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ON SPECTRAL METHODS FOR VOLTERRA INTEGRAL EQUATIONS AND THE CONVERGENCE ANALYSIS*

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Abstract

The main purpose of this work is to provide a novel numerical approach for the Volterra integral equations based on a spectral approach. A Legendre-collocation method is proposed to solve the Volterra integral equations of the second kind. We provide a rigorous error analysis for the proposed method, which indicates that the numerical errors decay exponentially provided that the kernel function and the source function are sufficiently smooth. Numerical results confirm the theoretical prediction of the exponential rate of convergence. The result in this work seems to be the first successful spectral approach (with theoretical justification) for the Volterra type equations.

Mathematics subject classification: 35Q99, 35R35, 65M12, 65M70.
Key words: Legendre-spectral method, Second kind Volterra integral equations, Convergence analysis.

1. Introduction

This paper is concerned with the second kind Volterra integral equations

$$y(t) + \int_0^t R(t,s)y(s)ds = f(t), \qquad t \in [0,T], \tag{1.1}$$

where the source function f and the kernel function R are given, and $y(t)$ is the unknown function.

For ease of analysis, we will transfer the problem (1.1) to an equivalent problem defined in $[-1,1]$. More specifically, we use the change of variables

$$t = T(1+x)/2, \qquad x = 2t/T - 1,$$

to rewrite the Volterra equation (1.1) as follows

$$u(x) + \int_0^{T(1+x)/2} R\left(\frac{T}{2}(1+x), s\right) y(s)ds = g(x), \tag{1.2}$$

where $x \in [-1,1]$, and

$$u(x) = y\left(\frac{T}{2}(1+x)\right), \qquad g(x) = f\left(\frac{T}{2}(1+x)\right). \tag{1.3}$$

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Furthermore, to transfer the integral interval $[0, T(1+x)/2]$ to the interval $[-1, x]$, we make a linear transformation: $s = T(1+\tau)/2$, $\tau \in [-1, x]$. Then, Eq. (1.2) becomes

$$u(x) + \int_{-1}^{x} K(x, \tau) u(\tau) d\tau = g(x), \qquad x \in [-1, 1], \tag{1.4}$$

where

$$K(x, \tau) = \frac{T}{2} R\left(\frac{T}{2}(1+x), \frac{T}{2}(1+\tau)\right). \tag{1.5}$$

We will consider the case that the solutions of (1.1) (or equivalently (1.4)) are sufficiently smooth — in this case it is necessary to consider very high-order numerical methods such as spectral methods for approximating the solutions. There are many existing numerical methods for solving the Volterra equation (1.1), such as collocation methods, product integration methods, see, e.g., Brunner [1] and references therein. However, very few works touched the spectral approximations to (1.1). In [6], Chebyshev spectral methods are developed to solve nonlinear Volterra-Hammerstein integral equations, and in [7], Chebyshev spectral methods are investigated for Fredholm integral equations of the first kind under multiple-precision arithmetic. However, no theoretical analysis is provided to justify the high accuracy obtained. In [13], a spectral method is developed for solving (1.1), but unfortunately spectral accuracy is not observed for most of computations.

It is known that the Fredholm type equations behave more or less like a boundary value problem (see, e.g., [5]). As a result, some efficient numerical methods useful for boundary values problems (such as spectral methods) can be used directly to handle the Fredholm type equations (again see [5]). However, the Volterra equation (1.1) behaves like an initial value problem. Therefore, it is unpopular to apply the spectral approximations to the Volterra type equations. The main reason is that (1.1) is a local equation while the spectral methods use global basis functions. One of the main difficulties is how to implement the method so that spectral accuracy can be eventually obtained. On the other hand, the numerical methods for (1.1) may be different with those for the standard initial values problems in the sense that the former requires storage of all values at grid points while the latter only requires information at a fixed number of previous grid points. The storage requirement for (1.1) also makes the use of the global basis functions of the spectral methods more acceptable.

The main purpose of this work is to provide a novel numerical approach for the Volterra integral equations based on a spectral approach. We will provide a rigorous error analysis which theoretically justifies the spectral rate of convergence. This paper is organized as follows. In Section 2, we introduce the spectral approaches for the Volterra integral equations of second kind. Some lemmas useful for the convergence analysis will be provided in Section 3, and the convergence analysis, in both l^∞ and l^2 spaces, will be given in Section 4. Numerical experiments are carried out in Section 5 to verify the theoretical results obtained in Section 4.

