Journal of Computational Mathematics
Vol.27, No.6, 2009, 697–712.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2009.09-m2899

A NEW APPROACH TO RECOVERY OF DISCONTINUOUS GALERKIN*

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Abstract

A new recovery operator $P : Q_n^{disc}(\mathcal{T}) \to Q_{n+1}^{disc}(\mathcal{M})$ for discontinuous Galerkin is derived. It is based on the idea of projecting a discontinuous, piecewise polynomial solution on a given mesh $\mathcal{T}$ into a higher order polynomial space on a macro mesh $\mathcal{M}$. In order to do so, we define local degrees of freedom using polynomial moments and provide global degrees of freedom on the macro mesh. We prove consistency with respect to the local L_2-projection, stability results in several norms and optimal anisotropic error estimates. As an example, we apply this new recovery technique to a stabilized solution of a singularly perturbed convection-diffusion problem using bilinear elements.

Mathematics subject classification: 65N12, 65N15, 65N30.
Key words: Discontinuous Galerkin, Postprocessing, Recovery.

1. Introduction

The importance of developing superconvergence recovery techniques for finite element approximations is two folded: firstly, the objective is to improve the approximation accuracy of low order finite elements on coarse meshes, which will significantly reduce the computational costs to achieve a certain accuracy. Secondly, the recovered solution values can be used in computation of a posteriori error estimators, which are essential for estimating the accuracy of finite element approximations and for guiding the mesh refinement in adaptive methods.

The main objective in this paper is the improvement of solution accuracy by using supercloseness results and an appropriate recovery technique (postprocessing). This type of superconvergence by recovery is well-known and has been extensively studied in the literature for different classes of problems, see, e.g., [5, 6, 9, 16]. The application of this technique to stabilized finite element discretization for solving singularly perturbed problems can be found in [11, 13, 14]. It has been shown that (in the two-dimensional case) the vertex-edge-cell interpolant, studied in [1], is superclose to the streamline-diffusion finite element solution on a Shishkin mesh. A recovery operator which is consistent with this special interpolant, allows to prove a superconvergence result for the postprocessed SDFEM solution.

* Received September 5, 2008 / Revised version received March 3, 2009 / Accepted April 3, 2009 /

An alternative stabilization method for singularly perturbed problems is the discontinuous Galerkin method for which a supercloseness result with respect to the discontinuous, local L_2-projection onto piecewise bilinear functions has been established in [12]. The discussed method therein is the so called NIPG, see also [3, 10].

Our recovery techniques applies to the more general case of the local L_2-projection onto the space of discontinuous, piecewise polynomials of arbitrary degree $n \in \mathbb{N}$ in each variable and in any space dimension. Therefore it can be applied to a more general class of discontinuous Galerkin methods. We choose for application the NIPG because here a supercloseness result is known. For convection-diffusion equations in 1d several supercloseness results using numerical traces and possible postprocessing methods are known, see [4, 15].

We also recommend the reader to the recent reference [18] on recovery techniques in finite elements with special emphasis on Zienkiewicz-Zhu's patch recovery and polynomial preserving recovery.

The outline of this article is as follows. We start in Section 2 with the $1d$-recovery operator. In Section 3 we construct the $2d$-recovery operator and prove stability and anisotropic error estimates. Finally, in Section 4 we connect our results to recently published results [12] in the case of bilinears on a Shishkin mesh for a singularly perturbed partial differential equation.

Notation: For a function $u : \mathcal{T} \to \mathbb{R}$ which belongs piecewise in L_2 we define the broken L_2-norm by

$$\|u\|_{0,\mathcal{T}} = \left(\sum_{K \in \mathcal{T}} \|u\|_{0,K}^2 \right)^{1/2} .$$

2. Basics in $1d$

We start the definition of the recovery operator in one space-dimension. In order to simplify the notation we will work on reference elements. Thus, let $I_L := [-1, 0]$ and $I_R := [0, 1]$ be the reference intervals.

Our operator will be a projection onto a higher order polynomial space on macro meshes. Let $I_M := I_L \cup I_R$ denote the reference macro element to a given macro element consisting of two intervals. The reference mesh consists of the two subintervals of I_M and is denoted by $\mathcal{T} := \{I_L, I_R\}$.

