{i=k+1}^J v_i\Big)_A\right).$$

Proof. Set $w_k=\sum_{i=k+1}^J v_i$, we have by (3.9) and (3.10),

$$\begin{aligned}
&\big(v_k+T_k^*w_k,v_k+T_k^*w_k\big)_{(\mathcal{Q}_k\bar{R}_k\mathcal{Q}_k)^\dagger}\\
\le&\frac{\omega_k}{\beta_k(2-\omega_k)}\big(v_k+T_k^*w_k,v_k+T_k^*w_k\big)_A\\
\le&\frac{2\omega_k}{\beta_k(2-\omega_k)}\Big((v_k,v_k)_A+(T_k^*w_k,T_k^*w_k)_A\Big)\\
\le&\frac{2\omega_k}{\beta_k(2-\omega_k)}\Big((v_k,v_k)_A+(T_k^*P_kw_k,T_k^*P_kw_k)_A\Big)\\
\le&\frac{2\omega_k}{\beta_k(2-\omega_k)}\Big((v_k,v_k)_A+\omega_k^2(P_kw_k,P_kw_k)_A\Big)\\
=&\frac{2\omega_k}{\beta_k(2-\omega_k)}\Big((v_k,v_k)_A+\omega_k^2(P_kw_k,w_k)_A\Big).
\end{aligned}$$

This completes the proof of the theorem. □

By Theorem 3.2, under the assumptions (H1) and (H2), the convergence rate estimate for subspace correction methods with inexact subspace solvers is shown to be similar from the counterpart with exact subspace solvers.

4. A Practical Illustration

In this section, we will provide the convergence analysis of the multigrid method for a semidefinite system from a simple linearized model of lithium-ion battery. We will construct a multigrid method for the singular system, and show that the multigrid method converges uniformly with respect to the mesh size h and some relevant parameters of the problem by the theories developed in the previous section.

The example is on a linearized simple model of lithium-ion battery [18, 19]. The system of equations can be prescribed on a bounded domain $\Omega \subset \mathbb{R}^d$ $(1 \le d \le 3)$ such that

$$\bar{\Omega} = \bar{\Omega}_a \cup \bar{\Omega}_s \cup \bar{\Omega}_c \subset \mathbb{R}^d,$$

where Ω_a, Ω_s and Ω_c are rectangular subdomains of Ω that correspond to the negative electrode, the separator and the positive electrode of the Lithium ion battery respectively, see Fig. 4.1.

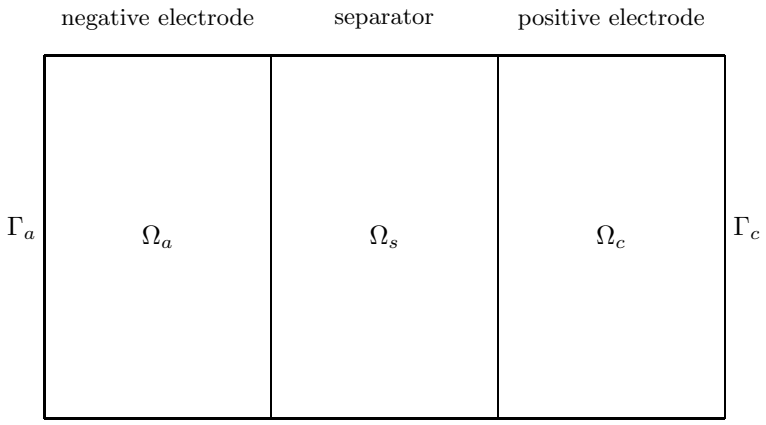

Fig. 4.1 The domain Ω.

Set $\Omega' = \Omega_a \cup \Omega_c$. The system of partial differential equations that we are interested takes the following form:

$$-\nabla \cdot (\nabla \Phi_e) = S, \qquad x \in \Omega, \tag{4.1}$$

$$-\nabla \cdot (\kappa \nabla \Phi_s) = -S, \quad x \in \Omega', \tag{4.2}$$

where Φ_e and Φ_s are electric potentials (in the electrolyte phase and solid phase respectively), $\kappa \gg 1$ is positive constants. The transfer current density is given by

$$S = \begin{cases} S_0 + \Phi_s - \Phi_e, & x \in \Omega', \\ 0, & \text{otherwise.} \end{cases}$$

where S_0 is piecewise constant. Proper Neumann boundary conditions are imposed for Φ_e, Φ_s, i.e.,

$$-\kappa \frac{\partial \Phi_s}{\partial n} = I, \quad \text{for } x \in \partial\Omega',$$

where I is a given function and

$$\frac{\partial \Phi_e}{\partial n} = 0, \quad \text{for } x \in \partial\Omega.$$

We assume that

$$\int_{\partial\Omega'} I dA = 0,$$

for the solvability of the system of the partial differential equations (4.1) and (4.2).