2. Legendre-collocation Method

As demonstrated in the last section, we can assume that the solution domain is $[-1, 1]$. The second kind linear integral equations in one-dimension is of the form (1.4), namely,

$$u(x) + \int_{-1}^{x} K(x, s) u(s) ds = g(x), \qquad x \in [-1, 1]. \tag{2.1}$$

Set the collocation points as the set of $(N+1)$ Legendre Gauss, or Gauss-Radau, or Gauss-Lobatto points, $\{x_i\}_{i=0}^N$. Assume that Eq. (2.1) holds at x_i:

$$u(x_i) + \int_{-1}^{x_i} K(x_i, s)u(s)ds = g(x_i), \qquad 0 \le i \le N. \tag{2.2}$$

The main difficulty in obtaining high order accuracy is to compute the integral term in (2.2) accurately. In particular, for small values of x_i, there is little information available for $u(s)$. To overcome this difficulty, we will transfer the integral interval $[-1, x_i]$ to a fixed interval $[-1, 1]$ and then make use of some appropriate quadrature rule. More precisely, we first make a simple linear transformation:

$$s(x, \theta) = \frac{1+x}{2}\theta + \frac{x-1}{2}, \qquad -1 \le \theta \le 1. \tag{2.3}$$

Then (2.2) becomes

$$u(x_i) + \frac{1+x_i}{2}\int_{-1}^{1} K(x_i, s(x_i, \theta))u(s(x_i, \theta))d\theta = g(x_i), \qquad 0 \le i \le N. \tag{2.4}$$

Using a $(N+1)$-point Gauss quadrature formula relative to the Legendre weights $\{\omega_k\}$ gives

$$u(x_i) + \frac{1+x_i}{2}\sum_{j=0}^{N} K(x_i, s(x_i, \theta_j))u(s(x_i, \theta_j))\omega_j = g(x_i), \qquad 0 \le i \le N, \tag{2.5}$$

where the set $\{\theta_j\}_{j=0}^N$ coincide with the collocation points $\{x_j\}_{j=0}^N$.

We now need to represent $u(s(x_i, \theta_j))$ using u_i, $0 \le i \le N$, i.e., the values at all the grid points. To this end, we expand u using Lagrange interpolation polynomials, i.e.,

$$u(\sigma) \approx \sum_{k=0}^{N} u_k F_k(\sigma), \tag{2.6}$$

where F_k is the k-th Lagrange basis function. Combining (2.6) and (2.5) yields

$$u_i + \frac{1+x_i}{2}\sum_{j=0}^{N} u_j \left(\sum_{p=0}^{N} K(x_i, s(x_i, \theta_p))F_j(s(x_i, \theta_p))\omega_p\right) = g(x_i), \qquad 0 \le i \le N. \tag{2.7}$$

Remark 2.1. It is seen from the numerical scheme (2.7) that to compute the approximation to $u(x_i)$, we require the entire solution information of $\{u(x_j)\}_{j=0}^N$ and the semi-local information of $\{K(x_i, s(x_i, \theta_j)\}_{j=0}^i$. Here, $-1 \le s(x_i, \theta_j) \le x_i$. This is different with the collocation methods or product integration methods which use the semi-local information of both the solution and K, namely, $\{u(x_j)\}_{j=0}^i$ and $\{K(x_i, \beta_j\}$ where $-1 \le \beta_j \le x_i$ are some collocation points. It is because of this difference that we will be able to obtain, as to be demonstrated in the next section, a spectral rate of accuracy instead of an algebraic order of accuracy for the proposed scheme (2.7).

Below we discuss some implementation issues of the spectral collocation algorithm. Denoting $U_N = [u_0, u_1, \cdots, u_N]^T$ and $g_N = [g(x_0), g(x_1), \cdots, g(x_N)]^T$, we can obtain an equation of the matrix form:

$$U_N + AU_N = g_N, \tag{2.8}$$

where the entries of the matrix A is given by

$$A_{i,j} = \frac{1+x_i}{2} \sum_{p=0}^{N} K(x_i, s(x_i, \theta_p)) F_j(s(x_i, \theta_p)) \omega_p.$$