We start the definition of the projector by defining local degrees of freedom on this mesh. Let

$$R_i(v) := \int_0^1 \eta_i(t)v(t)\,\mathrm{d}t \quad \text{and} \quad L_i(v) := \int_{-1}^0 \eta_i(t+1)v(t)\,\mathrm{d}t, \quad \forall i = 0, \dots, n \tag{2.1}$$

with $\{\eta_i\}_{i=0}^n$ denoting the Legendre polynomial basis of $\mathcal{P}_n(I_R)$, the space of polynomials of degree at most n. Due to the L_2-orthogonality of these polynomials, the sets $\{R_i\}_{i=0}^m$ and $\{L_i\}_{i=0}^m$ with $0 \le m \le n$ are $\mathcal{P}_m(I_R)$- resp. $\mathcal{P}_m(I_L)$-unisolvent, i.e. an element $v \in \mathcal{P}_m(I)$ is uniquely defined for given values $\{N_i^1 v\}_{i=0}^m$. Then, there is a local basis $\{\psi_i^1\}_{i=0}^n$ of $\mathcal{P}_n(I_R)$ with

$$R_i(\psi_j^1) = \delta_{ij},\; i, j = 0, \dots, n \tag{2.2}$$

where δ_{ij} is the Kronecker delta. Clearly our local basis functions are scaled Legendre polynomials with $\deg \psi_i^1 = i$, $i = 0, \dots, n$, and the interpolation operator defined by

$$\pi v \in P_n^{disc}(\mathcal{T}) : R_i(\pi v) = R_i(v), \quad L_i(\pi v) = L_i(v), \quad i = 0, \dots, n \tag{2.3}$$

is the local L_2-projection into $\mathcal{P}_n^{disc}(\mathcal{T}) := \{v \in L_2(I_M) : v|_K \in \mathcal{P}_n(K), \forall K \in \mathcal{T}\}$.

Now let us define the global degrees of freedom for our projection operator on the macro element combining local ones. Simple adding of the two sets of $n+1$ local degrees gives

$$N_i^1(v) := (R_i + L_i)(v), \quad i = 0, \dots, n. \tag{2.4a}$$

In order to define a function of $\mathcal{P}_{n+1}(I_M)$, we need one additional, independent degree. We use

$$N_{n+1}^1(v) := (R_n - L_n)(v). \tag{2.4b}$$

Lemma 2.1. *Let $0 \le m \le n+1$. The sets $\{N_i^1\}_{i=0}^m$ are $\mathcal{P}_m(I_M)$-unisolvent.*

Proof. Start with $m = n+1$. We have

$$\dim(\mathcal{P}_{n+1}(I_M)) = n+2 = \#\{N_i^1\}_{i=0}^{n+1}.$$

Thus, it is sufficient to show that for $p(t) = \sum_{i=0}^{n+1} p_i t^i \in \mathcal{P}_{n+1}(I_M)$ with $N_i^1(p) = 0$, $i = 0, \dots, n+1$ it follows $p \equiv 0$. Recall that the j-th order Legendre polynomial η_j is L_2-orthogonal to all lower order polynomials. Then we have

$$\begin{aligned}
0 &= N_{n+1}^1(p) \\
&= p_{n+1}\left(\int_0^1 \eta_n(t)t^{n+1}\,\mathrm{d}t - \int_{-1}^0 \eta_n(t+1)t^{n+1}\,\mathrm{d}t\right) + p_n\left(\int_0^1 \eta_n(t)t^n\,\mathrm{d}t - \int_{-1}^0 \eta_n(t+1)t^n\,\mathrm{d}t\right) \\
&= p_{n+1}\int_0^1 \Big(\eta_n(t)t^{n+1} - \eta_n(t)(t^{n+1} - (n+1)t^n)\Big)\,\mathrm{d}t + p_n\int_0^1 \Big(\eta_n(t)t^n - \eta_n(t)t^n\Big)\,\mathrm{d}t \\
&= (n+1)p_{n+1}\underbrace{\int_0^1 \eta_n(t)t^n\,\mathrm{d}t}_{\neq 0} \Rightarrow p_{n+1} = 0,
\end{aligned}$$

where we used the substitution $t+1 \to t$ in the second line. Thus, $p \in \mathcal{P}_n(I_M)$.

$$\begin{aligned}
0 = N_n^1(p) &= p_n\left(\int_{-1}^0 \eta_n(t+1)t^n\,\mathrm{d}t + \int_0^1 \eta_n(t)t^n\,\mathrm{d}t\right) \\
&= 2p_n\underbrace{\int_0^1 \eta_n(t)t^n\,\mathrm{d}t}_{\neq 0} \Rightarrow p_n = 0.
\end{aligned}$$

Thus, $p \in \mathcal{P}_{n-1}(I_M)$. Recursively, we conclude $p_{n-1} = \cdots = p_0 = 0$ and therefore $p \equiv 0$.

For $m \le n$ the proof can be shown similarly without the first step. □

Let us construct a polynomial basis $\{\zeta_i^1\}_{i=0}^{n+1}$ of $\mathcal{P}_{n+1}(I_M)$ by

$$N_i^1(\zeta_j^1) = \delta_{ij}, \quad i,j = 0, \dots, n+1.$$

Remark 2.2. The global degrees of freedom are orthogonal to all lower order polynomials, i.e.

$$N_i^1(p_{i-1}) = 0, \quad p_{i-1} \in \mathcal{P}_{i-1}(I_M), \quad i = 1, \dots, n+1.$$

As a consequence each basis function ζ_i^1 is a polynomial of degree i.