Denote $\Phi = (\Phi_e, \Phi_s)$, $\Psi = (\Psi_e, \Psi_s)$. Then we define a bilinear form $a(\Phi, \Psi)$:

$$a(\Phi, \Psi) = \int_{\Omega} \nabla\Phi_e \nabla\Psi_e dx + \int_{\Omega'} \kappa\nabla\Phi_s\nabla\Psi_s + (\Phi_s - \Phi_e)(\Psi_s - \Psi_e)dx \tag{4.3}$$

where $\Phi, \Psi \in H^1(\Omega) \times H^1(\Omega')$. Then the weak formula of (4.1) and (4.2) is given by: Find $\Phi \in H^1(\Omega) \times H^1(\Omega')$ such that

$$a(\Phi, \Psi) = (F, \Psi), \quad \forall\Psi \in H^1(\Omega) \times H^1(\Omega'), \tag{4.4}$$

where

$$(F, \Psi) = \int_{\Omega'} S_0(\Psi_e - \Psi_s)dx + \int_{\partial\Omega'} I\Phi_s dS.$$

The solution to (4.4) is not unique. More precisely, if (Φ_e, Φ_s) is a solution to (4.4), so is $(\Phi_e + C, \Phi_s + C)$ where C is a constant function. The finite element discretization of (4.4) will lead to a singular system of equations.

Now we consider the finite element approximation and multigrid setting. Throughout this section, we assume that Ω is triangulated with a nested sequence of quasi-uniform triangles $\mathcal{T}_k = \{\tau_k^i\}$ of size h_k, where the quasi-uniformity constants are independent of k and $h_k \sim \gamma^k$ with $\gamma \in (0, 1)$ for $k = 1, \cdots, J$. We also assume that for any $\tau_k^i \in \mathcal{T}_k$, $\tau_k^i \cap \bar{\Omega}_a(\ \bar{\Omega}_s,\ \bar{\Omega}_c)$ equals to τ_k^i or $\emptyset$. Associated with each $\mathcal{T}_k$, we have the finite element space of continuous piecewise linear functions $V_k \subset H^1(\Omega) \times H^1(\Omega')$. In this setting, it is clear that

$$V_1 \subset \cdots \subset\ V_k \subset \cdots \subset V_J.$$

The finite element approximation is given as follows: Find $\Phi^h \in V_J = V$, such that

$$a\big(\Phi^h, \Psi^h\big) = \big(F, \Psi^h\big), \quad \forall\Psi^h \in V_J = V. \tag{4.5}$$

Define a function $\pi(x)$ $(x \in \bar{\Omega})$ satisfying that $\pi(x) = 1$ for $x \in \bar{\Omega}_a \cup \bar{\Omega}_c$ and $\pi(x) = 0$ for $x \in \Omega_s$. Clearly we have the following decomposition,

$$V_k = \sum_{i=1}^{n_k} V_k^i,$$

where $V_k^i = \text{span}\{(\phi_k^i, 0), (0, \pi\phi_k^i)\}$ and each ϕ_k^i is the usual nodal basis function that is one at the node x_k^i and zero at the other nodes and n_k the number of grid nodes, $\{x_k^l\}_{l=1}^{n_k}$ of $\mathcal{T}_k$.

Now we try to construct a multigrid method which converges independent of h and the constants κ. For this purpose, we need to define an additional space

$$V_0^1 = \text{span}\big\{(\mathbf{1}, 0), (0, \mathbf{1}_a), (0, \mathbf{1}_c)\big\},$$

where $\mathbf{1}$ denotes the function that is 1 on Ω, $\mathbf{1}_a$ and $\mathbf{1}_c$ denote the functions that are 1 on $\bar{\Omega}_a$ and $\bar{\Omega}_c$ respectively, and 0 otherwise. Set $n_0 = 1$, we have the following space decomposition,

$$V = \sum_{k=0}^{J} \sum_{i=1}^{n_k} V_k^i. \tag{4.6}$$

The additional subspace V_0^1 is to guarantee the uniform convergence with respect to κ (see (4.11)).