We now discuss an efficient computation of $F_j(s(x_i, \theta_p))$. The idea is to express $F_j(s)$ in terms of the Legendre functions:

$$F_j(s) = \sum_{p=0}^{N} \alpha_{p,j} L_p(s), \tag{2.9}$$

where $\alpha_{p,j}$ is called the discrete polynomial coefficients of F_j. The inverse relation is (see, e.g., [4]):

$$\alpha_{p,j} = \frac{1}{\gamma_p} \sum_{i=0}^{N} F_j(x_i) L_p(x_i) \omega_i = L_p(x_j) \omega_j / \gamma_p, \tag{2.10}$$

where

$$\gamma_p = \sum_{i=0}^{N} L_p^2(x_i) \omega_i = (p + \frac{1}{2})^{-1}, \quad \text{for} \quad p < N \tag{2.11}$$

and $\gamma_N = (N + 1/2)^{-1}$ for the Gauss and Gauss-Radau formulas, and $\gamma_N = 2/N$ for the Gauss-Lobatto formula. It follows from (2.9) and (2.10) that

$$F_j(s) = \sum_{p=0}^{N} L_p(x_j) L_p(s) \omega_j / \gamma_p, \tag{2.12}$$

which, together with the known recurrence formulas for $L_p(s)$, can be used to evaluate $F_j(s(x_i, \theta_p))$ in an efficient way.

3. Some Useful Lemmas

In this section, a convergence analysis for the numerical schemes for the Volterra equation (2.1) will be provided. The goal is to show that the rate of convergence is exponential, i.e., the spectral accuracy can be obtained for the proposed spectral approximations.

Lemma 3.1. ([4], p.290. *Integration error from Gauss quadrature.*) *Assume that a $(N+1)$-point Gauss, or Gauss-Radau, or Gauss-Lobatto quadrature formula relative to the Legendre weight is used to integrate the product $u\phi$, where $u \in H^m(I)$ with $I := (-1, 1)$ for some $m \geq 1$ and $\phi \in \mathcal{P}_N$. Then there exists a constant C independent of N such that*

$$\left| \int_{-1}^{1} u(x)\phi(x)dx - (u, \phi)_N \right| \leq C N^{-m} |u|_{\tilde{H}_{m,N}(I)} \|\phi\|_{L^2(I)}, \tag{3.1}$$

where

$$|u|_{\tilde{H}_{m,N}(I)} = \left(\sum_{j=min(m,N+1)}^{m} \|u^{(j)}\|_{L^2(I)}^2 \right)^{1/2}, \tag{3.2}$$

$$(u, \phi)_N = \sum_{j=0}^{N} \omega_j u(x_j) \phi(x_j). \tag{3.3}$$

Lemma 3.2. ([4], p.289. *Estimates for the interpolation error.*) *Assume that* $u \in H^m(I)$ *and denote* $I_N u$ *its interpolation polynomial associated with the* $(N+1)$*-point Gauss, or Gauss-Radau, or Gauss-Lobatto points* $\{x_j\}_{j=0}^N$*, namely,*

$$I_N u = \sum_{i=0}^{N} u(x_i) F_i(x). \tag{3.4}$$

Then the following estimates hold

$$\|u - I_N u\|_{L^2(I)} \le C N^{-m} |u|_{\tilde{H}_{m,N}(I)}, \tag{3.5}$$

$$\|u - I_N u\|_{H^l(I)} \le C N^{2l-1/2-m} |u|_{\tilde{H}_{m,N}(I)}, \quad 1 \le l \le m. \tag{3.6}$$

Lemma 3.3. ([11] *Lebesgue constant for the Legendre series.*) *Assume that* $F_j(x)$ *is the* N*-th Lagrange interpolation polynomials associated with the Gauss, or Gauss-Radau, or Gauss-Lobatto points. Then*

$$\max_{x\in(-1,1)} \sum_{j=0}^{N} |F_j(x)| = 1 + \frac{2^{3/2}}{\sqrt{\pi}} N^{1/2} + B_0 + \mathcal{O}(N^{-1/2}), \tag{3.7}$$

where B_0 *is a bounded constant.*

Lemma 3.4. (*Gronwall inequality.*) *If a non-negative integrable function* $E(t)$ *satisfies*

$$E(t) \le C_1 \int_{-1}^{t} E(s)ds + G(t), \quad -1 < t \le 1, \tag{3.8}$$

where $G(t)$ *is an integrable function, then*

$$\|E\|_{L^p(I)} \le C \|G\|_{L^p(I)}, \quad p \ge 1. \tag{3.9}$$

4. Convergence Analysis

In this section, we will carry our convergence analysis in both L^2 and L^∞ spaces.