Now, we define the $1d$-recovery operator $\mathbf{P}^1 : L_1(I_M) \to \mathcal{P}_{n+1}(I_M)$ by means of the nodal functionals (2.4) in the canonical way

$$\mathbf{P}^1 v \in \mathcal{P}_{n+1}(I_M) : \; N_k^1(\mathbf{P}^1 v) = N_k^1(v), \quad k = 0, \ldots, n+1. \tag{2.5}$$

Lemma 2.3. *The recovery operator is consistent in the sense of*

$$\mathbf{P}^1 v = \mathbf{P}^1 \pi v, \quad \textit{for all } v \in L_1([-1,1]).$$

Proof. We show, that $N_k^1(\mathbf{P}^1 v) = N_k^1(\pi v)$, $k = 0, \ldots, n+1$. Then by definition (2.5) the lemma follows. Using (2.3)–(2.5) we get

$$\begin{aligned} N_k^1(\mathbf{P}^1 v) = N_k^1(v) &= R_k(v) \; + L_k(v) \\ &= R_k(\pi v) + L_k(\pi v) = N_k^1(\pi v), \quad k = 0, \ldots, n \\ N_{n+1}^1(\mathbf{P}^1 v) = N_{n+1}^1(v) &= R_{n+1}(v) \; - L_{n+1}(v) \\ &= R_{n+1}(\pi v) - L_{n+1}(\pi v) = N_{n+1}^1(\pi v). \end{aligned} \qquad \square$$

Remark 2.4. Although the definition of $\mathbf{P}^1 u$ in (2.5) and πu in (2.3) are similar, the recovery operator is not the L_2-projection onto $\mathcal{P}_{n+1}(I_M)$. Indeed, let $u = \operatorname{sign}(x) \in \mathcal{P}_0^{disc}(\mathcal{T})$. Then, a direct calculation shows that

$$\mathbf{P}^1 u = \begin{cases} 2x, & n = 0, \\ 0, & n > 0, \end{cases}$$

whereas the L_2-projection for $n = 0, \ldots, 3$ satisfies

$$\pi u = \begin{cases} \frac{3}{2} x, & n = 0, 1 \\ \frac{5}{16} x(9 - 7x^2), & n = 2, 3. \end{cases}$$

Moreover, we can show (as in Theorem 3.2 below) that $\mathbf{P}^1 : P_n^{disc}(\mathcal{T}) \to P_{n+1}^{disc}(\mathcal{M})$ is H^1-stable for $n \geq 1$ in the sense

$$\|(\mathbf{P}^1 u)'\|_{0,I_M} \leq C \|u'\|_{0,\mathcal{T}} \quad \forall u \in P_n^{disc}(\mathcal{T}).$$

The L_2-projection does not satisfy this property as one can see from the same example because for $u = \operatorname{sign}(x) \in \mathcal{P}_0^{disc}(\mathcal{T}) \subset \mathcal{P}_1^{disc}(\mathcal{T})$ we have

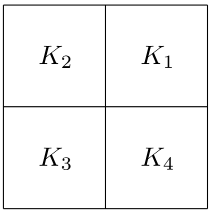
$$\|u'\|_{0,\mathcal{T}} = 0 \quad \text{and} \quad \|(\pi u)'\|_{0,I_M} > 0.$$

3. Postprocessing in $2d$

In this section the main aspect of this article is developed. We start by defining the $2d$-reference elements, see Figure 3.1 and denote the reference mesh by $\mathcal{T} := \{K_1, K_2, K_3, K_4\}$.

The local L_2-projection π into $\mathcal{Q}_n^{disc}(\mathcal{T}) := \{v \in L_2(M) : v|_K \in \mathcal{Q}_n(K), \forall K \in \mathcal{T}\}$ can be defined similarly to (2.3) by introducing appropriate nodal functionals.