Under the settings outlined above, the abstract convergence theory shall start with the following observation that the error transfer operator, E of the subspace correction method with exact solver in each subspace V_k^l can be written as follows

$$E = \prod_{k=0}^{J} \prod_{l=1}^{n_k} (I - P_k^l), \tag{4.7}$$

where P_k^l is the exact solver on V_k^l, (see also [20]).

Denote $|\Phi|_a = a(\Phi, \Phi)^{1/2}$ and $\mathcal{N} = \text{span}(\mathbf{1}, \pi\mathbf{1})$. By a direct application of Corollary 3.1, we obtain the following relation:

$$|E|_a^2 = 1 - K^{-1}, \tag{4.8}$$

where

$$K = \sup_{\Phi \in \mathcal{N}^\perp} \inf_{c \in \mathcal{N}} \inf_{\sum_{k=0}^{J} \sum_{i=1}^{n_k} \Phi_k^i = \Phi + c} \frac{\sum_{k=0}^{J} \sum_{i=1}^{n_k} \left| P_k^i \left(\sum_{(l,j) \geq (k,i)} \Phi_l^j \right) \right|_a^2}{|\Phi|_a}. \tag{4.9}$$

Theorem 4.1. *The norm of the error transfer operator, given by* (4.9) *can be bounded as follows:*

$$|E|_a^2 \leq \delta < 1, \tag{4.10}$$

with δ independent of the mesh size h, the number of levels J and the constants κ.

Proof. For any $\Phi \in \mathcal{N}^\perp$, we have the decomposition $\Phi = \Psi + c$ where $\Psi = (\Psi_e, \Psi_s) \in (V_0^1)^\perp$ and $c \in V_0^1 \cap \mathcal{N}^\perp$. Noting that $\int_\Omega \Psi_e dx = 0$, $\int_{\Omega_a} \Psi_s dx = 0$ and $\int_{\Omega_c} \Psi_s dx = 0$; we get

$$|\Phi|_a^2 \geq |\Psi_e|_{1,\Omega}^2 + \kappa |\Psi_s|_{1,\Omega'}^2 \gtrsim \|\Psi_e\|_{1,\Omega}^2 + \kappa \|\Psi_s\|_{1,\Omega'}^2 \gtrsim |\Psi|_a^2,$$

and

$$|\Phi|_a^2 \gtrsim |\Phi|_a^2 + |\Psi|_a^2 \gtrsim |\Phi - \Psi|_a^2 = |c|_a^2.$$

Then

$$\begin{aligned}
K &\leq \sup_{\Phi \in \mathcal{N}^\perp} \inf_{\sum_{k=0}^{J} \sum_{i=1}^{n_k} \Phi_k^i = \Phi} \frac{\sum_{k=0}^{J} \sum_{i=1}^{n_k} \left| P_k^i \left(\sum_{(l,j) \geq (k,i)} \Phi_l^j \right) \right|_a^2}{|\Phi|_a} \\
&\lesssim \sup_{\Psi \in (V_0^1)^\perp} \inf_{\sum_{k=0}^{J} \sum_{i=1}^{n_k} \Psi_k^i = \Psi} \frac{\sum_{k=0}^{J} \sum_{i=1}^{n_k} \left| P_k^i \left(\sum_{(l,j) \geq (k,i)} \Psi_l^j \right) \right|_a^2}{|\Psi|_a^2} \\
&\quad + \sup_{c \in V_0^1 \cap \mathcal{N}^\perp} \inf_{\sum_{k=0}^{J} \sum_{i=1}^{n_k} c_k^i = c} \frac{\sum_{k=0}^{J} \sum_{i=1}^{n_k} \left| P_k^i \left(\sum_{(l,j) \geq (k,i)} c_l^j \right) \right|_a^2}{|c|_a^2} \\
&=: \mathbf{I} + \mathbf{II}.
\end{aligned}$$

The estimate on **II** is simply given by setting $c_0^1 = c$, and other $c_k^i = 0$,

$$\mathbf{II} \leq 1.$$

For $\Psi = (\Psi_e, \Psi_s) \in (V_0^1)^\perp$, we have

$$|\Psi|_a^2 \gtrsim \|\Psi_e\|_{1,\Omega}^2 + \kappa \|\Psi_s\|_{1,\Omega'}^2 + \|\Psi_e - \Psi_s\|_{0,\Omega'}^2. \tag{4.11}$$