4.1. Error analysis in L^2

Theorem 4.1. *Let* u *be the exact solution of the Volterra equation* (2.1) *and assume that*

$$U^N(x) = \sum_{j=0}^{N} u_j F_j(x), \tag{4.1}$$

where u_j *is given by* (2.7) *and* $F_j(x)$ *is the* j*-th Lagrange basis function associated with the Gauss-points* $\{x_j\}_{j=0}^N$*. If* $u \in H^m(I)$*, then for* $m \ge 1$*,*

$$\begin{aligned} &\|u - U\|_{L^\infty(I)} \\ &\le C N^{1/2-m} \max_{-1\le x\le 1} |K(x, s(x,\cdot))|_{\tilde{H}_{m,N}(I)} \|u\|_{L^2(I)} + C N^{-m} |u|_{\tilde{H}_{m,N}(I)}, \end{aligned} \tag{4.2}$$

provided that N *is sufficiently large, where* $s(x_i, \theta)$ *is defined by* (2.3) *and* C *is a constant independent of* N*.*

Proof. Following the notations of (3.3), we let

$$(K(x,s),\phi(s))_{N,s}=\sum_{j=0}^{N}K(x,s(x,\theta_j))\phi(s(x,\theta_j))\omega_j. \tag{4.3}$$

Then the numerical scheme (2.7) can be written as

$$u_i+\frac{1+x_i}{2}(K(x_i,s),U^N(s))_{N,s}=g(x_i), \tag{4.4}$$

which gives

$$\begin{aligned}&u_i+\frac{1+x_i}{2}\int_{-1}^{1}K(x_i,s(x_i,\theta))U^N(s(x_i,\theta))d\theta\\&=g(x_i)+J_1(x_i),\quad 1\le i\le N,\end{aligned} \tag{4.5}$$

where

$$J_1(x)=\frac{1+x}{2}\int_{-1}^{1}K(x,s(x,\theta))U^N(s(x,\theta))d\theta-\frac{1+x}{2}(K(x,s),U^N(s))_{N,s}. \tag{4.6}$$

Using Lemma 3.1 gives

$$|J_1(x)|\le CN^{-m}|K(x,s(x,\cdot))|_{\tilde{H}_{m,N}(I)}\|U^N\|_{L^2(I)}. \tag{4.7}$$

It follows from (4.5), (2.2) and (2.4) that

$$u_i+\int_{-1}^{x_i}K(x_i,s)U^N(s)ds=g(x_i)+J_1(x_i),\quad 1\le i\le N. \tag{4.8}$$

Multiplying $F_j(x)$ on both sides of (4.8) and summing up from 0 to N yield

$$\begin{aligned}&U^N(x)+I_N\left(\int_{-1}^{x}K(x,s)u(s)ds\right)+I_N\left(\int_{-1}^{x}K(x,s)e(s)ds\right)\\&=I_N(g)+I_N(J_1),\end{aligned} \tag{4.9}$$

where U^N is defined by (4.1), the interpolation operator I_N is defined by (3.4), e denotes the error function, i.e.,

$$e(x)=U^N(x)-u(x),\qquad x\in[-1,1]. \tag{4.10}$$

It follows from (4.9) and (2.1) that

$$\begin{aligned}&U^N(x)+I_N(g-u)+I_N\left(\int_{-1}^{x}K(x,s)e(s)ds\right)\\&=I_N(g)+I_N(J_1),\end{aligned}$$

which gives

$$e(x)+(u-I_Nu)(x)+I_N\left(\int_{-1}^{x}K(x,s)e(s)ds\right)=I_N(J_1). \tag{4.11}$$

Consequently,

$$e(x)+\int_{-1}^{x}K(x,s)e(s)ds=I_N(J_1)+J_2(x)+J_3(x), \tag{4.12}$$

where

$$J_2 = I_N u(x) - u(x), \quad J_3 = \int_{-1}^{x} K(x,s)e(s)ds - I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right). \tag{4.13}$$

It follows from the Gronwall inequality (see Lemma 3.4) with $p = 2$ that

$$\|e\|_{L^2(I)} \le C\Big(\|I_N(J_1)\|_{L^2(I)} + \|J_2\|_{L^2(I)} + \|J_3\|_{L^2(I)}\Big). \tag{4.14}$$