For simplifying the following notation, define a bijective index mapping $k : \{0, \ldots, n+1\}^2 \to \{1, \ldots, (n+2)^2\}$. Furthermore, we modify the notations R_i, L_i and N_i from Section 2, such that an additional x or y in the subindex indicates the direction of integration.



$$M = I_M \times I_M = [-1,1]^2,$$
$$K_1 = I_R \times I_R, \quad K_2 = I_L \times I_R,$$
$$K_3 = I_L \times I_L, \quad K_4 = I_R \times I_L$$

Fig. 3.1. Subdomains of the reference macro element M in $2d$

Using tensor products of the $1d$-degrees of freedom in (2.4), we propose the $2d$-degrees of freedom for $i,j = 0,\ldots,n+1$ by

$$N^2_{k(i,j)}(v) = (N^1_{i,x} \circ N^1_{j,y})(v) = (N^1_{j,y} \circ N^1_{i,x})(v). \tag{3.1}$$

For example, we have

$$(N^1_{1,x} \circ N^1_{0,y})(v) = \int_{-1}^{+1}\int_{0}^{+1} (2x-1)v(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{-1}^{+1}\int_{-1}^{0} (2x+1)v(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Lemma 3.1. *The set $\{N^2_k\}_{k=1}^{(n+2)^2}$ is $\mathcal{Q}_{n+1}(M)$-unisolvent and there exists a polynomial basis $\{\zeta^2_l\}_{l=0}^{(n+2)^2}$ of $\mathcal{Q}_{n+1}(M)$ with*

$$N^2_k(\zeta^2_l) = \delta_{kl}, \quad k,l = 1,\ldots,(n+2)^2.$$

Proof. The unisolvence follows directly from Lemma 2.1 and definition (3.1). Thus, a polynomial basis with the proclaimed property exists and moreover, its basis functions can be rewritten using the $1d$ counterparts

$$\zeta^2_{k(i,j)} = \zeta^1_i(x)\zeta^1_j(y), \quad i,j = 0,\ldots,n+1. \tag{3.2}$$

Now we are ready to define the $2d$-recovery operator $\mathbf{P}^2 : L_1(M) \to \mathcal{Q}_{n+1}(M)$ by

$$\mathbf{P}^2 v \in \mathcal{Q}_{n+1}(M) : \; N^2_k(\mathbf{P}^2 v) = N^2_k(v), \quad k = 1,\ldots,(n+2)^2. \qquad \square \tag{3.3}$$

Theorem 3.2 (Consistency and stability) *The operator $\mathbf{P}^2$ is consistent in the sense of*

$$\mathbf{P}^2 v = \mathbf{P}^2 \pi v, \quad \textit{for all } v \in L_1(M)$$

and stable for a differential operator D^γ with $|\gamma| = m \le n$, i.e.

$$\|D^\gamma \mathbf{P}^2 v\|_{0,\mathcal{T}} \le C\|D^\gamma v\|_{0,\mathcal{T}}, \quad \forall v \in \mathcal{Q}^{disc}_n(\mathcal{T}).$$

Proof. Consistency for the $2d$-projection follows directly from the $1d$-projection (Lemma 2.3). In order to prove stability, we introduce

$$Q_i := \{v \in \mathcal{Q}^{disc}_n(\mathcal{T}) : v|_{K_i} \in \mathcal{Q}_n(K_i),\ v|_{M\setminus K_i} = 0\},$$
$$\widetilde{Q}_i := \{v \in Q_i : D^\gamma v \equiv 0\}.$$

According to the direct sum $\mathcal{Q}^{disc}_n(\mathcal{T}) = Q_1 \oplus \cdots \oplus Q_4$, we decompose a function $v \in \mathcal{Q}^{disc}_n(\mathcal{T})$ into $v = v_1 + v_2 + v_3 + v_4$ with $v_i \in Q_i$. Then, we get

$$N^2_{k(i,j)}(v_1) = (R_{i,x} \circ R_{j,y})(v_1)$$

and using the local basis (2.2)

$$\begin{aligned} v_1(x,y) &= \sum_{i,j=0}^{n} (R_{i,x} \circ R_{j,y})(v_1)\psi_i(x)\psi_j(y) \\ &= \sum_{i,j=0}^{n} N^2_{k(i,j)}(v_1)\psi_i(x)\psi_j(y) \quad \forall (x,y) \in K_1. \end{aligned}$$

Splitting the differential operator D^γ into

$$D^\gamma \zeta^2_{k(i,j)}(x,y) = D^{\gamma_1}\zeta^1_i(x) D^{\gamma_2}\zeta^1_j(y), \quad i,j = 0,\dots,n+1$$

and taking into consideration

$$N^2_{k(i,j)}(\mathbf{P}^2 v_1) = N^2_{k(i,j)}(v_1),$$

we obtain

$$\begin{aligned} D^\gamma \mathbf{P}^2 v_1 = 0 &\Rightarrow \sum_{i=0}^{n+1}\sum_{j=0}^{n+1} N^2_{k(i,j)}(v_1) D^{\gamma_1}\zeta^1_i(x) D^{\gamma_2}\zeta^1_j(y) = 0 \\ &\Rightarrow \sum_{i=|\gamma_1|}^{n+1}\sum_{j=|\gamma_2|}^{n+1} N^2_{k(i,j)}(v_1) D^{\gamma_1}\zeta^1_i(x) D^{\gamma_2}\zeta^1_j(y) = 0 \end{aligned}$$

because of $D^\ell \zeta^1_i = 0$ for $\ell > i$. From the linear independence of the basis functions and their derivatives we conclude

$$N^2_{k(i,j)}(v_1) = 0, \quad i = |\gamma_1|,\dots,n+1,\ j = |\gamma_2|,\dots,n+1.$$