Then, we obtain that by setting $\Psi_k^0 = 0$,

$$\begin{aligned}\mathbf{I} &\lesssim \sup_{\Psi\in(V_0^1)^\perp}\ \inf_{\sum_{k=1}^J\sum_{i=1}^{n_k}\Psi_k^i=\Psi} \frac{\sum_{k=1}^J\sum_{i=1}^{n_k}\left|P_k^i\left(\sum_{(l,j)\geq(k,i)}\Psi_l^j\right)\right|_a^2}{\|\Psi_e\|_{1,\Omega}^2+\kappa\|\Phi_s\|_{1,\Omega'}^2+\|\Psi_e-\Psi_s\|_{0,\Omega'}^2}\\ &\leq \sup_{\Psi\in V}\ \inf_{\sum_{k=1}^J\sum_{i=1}^{n_k}\Psi_k^i=\Psi} \frac{\sum_{k=1}^J\sum_{i=1}^{n_k}\left|P_k^i\left(\sum_{(l,j)\geq(k,i)}\Psi_l^j\right)\right|_a^2}{\|\Psi_e\|_{1,\Omega}^2+\kappa\|\Phi_s\|_{1,\Omega'}^2}\\ &\lesssim 1.\end{aligned}$$

The last inequality above is the standard estimate for multigrid method, we refer [8,21] for the detail of the estimate. □

5. Proof of Lemma 3.3

In this section, we will show Lemma 3.3. First we introduce the symmetrization $\bar{T}_i : V \mapsto V_i$ of T_i as follow,

$$\bar{T}_i = T_i + T_i^* - T_i^* T_i. \tag{5.1}$$

Then

$$\bar{T}_i = \bar{R}_i Q_i A.$$

Because of the assumption (3.7), we have $\mathcal{N}(R_i^t) = \mathcal{N}_i$ and $\mathcal{R}(R_i^t) = \mathcal{N}_i^\perp$; and for $\bar{T}_i : V_i \mapsto V_i$,

$$\mathcal{R}(\bar{T}_i) = \mathcal{N}_i^\perp \quad \text{and} \quad \mathcal{N}(\bar{T}_i) = \mathcal{N}_i.$$

One key idea of the convergence analysis of Algorithm 3.1 is to use appropriate restrictions of subspaces V_i's and the subspace solvers T_i's onto $\mathcal{R}(A)$ and then apply the theory for the positive definite case. In order to do that, we will introduce an additional projection. The orthogonal projection with respect to $(\cdot,\cdot)$, $\mathcal{P} : V \mapsto \mathcal{R}(A)$ is defined as

$$(\mathcal{P}v, w) = (v, w), \quad \forall v \in V, w \in \mathcal{R}(A).$$

In what follows, we denote the space $\mathcal{P}V_i$ by $\widetilde{V}_i$ for simplicity.

Lemma 5.1. *Suppose that* (H1) *and* (H2) *hold. Then the followings hold*

(A0) $\mathcal{R}(A) = \sum_{i=1}^J \widetilde{V}_i$,

(A1) $\exists\omega \in (0,2)$ *such that* $(\mathcal{P}T_i v_i, \mathcal{P}T_i v_i)_A \leq \omega a(\mathcal{P}T_i v_i, v_i)_A, \quad \forall v_i \in \widetilde{V}_i$,

(A2) $\mathcal{P}T_i : \widetilde{V}_i \mapsto \widetilde{V}_i$ *is an isomorphism.*

Proof. Noting that V is finite dimensional and $T_i\mathcal{P}v_i = T_i v_i$ for any $v_i \in V_i$, **(A0)**, **(A1)** and **(A2)** are obtained directly by (3.1), (H1) and (H2), respectively. □

Now we consider $|E_J v|_A$ which is defined by (3.5), we have

$$\begin{aligned}|E_J v|_A^2 &= |\mathcal{P}E_J v|_A^2\\ &= |(I-\mathcal{P}T_J)(I-T_{J-1})\cdots(I-T_1)v|_A^2\\ &= |(I-\mathcal{P}T_J)(I-\mathcal{P}T_{J-1})\cdots(I-T_1)v|_A^2\\ &= \cdots\\ &= |(I-\mathcal{P}T_J)(I-\mathcal{P}T_{J-1})\cdots(I-\mathcal{P}T_1)v|_A^2\end{aligned}$$