Using (4.7) and Lemma 3.3 gives

$$\begin{aligned}
&\|I_N(J_1)\|_{L^2(I)} \le CN^{-m} \max_{x\in I} |K(x,s(x,\cdot))|_{\tilde{H}_{m,N}(I)} \|U^N\|_{L^2(I)} \max_{x\in I} \sum_{j=0}^{N} |F_j(x)| \\
&\le CN^{1/2-m} \max_{x\in I} |K(x,s(x,\cdot))|_{\tilde{H}_{m,N}(I)} \|U^N\|_{L^2(I)} \\
&\le CN^{1/2-m} \max_{x\in I} |K(x,s(x,\cdot))|_{\tilde{H}_{m,N}(I)} \left(\|e\|_{L^2(I)} + \|u\|_{L^2(I)}\right).
\end{aligned} \tag{4.15}$$

Using the L^2-error bounds for the interpolation polynomials (i.e., Lemma 3.2) gives

$$\|J_2\|_{L^2(I)} \le CN^{-m} |u|_{\tilde{H}_{m,N}(I)}, \tag{4.16}$$

and, by letting $m = 1$ in (3.5), yields

$$\|J_3\|_{L^2(I)} \le CN^{-1} \left\| K(x,x)e(x) + \int_{-1}^{x} K_x(x,s)e(s)ds \right\|_{L^2(I)} \le CN^{-1}\|e\|_{L^2(I)}. \tag{4.17}$$

The above estimates, together with (4.14), yield

$$\begin{aligned}
\|e\|_{L^2(I)} \le{}& CN^{1/2-m} \max_{x\in I} |K(x,s(x,\cdot))|_{\tilde{H}_{m,N}(I)} \Big(\|e\|_{L^2} + \|u\|_{L^2(I)}\Big) \\
&+ CN^{-m}|u|_{\tilde{H}_{m,N}(I)} + CN^{-1}\|e\|_{L^2(I)},
\end{aligned} \tag{4.18}$$

which leads to (4.2) provided that N is sufficiently large. This completes the proof of this theorem. □

4.2. Error analysis in L^∞

Below we will extend the L^2 error estimate in the last subsection to the L^∞ space. The key technique is to use an extrapolation between L^2 and H^1.

Theorem 4.2. *Let u be the exact solution of the Volterra equation* (2.1) *and U^N be defined by* (4.1). *If $u \in H^m(I)$, then for $m \ge 1$,*

$$\begin{aligned}
&\|u - U^N\|_{L^\infty(I)} \\
&\le CN^{1/2-m} \max_{x\in I} |K(x,s(x,\cdot))|_{\tilde{H}_{m,N}(I)} \|u\|_{L^2(I)} + CN^{3/4-m}|u|_{\tilde{H}_{m,N}(I)},
\end{aligned} \tag{4.19}$$

provided that N is sufficiently large, where $s(x_i,\theta)$ is defined by (2.3) *and C is a constant independent of N.*

Proof. Following the same procedure as in the proof of Theorem 4.1, we have

$$e(x) + \int_{-1}^{x} K(x,s)e(s)ds = I_N(J_1) + J_2(x) + J_3(x), \tag{4.20}$$

where $I_N(J_1), J_2$ and J_3 are defined by (4.6) and (4.13), respectively. It follows from the Gronwall inequality (see Lemma 3.4) that

$$\|e\|_{L^\infty(I)} \le C\Big(\|I_N(J_1)\|_{L^\infty(I)} + \|J_2\|_{L^\infty(I)} + \|J_3\|_{L^\infty(I)}\Big). \tag{4.21}$$

Using (4.7) and Lemma 3.3 gives

$$\begin{aligned}
\|I_N(J_1)\|_{L^\infty(I)} &\le CN^{-m} \max_{x\in I} |K(x,s(x,\cdot))|_{\tilde{H}_{m,N}(I)} \|U^N\|_{L^2(I)} \max_{x\in I} \sum_{j=0}^{N} |F_j(x)| \\
&\le CN^{1/2-m} \max_{x\in I} |K(x,s(x,\cdot))|_{\tilde{H}_{m,N}(I)} \|U^N\|_{L^2(I)} \\
&\le CN^{1/2-m} \max_{x\in I} |K(x,s(x,\cdot))|_{\tilde{H}_{m,N}(I)} \left(\|e\|_{L^\infty(I)} + \|u\|_{L^2(I)}\right).
\end{aligned} \tag{4.22}$$