Therefore, we have that

$$D^\gamma \mathbf{P}^2 v_1 = 0 \Rightarrow D^\gamma v_1 = \sum_{i=0}^{n}\sum_{j=0}^{n} N^2_{k(i,j)}(v_1) D^{\gamma_1}\psi^1_i(x) D^{\gamma_2}\psi^1_j(y) = 0,$$

and

$$\|D^\gamma \mathbf{P}^2 v_1\|_{0,\mathcal{T}} = 0 = \|D^\gamma v_1\|_{0,K_1}, \quad \forall v_1 \in \widetilde{Q}_1.$$

The mapping $v_1 \mapsto \|D^\gamma \mathbf{P}^2 v_1\|_{0,\mathcal{T}}$ is consequently a norm on the factor space $Q_1 \setminus \widetilde{Q}_1$. The equivalence of norms in finite dimensional spaces yields

$$\|D^\gamma \mathbf{P}^2 v_1\|_{0,\mathcal{T}} \le C\|D^\gamma v_1\|_{0,\mathcal{T}} = C\|D^\gamma v_1\|_{0,K_1}, \quad \forall v_1 \in Q_1 \setminus \widetilde{Q}_1.$$

In a similar manner we get

$$\|D^\gamma \mathbf{P}^2 v_i\|_{0,\mathcal{T}} \le C\|D^\gamma v_i\|_{0,K_i}, \quad i = 2,3,4.$$

Thus

$$\|D^\gamma \mathbf{P}^2 v\|^2_{0,\mathcal{T}} \le 4\sum_{i=1}^{4} \|D^\gamma \mathbf{P}^2 v_i\|^2_{0,\mathcal{T}} \le C\sum_{i=1}^{4} \|D^\gamma v_i\|^2_{0,K_i} = C\|D^\gamma v\|^2_{0,\mathcal{T}}. \qquad \square$$

Theorem 3.3 (Anisotropic error estimates) *Let γ be a multi-index with $|\gamma| = m \leq l \leq n+2$. If $v \in H^\gamma(\mathcal{T})$ with $D^\gamma v \in H^{l-m}(\mathcal{T})$, then the interpolation error of the recovery operator $\mathbf{P}^2$ can be estimated by*

$$\|D^\gamma(v - \mathbf{P}^2 v)\|_{0,\mathcal{T}} \leq C[D^\gamma v]_{l-m,\mathcal{T}}$$

where $[.]_{l-m,\mathcal{T}}$ denotes the seminorm in $H^{l-m}(\mathcal{T})$ including only pure derivatives.

Proof. In order to apply [2, Lemma 2.14], we define d=dim$(D^\gamma Q_{n+1}(M))$ linear functionals F_k fulfilling

$$F_k \in (H^{l-m}(M))', \qquad k = 1, \ldots, d \tag{3.4}$$

$$F_k(D^\gamma(u - Iu)) = 0, \quad k = 1, \ldots, d \tag{3.5}$$

$$\{w \in \mathcal{Q}_{n+1}(M) \,:\, F_k(D^\gamma w) = 0, \quad k = 1, \ldots, d\} \Rightarrow D^\gamma w = 0. \tag{3.6}$$

For the given differential operator $D^\gamma = D_x^{\gamma_1} D_y^{\gamma_2}$, let $J^\gamma = J_x^{\gamma_1} J_y^{\gamma_2}$ be an integrational operator, defined by

$$D^\gamma J^\gamma = (D_x^{\gamma_1} J_x^{\gamma_1})(D_y^{\gamma_2} J_y^{\gamma_2}) = id.$$

Note that

$$\begin{aligned} J^\gamma D^\gamma v &= (J_x^{\gamma_1} D_x^{\gamma_1})(J_y^{\gamma_2} D_y^{\gamma_2}) v = (J_x^{\gamma_1} D_x^{\gamma_1})(v + f_1(x) p_{\gamma_2 - 1}(y)) \\ &= v + f_1(x) p_{\gamma_2 - 1}(y) + f_2(y) p_{\gamma_1 - 1}(x) \end{aligned} \tag{3.7}$$

with arbitrary functions $f_1, f_2 \in C(I_M)$ and polynomials $p_i \in \mathcal{P}_i(I_M)$.