Hence

$$|E_J|_A = |(I - \mathcal{P}T_J)\cdots(I - \mathcal{P}T_1)|_A. \tag{5.2}$$

From the relation (5.2), the fact that $(\cdot,\cdot)_A : \mathcal{R}(A) \times \mathcal{R}(A) \mapsto \mathbb{R}$ is positive definite and Lemma 5.1, we have the following important auxiliary result by the identity in [21] and [8].

Lemma 5.2. *Suppose that* (H1) *and* (H2) *hold. Then*

$$|E_J|_A^2 = (I - \mathcal{P}T_J)\cdots(I - \mathcal{P}T_1)|_A^2 = 1 - \frac{1}{K},$$

where $K = \sup_{|v|_A=1, v\in\mathcal{R}(A)} K(v)$ *and with* $\tilde{w}_k := \sum_{i=k+1}^{J} \tilde{v}_i$ *and* $\tilde{v}_i \in \widetilde{V}_i$,

$$K(v) = \inf_{\sum \tilde{v}_k = v} \sum_{k=1}^{J} \left((\mathcal{P}\bar{T}_k)^{-1}(\tilde{v}_k + \mathcal{P}T_k^*\tilde{w}_k), (\tilde{v}_k + \mathcal{P}T_k^*\tilde{w}_k)\right)_A.$$

Now we try to rewrite the expression of K in Lemma 5.2 in terms of real subspace operators R_i, T_i^* and $\bar{T}_i$. We first prove a lemma relating $\bar{T}_i^\dagger$.

Lemma 5.3. *Suppose that* (3.7), (H1) *and* (H2) *hold. Then for each* $1 \le i \le J$, *we have*

$$\mathcal{P}\bar{T}_i^\dagger \mathcal{P}_i^\times = (\mathcal{P}\bar{T}_i)^{-1} \quad \text{on} \quad \widetilde{V}_i. \tag{5.3}$$

where $\mathcal{P}_i^\times : \widetilde{V}_i \mapsto \mathcal{N}_i^\perp$, *such that* $\mathcal{P}\mathcal{P}_i^\times = I$ *on* $\widetilde{V}_i$.

Proof. For any given $v_i \in V_i$, assume that $v_i = w_i + c_i$ where $w_i \in \mathcal{N}_i^\perp$ and $c_i \in \mathcal{N}_i$. Notice that $\mathcal{P}_i^\times \mathcal{P} = I$ on $\mathcal{N}_i^\perp$. Then

$$\begin{aligned} \mathcal{P}\bar{T}_i^\dagger \mathcal{P}_i^\times \mathcal{P}\bar{T}_i(\mathcal{P}v_i) &= \mathcal{P}\bar{T}_i^\dagger \bar{T}_i(\mathcal{P}v_i) \\ &= \mathcal{P}\bar{T}_i^\dagger \bar{T}_i(w_i) = \mathcal{P}w_i = \mathcal{P}v_i. \end{aligned}$$

Moreover

$$\mathcal{P}\bar{T}_i\mathcal{P}\bar{T}_i^\dagger\mathcal{P}_i^\times(\mathcal{P}v_i) = \mathcal{P}\bar{T}_i\bar{T}_i^\dagger\mathcal{P}_i^\times\mathcal{P}w_i = \mathcal{P}w_i = \mathcal{P}v_i.$$

These equalities clearly show (5.3). □

Lemma 5.4. *Under the assumptions,* (3.7), (H1) *and* (H2), *the* K *in Lemma* 5.2 *is given by the following:* $K = \sup_{|v|_A=1, v\in\mathcal{R}(A)} K(v)$ *and*

$$K(v) = \inf_{c\in\mathcal{N}} \inf_{\sum v_k = v+c} \sum_{k=1}^{J} \left(\bar{T}_k^\dagger(v_k + T_k^* w_k), v_k + T_k^* w_k\right)_A,$$

where $w_k = \sum_{i=k+1}^{J} v_i$.