Using the inequality in the Sobolev Space ([4], p.496)

$$\|w\|_{L^\infty(a,b)} \le \big(\frac{1}{b-a} + 2\big)^{\frac{1}{2}} \|w\|^{1/2}_{L^2(a,b)} \|w\|^{1/2}_{H^1(a,b)}, \qquad \forall u \in H^1(a,b), \tag{4.23}$$

and Lemma 3.2, we have

$$\begin{aligned}
\|J_2\|_{L^\infty(I)} &= \|u - I_N u\|_{L^\infty(I)} \\
&\le C\|u - I_N u\|^{1/2}_{L^2(I)} \|u - I_N u\|^{1/2}_{H^1(I)} \le CN^{3/4-m} |u|_{\tilde{H}_{m,N}(I)}.
\end{aligned} \tag{4.24}$$

It follows again from Lemma 3.2 and (4.17) that

$$\|J_3\|_{L^2(I)} \le CN^{-1}\|e\|_{L^2(I)} \le CN^{-1}\|e\|_{L^\infty(I)}, \tag{4.25}$$

and

$$\begin{aligned}
\|J_3\|_{H^1(I)} &\le CN^{1/2} \left\| K(x,x)e(x) + \int_{-1}^{x} K_x(x,s)e(s)ds \right\|_{L^2(I)} \\
&\le CN^{1/2}\|e\|_{L^2(I)} \le CN^{1/2}\|e\|_{L^\infty(I)},
\end{aligned} \tag{4.26}$$

Using the Sobolev inequality (4.23) gives

$$\|J_3\|_{L^\infty(I)} \le CN^{-1/4}\|e\|_{L^\infty(I)}. \tag{4.27}$$

The desired estimate (4.19) follows from the above estimates and (4.21). □

5. Numerical Experiments

Without lose of generality, we will only use the Legendre-Gauss-Lobatto points (i.e., the zeros of $(1-x^2)L'_N(x)$) as the collocation points. Our numerical evidences show that the other two kinds of Legendre-Gauss points produce results with similar accuracy. For the Legendre-Gauss-Lobatto points, the corresponding weights are

$$\omega_j = \frac{2}{(1-x_j^2)[L'_{N+1}(x_j)]^2}, \quad 0 \le j \le N.$$

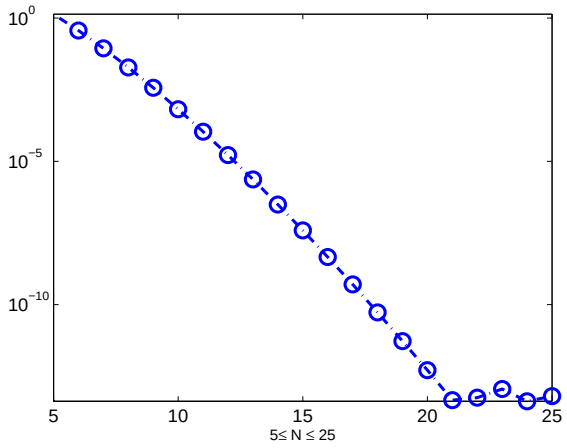

Fig. 5.1. Example 5.1: maximum error for the 1-D linear Volterra equation.

Example 5.1. Our first example is concerned with an one-dimensional Volterra equation of the second kind. More precisely, consider the Volterra equation (2.1) with

$$K(x,s) = e^{xs}, \quad g(x) = e^{4x} + \frac{1}{x+4}(e^{x(x+4)} - e^{-(x+4)}).$$

The corresponding exact solution is given by $u(x) = e^{4x}$.

We use the numerical scheme (2.7). Numerical errors with several values of N are displayed in Table 5.1 and Fig. 5.1. These results indicate that the desired spectral accuracy is obtained.

Table 5.1: Example 5.1: The maximum point-wise error.

N	6	8	10	12	14
error	3.66e-01	1.88e-02	6.57e-04	1.65e-05	3.11e-07
N	16	18	20	22	24
error	4.57e-09	5.37e-11	5.19e-13	5.68e-14	4.26e-14

In practice, many Volterra equations are usually nonlinear. However, the nonlinearity adds rather little to the difficulty of obtaining a numerical solution. The methods described above remain applicable. Although in this work our convergence theory does not cover the nonlinear case, it should be quite straightforward to establish a convergence result similar to Theorem 4.1 provided that the kernel k in (5.1) is Lipschitz continuous with its third argument. A similar technique for the collocation methods to the nonlinear Volterra equations was used by Brunner and Tang [3]. Below we will provide a numerical example using the spectral technique proposed in this work.