Furthermore, we need an index set of those basis functions that do not vanish under D^γ. Using (3.2) we have

$$D^\gamma \zeta^2_{k(i,j)} = D_x^{\gamma_1} \zeta_i^1(x) D_y^{\gamma_2} \zeta_j^1(y).$$

With ζ_i^1 being a polynomial of degree i we can define the set by

$$D^\gamma \zeta^2_{k(i,j)} \not\equiv 0 \;\Leftrightarrow\; (i,j) \in I := \{(l,m) \in (0, \ldots, n+1)^2 : \gamma_1 \leq l \leq n+1, \gamma_2 \leq m \leq n+1\}.$$

Note that I has exactly d components. We formally define the d linear functionals to be given by

$$F_{i,j}(v) := N^2_{k(i,j)}(J^\gamma v), \quad (i,j) \in I. \tag{3.8}$$

Integrating by parts we will see that the $F_{i,j}$ are uniquely defined, linear and continuous on $L_1(M)$ such that (3.4) is satisfied. We illustrate this in the one-dimensional case.

Due to (2.4) it is enough to show that $v \to R_i J^{\gamma_1} v$ and $v \to L_i J^{\gamma_1} v$ are linear and continuous for $i \geq \gamma_1$. This can be shown using the formula of Rodriquez for Legendre polynomials of order i on $(-1,1)$

$$L_i(s) = \frac{1}{2^i i!} \frac{d^i}{ds^i}((s^2-1)^i).$$

Then with $\eta_i(t) = L_i(2t-1)$ we have for $i \geq \gamma_1$

$$\begin{aligned} R_i(J^{\gamma_1} v) &= \int_0^1 L_i(2t-1)(J^{\gamma_1} v)(t)\,\mathrm{d}t = \frac{1}{i!} \int_0^1 \frac{d^i}{dt^i} \left((t(t-1))^i\right) (J^{\gamma_1} v)(t)\,\mathrm{d}t \\ &= \frac{(-1)^{\gamma_1}}{i!} \int_0^1 \frac{d^{i-\gamma_1}}{dt^{i-\gamma_1}} \left((t(t-1))^i\right) \left(\frac{d^{\gamma_1}}{dt^{\gamma_1}} J^{\gamma_1} v\right)(t)\,\mathrm{d}t \\ &= \frac{(-1)^{\gamma_1}}{i!} \int_0^1 \frac{d^{i-\gamma_1}}{dt^{i-\gamma_1}} \left((t(t-1))^i\right) v(t)\,\mathrm{d}t. \end{aligned}$$

Thus $R_i J^{\gamma_1}$ is linear and continuous on $L_1(I_R)$. Note that the argument relies on the fact that we can integrate by parts and the factor $D_t^{i-\gamma_1}(t(t-1))^i$ vanishes at the boundaries of the integral for $0 \le \gamma_1 - 1 \le i$. In the 2d case the d linear functionals $F_{i,j}$ can also be defined directly as sums of weighted mean values over the subdomains $K_1, \dots, K_4$ such that

$$F_{i,j}(D^\gamma v) = N^2_{k(i,j)}(v) \quad \forall (i,j) \in I,\ \forall v \in L_1(M) \text{ with } D^\gamma v \in L_1(M). \tag{3.9}$$

For example, if $\gamma = (1,0)$, then

$$F_{1,0}(v) = -\int_{-1}^{+1}\int_{0}^{+1} x(x-1)v(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_{-1}^{+1}\int_{-1}^{0} x(x+1)v(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$F_{1,0}(D^{(1,0)}v) = N^2_{k(1,0)}(v) \quad \forall v \in L_1(M) \text{ with } D^{(1,0)}v \in L_1(M).$$

For the formally defined functionals (3.8) we have (3.9) by (3.7) and Remark 2.2.

Now we are ready to prove (3.5). We have

$$F_{i,j}(D^\gamma(v - \mathbf{P}^2 v)) = N^2_{k(i,j)}(v) - N^2_{k(i,j)}(\mathbf{P}^2 v) = 0 \quad \text{ for all } (i,j) \in I$$

due to (3.9) and the interpolation property.

Now let $v = \sum_{k=1}^{(n+2)^2} v_k \zeta_k^2 \in \mathcal{Q}_{n+1}(M)$ be an arbitrary polynomial in $\mathcal{Q}_{n+1}(M)$. Then we have with (3.9)

$$F_{i,j}(D^\gamma v) = N^2_{k(i,j)}(v) = v_{k(i,j)} \quad \text{ for all } (i,j) \in I.$$

Therefore, (3.6) holds with

$$\{F_{i,j}(D^\gamma v) = 0,\ (i,j) \in I\} \Rightarrow D^\gamma v = \sum_{(i,j)\in I} \underbrace{v_{k(i,j)}}_{=0} D^\gamma \zeta^2_{k(i,j)} + \sum_{(i,j)\notin I} v_{k(i,j)} \underbrace{D^\gamma \zeta^2_{k(i,j)}}_{=0} = 0.$$

With [2, Lemma 2.14] we are done. □

Remark 3.4. Stability and error estimates for the 1d case considered in Section 2 can be derived similarly. Moreover, using the ideas presented in this section, recovery can be constructed to any space dimension. This is different from the case when consistency of the recovery with respect to a continuous interpolant is required [1].