Proof. First, we show that

$$\sum_{k=1}^{J} \left((\mathcal{P}\bar{T}_k)^{-1}(\tilde{v}_k + \mathcal{P}T_k^*\tilde{w}_k), \tilde{v}_k + \mathcal{P}T_k^*\tilde{w}_k\right)_A = \sum_{k=1}^{J} \left(\bar{T}_k^\dagger(v_k + T_k^* w_k), v_k + T_k^* w_k\right)_A, \tag{5.4}$$

where $\tilde{v}_k = \mathcal{P}v_k$, $w_k = \sum_{i=k+1}^{J} v_i$ and $\tilde{w}_k = \sum_{i=k+1}^{J} \tilde{v}_i$ for all $1 \le k \le J$. Observe that

$$\begin{aligned}
&\big((\mathcal{P}\bar{T}_k)^{-1}(\tilde{v}_k + \mathcal{P}T_k^*\tilde{w}_k), \tilde{v}_k + \mathcal{P}T_k^*\tilde{w}_k\big)_A \\
&= \big(\mathcal{P}\bar{T}_k^\dagger\mathcal{P}_i^\times(\tilde{v}_k + \mathcal{P}T_k^*\tilde{w}_k), \tilde{v}_k + \mathcal{P}T_k^*\tilde{w}_k\big)_A \quad \text{(by Lemma 5.3)} \\
&= \big(\bar{T}_k^\dagger\mathcal{P}_i^\times(\mathcal{P}v_k + \mathcal{P}T_k^*w_k), \mathcal{P}v_k + \mathcal{P}T_k^*w_k\big)_A \\
&= \big(\bar{T}_k^\dagger\mathcal{P}_i^\times\mathcal{P}(v_k + T_k^*w_k), v_k + T_k^*w_k\big)_A \\
&= \big(\bar{T}_k^\dagger(v_k + T_k^*w_k), v_k + T_k^*w_k\big)_A.
\end{aligned}$$

In the last equality above, we used the fact that $\mathcal{P}_i^\times\mathcal{P} : V_i \mapsto \mathcal{N}_i^\perp$ is an orthogonal projection and the property of $\bar{T}_i^\dagger$ that $\bar{T}_i^\dagger c_i = 0$ for all $c_i \in \mathcal{N}_i$. (5.4) follows by the above equalities.

Due to Lemmas 5.1 and 5.2, we may begin the proof with the following expression of $K(v)$,

$$K(v) = \inf_{\sum \tilde{v}_k = v} \sum_{k=1}^{J} \big((\mathcal{P}\bar{T}_k)^{-1}(\tilde{v}_k + \mathcal{P}T_k^*\tilde{w}_k), (\tilde{v}_k + \mathcal{P}T_k^*\tilde{w}_k)\big)_A. \tag{5.5}$$

To complete the proof, we have to show that $K(v)$ defined in (5.5) is equal to the following quantity

$$\tilde{K}(v) = \inf_{c\in\mathcal{N}} \inf_{\sum v_k = v+c} \sum_{k=1}^{J} \big(\bar{T}_k^\dagger(v_k + T_k^*w_k), v_k + T_k^*w_k\big)_A. \tag{5.6}$$

It is straightforward to see that $\tilde{K}(v) \le K(v)$ by (5.4). The reverse inequality also follows by (5.4) from the fact that for a given $v \in \mathcal{R}(A)$, any choice $c \in \mathcal{N}$ and a decomposition of $v + c$ such that $\sum_i v_i = v + c$, there exist a decomposition of v such that $\sum_i \tilde{v}_i = v$ where $\tilde{v}_i = \mathcal{P}v_i$ (see [9] for the detail). This completes the proof. □

From the definition of A_i, we know that $\bar{T}_i = \bar{R}_i A_i$ on V_i. Then

$$\bar{T}_i^\dagger = (\bar{R}_i A_i)^\dagger = A_i^\dagger \bar{R}_i^\dagger \quad \text{on } V_i.$$

So we get that

$$\begin{aligned}
\big(\bar{T}_k^\dagger(v_k + T_k^*w_k), (v_k + T_k^*w_k)\big)_A &= \big(A_kA_k^\dagger\bar{R}_k^\dagger(v_k + T_k^*w_k), (v_k + T_k^*w_k)\big) \\
&= \big(\bar{R}_k^\dagger(v_k + T_k^*w_k), (v_k + T_k^*w_k)\big).
\end{aligned}$$

Combined Lemma 5.2, Lemma 5.4 and the above equalities, we get Lemma 3.3.

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