Example 5.2. Our second example is about a nonlinear problem in one-dimension. Consider the Volterra equation (5.1) with

$$g(x) = -\frac{1}{2(1+36\pi^2)}\Big(e^{-x} + 36\pi^2 e^{-x} - e^{-x}\cos 6\pi x + 6\pi e^{-x}\sin 6\pi x$$
$$-36e\pi^2)e^x + e^x \sin 3\pi x, \quad K(x,s,u(s)) = e^{x-3s}u^2(s).$$

The exact solution is $u(x) = e^x \sin 3\pi x$.

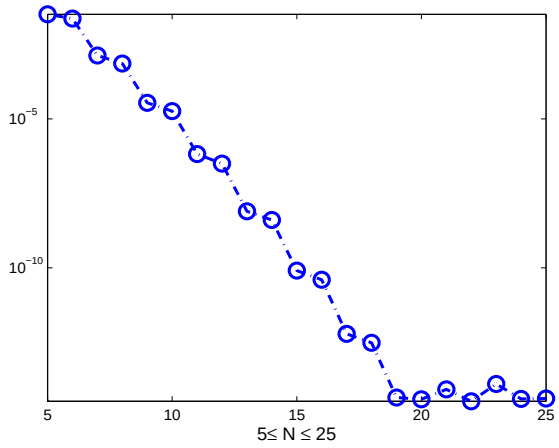

Fig. 5.2. Example 5.2: maximum error for the 1-D nonlinear Volterra equation.

For the nonlinear Volterra equations of the second kind in the form

$$u(x) + \int_{-1}^{x} k(x, s, u(s))ds = g(x), \qquad x \in [-1, 1], \tag{5.1}$$

we can design a spectral collocation method similar to the linear case. More precisely, we assume that (2.8) holds at the Legendre collocation points and transform the interval $[-1, x]$ to $[-1, 1]$. This gives

$$u(x_i) + \frac{1 + x_i}{2} \int_{-1}^{1} k(x_i, s(x_i, \theta), u(s(x_i, \theta)))d\theta = g(x_i), \qquad 0 \le i \le N. \tag{5.2}$$

Similar to (2.7), we obtain

$$u_i + \frac{1 + x_i}{2} \sum_{j=0}^{N} k\left(x_i, s(x_i, \theta_j), \sum_{p=0}^{N} u_p F_p(s(x_i, \theta_j))\right) \omega_j = g(x_i), \qquad 0 \le i \le N. \tag{5.3}$$

It is noticed that the numerical scheme (5.3) leads to a nonlinear system for $\{u_i\}_{i=1}^{N}$, and a proper solver for the nonlinear system (e.g., Newton method) should be used. In our computations, we just use a simple Jacobi type iteration method to solve the nonlinear system, which takes about 5 to 6 iterations. The numerical results can be seen from Table 5.2 and Fig. 5.2. Again the exponential rate of convergence is observed for the nonlinear problem.

Table 5.2: Example 5.2: The maximum point-wise error.

N	6	8	10	12	14
error	2.33e-02	7.22e-04	1.82e-05	3.15e-07	4.06e-09
N	16	18	20	22	24
error	3.98e-11	3.05e-13	3.86e-15	3.33e-15	3.98e-15

There are some recent studies for using the collocation methods and the product integration methods for solving multi-dimensional Volterra integral equations, see, e.g., [2, 10]. In two-dimensions, we have

$$u(x, y) + \int_{-1}^{x} \int_{-1}^{y} k(x, y, s, t, u(s, t))dsdt = g(x, y), \qquad (x, y) \in [-1, 1]^2. \tag{5.4}$$

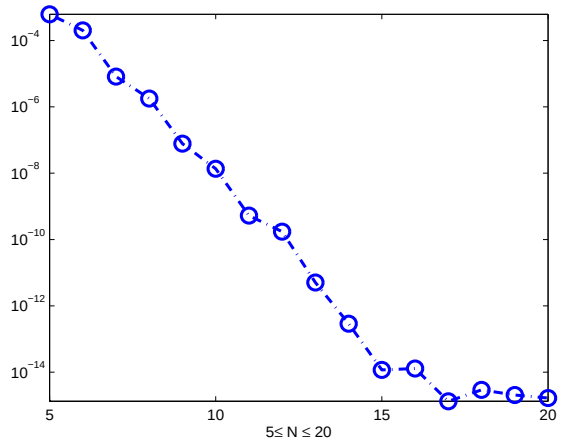

Fig. 5.3. Example 5.3: maximum error for the 2-D linear Volterra equation.