Remark 3.5. The recovery operator presented operates on a highly structured tensor-product mesh. The idea of using locally defined momentum-degrees of freedom and combining them in an appropriate way can also be applied to unstructured meshes. Nevertheless, in this case the analysis becomes much more complicated and we are not aware of any supercloseness result on unstructured meshes.

4. Application: Recovery of Q_1^{disc}-functions

In this section we apply our recovery operator to the supercloseness results of [12]. The model problem therein is the singularly perturbed equation

$$-\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu = f, \quad \text{in } \Omega = (0,1)\times(0,1) \tag{4.1a}$$

$$u = 0, \quad \text{on } \Gamma = \partial\Omega \tag{4.1b}$$

with a small perturbation parameter $0 < \varepsilon \ll 1$ and sufficiently smooth data $\mathbf{b}$, c and f. If $c_0^2 = c - \frac{1}{2}\nabla \cdot \mathbf{b} \geq \gamma_0 > 0$ then a unique solution $u \in H_0^1(\Omega)$ of (4.1) exists.

Let us assume $\mathbf{b} \geq (\beta_1, \beta_2) > 0$. Then, u exhibits exponential boundary layers at $x = 1$ and $y = 1$. In order to construct layer-adapted meshes and prove uniform convergence it is convenient to have a solution decomposition. Under a certain compatibility conditions on the data, the following assumption is satisfied, see [8, Theorem 5.1].

Assumption 4.1. *Assume that the solution u of* (4.1) *can be decomposed as*

$$u = S + E_1 + E_2 + E_3$$

where for all $(x, y) \in \Omega$ and $0 \leq i + j \leq 2$ we have the pointwise estimates

$$\left.\begin{aligned} \left|\partial_x^i \partial_y^j S(x,y)\right| \leq C, \qquad & \left|\partial_x^i \partial_y^j E_1(x,y)\right| \leq C\varepsilon^{-i} e^{-\beta_1(1-x)/\varepsilon}, \\ & \left|\partial_x^i \partial_y^j E_2(x,y)\right| \leq C\varepsilon^{-i} e^{-\beta_2(1-y)/\varepsilon}, \\ & \left|\partial_x^i \partial_y^j E_3(x,y)\right| \leq C\varepsilon^{-(i+j)} e^{-\beta_1(1-x)/\varepsilon} e^{-\beta_2(1-y)/\varepsilon}, \end{aligned}\right\} \tag{4.2}$$

and for $0 \leq i + j \leq 3$ the L_2 bounds

$$\left.\begin{aligned} \left\|\partial_x^i \partial_y^j S\right\|_{0,\Omega} \leq C, \qquad & \left\|\partial_x^i \partial_y^j E_1\right\|_{0,\Omega} \leq C\varepsilon^{-i+1/2}, \\ \left\|\partial_x^i \partial_y^j E_2\right\|_{0,\Omega} \leq C\varepsilon^{-j+1/2}, \qquad & \left\|\partial_x^i \partial_y^j E_3\right\|_{0,\Omega} \leq C\varepsilon^{-i-j+1}. \end{aligned}\right\} \tag{4.3}$$

For convenience of the reader, we now recall the properties of the recovery operator in the case of bilinears.

Lemma 4.2. *Let M be a given macroelement with width h_M and height k_M and $\mathcal{T}$ the mesh consisting of the four rectangles of M. Then for bilinears on M, the recovery operator $\mathbf{P}^2 : L_1(M) \to Q_2(M)$ defined in* (3.3) *has the following properties.*