Letting the above equation hold at the Legendre point pairs (x_i, y_j), and then using the linear transformation and tricks used in 1D case yields

$$u_{i,j} + \frac{1+x_i}{2}\frac{1+x_j}{2}\sum_{p=0}^{N}\sum_{l=0}^{N} k\left(x_i, y_j, s(x_i,\theta_p), s(x_i,\theta_l), u(s(x_i,\theta_p), s(x_i,\theta_l))\right)\omega_p\omega_l = g(x_i, y_j). \tag{5.5}$$

The values of $u(s(x_i,\theta_p), s(x_i,\theta_l))$ can also be approximated by $u_{i,j}$ with the use of the relationship between the Lagrange interpolation polynomials associate with the Legendre collocation points, as demonstrated in the one-dimensional case. It is expected that the analysis techniques proposed in this work can be used to extend Theorem 4.1 to obtain a spectral convergence rate for (5.5).

Example 5.3. The third example is concerned with a 2D linear Volterra equation with second kind. Consider the equation (5.4) with

$$K(x, y, s, t, u(s,t)) = e^{x+y}\cos(2s+t)u(s,t),$$
$$g(x,y) = \frac{1}{16}e^{x+y}(\sin(4x+2y) + \sin(2y-4) + \sin(4x-2) + \sin 6) + \sin(2x+y).$$

This problem has a unique solution: $u(x,y) = \sin(2x+y)$.

Table 5.3 and Fig. 5.3 present the maximum pointwise errors with difference values of N. Again, it is observed clearly that the errors decay exponentially.

Table 5.3: Example 5.3: The maximum point-wise error for the 2D linear problem.

N	5	6	7	8	9
error	6.21e-04	2.02e-04	8.16e-006	1.78e-06	7.77e-08
N	12	14	16	18	20
error	1.73e-10	2.89e-13	1.30e-14	2.94e-15	1.67e-15

6. Concluding Remarks

This paper proposes a numerical method for the Volterra type equations based on spectral methods. The most important contribution of this work is that we are able to demonstrate rigorously that the errors of the spectral approximations decay exponentially. More precisely, it is proved that if the kernel function and solutions of the underlying Volterra equations are smooth, then errors obtained by the proposed spectral method decay exponentially which is a desired feature for a spectral method.

This work seems to be the first successful numerical method for the Volterra integral equations having exponential rate of convergence, which can be demonstrated theoretically and numerically. The tools used in establishing the error estimates include the standard estimates for the quadrature rule and the L^2-error bounds for the interpolation function.

We conclude this work by making some discussions following a suggestion by one of the referees. In an earlier version of this paper, we conjectured that the optimal rate of convergence in Theorem 4.1 should be $\mathcal{O}(N^{-m})$ instead of $\mathcal{O}(N^{1/2-m})$. The referee suggested that $\|J_1\|_{L^2(I)}$, where J_1 is defined by (4.6), can be bounded by the stability of the Legendre interpolation. For instance, let I_N be the Legendre-Gauss interpolation operator, then one can improve (4.15) using [8, 9]

$$\|I_N(I_1)\| \leq C\left(\|I_1\|_{L^2(I)} + N^{-1}\|\partial_x I_1\|_{L^2(I)}\right), \tag{6.1}$$

for any $I_1 \in H^1(I)$. However, by a close observation of the form of J_1, it is noticed that $U^N(s(x,\theta))$ depends not only on θ but also on x. This is quite different with the usual quadrature rule where the polynomial in the inner product depends only on one variable. It can be verified that

$$N^{-1}\|\partial_x J_1\|_{L^2(I)} \sim N^{-m-1}\|\partial_x U^N\|_{L^2(I)} \geq CN^{-m+1}\|U^N\|_{L^2(I)}, \tag{6.2}$$

where in the last step we have used the inverse inequality given in [8,9]. The result (6.2) is even worse than $\mathcal{O}(N^{1/2-m})$. It can be verified that the first term on the right-hand side of (4.6) equals to $\int_{-1}^{x} K(x,s)U^N(s)ds$ whose derivative with respect to x is well-behaved in L^2; but the last term in (4.6), i.e., $(K(x,s),U^N(s))$ involves $U^N(x,s(x,\theta)$ which gives difficulty in using (6.1). It may need other ways, even with some modification on the spectral method proposed in Section 2, to obtain the (possible) optimal rate of convergence.

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