1. *consistency:*
   $\mathbf{P}^2 u = \mathbf{P}^2 \pi u$ *for all $u \in L_1(M)$, with the local L_2-projection $\pi u \in Q_1^{disc}(\mathcal{T})$*
2. *stability:*
   $\|\mathbf{P}^2 u^N\|_{0,\mathcal{T}} \leq \|u^N\|_{0,\mathcal{T}}$,
   $\|\nabla(\mathbf{P}^2 u^N)\|_{0,\mathcal{T}} \leq \|\nabla(u^N)\|_{0,\mathcal{T}}$, *for all $u^N \in Q_1^{disc}(\mathcal{T})$*
3. *anisotropic error-estimates:*
$$\begin{aligned} \|\mathbf{P}^2 u - u\|_{0,\mathcal{T}} &\leq C\left(h_M^3 \|u_{xxx}\|_{0,\mathcal{T}} + k_M^3 \|u_{yyy}\|_{0,\mathcal{T}}\right), \\ \|(\mathbf{P}^2 u - u)_x\|_{0,\mathcal{T}} &\leq C\left(h_M^2 \|u_{xxx}\|_{0,\mathcal{T}} + k_M^2 \|u_{xyy}\|_{0,\mathcal{T}}\right), \\ \|(\mathbf{P}^2 u - u)_y\|_{0,\mathcal{T}} &\leq C\left(h_M^2 \|u_{xxy}\|_{0,\mathcal{T}} + k_M^2 \|u_{yyy}\|_{0,\mathcal{T}}\right) \textit{ for all } u \in H^3(M). \end{aligned}$$

4.1. Content of [12] and Supercloseness

In order to resolve the layer-structure, let $\mathcal{T}(\Omega)$ denote a Shishkin mesh over Ω with transition points

$$\lambda_x = \min\left\{\frac{1}{2}, \frac{\alpha}{\beta_1}\varepsilon \ln N\right\} \quad \text{and} \quad \lambda_y = \min\left\{\frac{1}{2}, \frac{\alpha}{\beta_2}\varepsilon \ln N\right\}.$$

Here α is a user-chosen parameter specified later and N the number of cells in each direction. Usually, ε is very small and it is appropriate to assume $\lambda_x < 1/2$ and $\lambda_y < 1/2$. Moreover, let $\mathcal{T}(D)$ be the part of the mesh, that covers $D \subset \Omega$.

Denote by $\Omega_c := (0, 1-\lambda_x) \times (0, 1-\lambda_y)$ and $\Omega_f := \Omega \setminus \Omega_c$ the parts of Ω, where the mesh is coarse (large isotropic rectangles) resp. fine (anisotropic rectangles with at least one small side).

The standard Galerkin method on a Shishkin mesh with $\alpha \geq 5/2$ is superconvergent in a discrete weighted H^1-norm [17] but results into linear systems that are hard to solve. Therefore, stabilisation methods are applied. In [12] the non-symmetric version of discontinuous Galerkin (dG) on Ω_c is used to stabilise. In the layer region, i.e. in Ω_f, the mesh is fine enough and no stabilisation is needed. In order to review the method and its properties, we give some notations from [12].

Let us define the broken Sobolev space over $D \subset \Omega$

$$H^1(D, \mathcal{T}(D)) = \{v \in L_2(D) : v|_K \in H^1(K), \forall K \in \mathcal{T}(D)\}.$$

For an element $K \in \mathcal{T}$, let ∂K denote the union of all open edges of K and μ_K the unit outward normal vector. We also define the inflow and outflow parts of ∂K by

$$\partial_{-K} = \{(x,y) \in \partial K : \mathbf{b}(x,y) \cdot \mu_K(x,y) < 0\},$$
$$\partial_{+K} = \{(x,y) \in \partial K : \mathbf{b}(x,y) \cdot \mu_K(x,y) \geq 0\},$$

respectively.

For our notations, assume $v \in H^1(\Omega, \mathcal{T})$. Let $\mathcal{E}$ be the set of all edges of $\mathcal{T}$, and $\mathcal{E}_{int} \subset \mathcal{E}$ the set of all inner edges. For each $e \in \mathcal{E}_{int}$ there are indices i and j, such that $i > j$, and $K := K_i$ and $K' := K_j$ share the interface e. On $e \in \mathcal{E}_{int}$, we define the jump across e, the mean value on e and the unit outward normal vector by

$$[v]_e = v|_{\partial K \cap e} - v|_{\partial K' \cap e}, \quad \langle v \rangle_e = \frac{1}{2}(v|_{\partial K \cap e} + v|_{\partial K' \cap e}) \quad \text{and} \quad \nu = \mu_K = -\mu_{K'}$$

respectively. For $e \in \partial K \cap \Gamma$ set

$$[v]_e = \langle v \rangle_e = v \quad \text{and} \quad \nu = \mu_K.$$

For any element $K \in \mathcal{T}$, we denote by v_K^+ the inner trace of $v|_K$ on ∂K. If $\partial_{-K} \setminus \Gamma \neq \emptyset$ then there is a $K' \in \mathcal{T}$ such that $\partial_{-K} = \partial_{+K'}$. In this case, we define the outer trace by $v_K^- = v_{K'}^+$ and the orientated jump of v across $\partial_{-K} \setminus \Gamma$ by

$$\lfloor v \rfloor_K = v_K^+ - v_K^-.$$

In order to simplify the notation, we omit the indices in the terms $[v]_e$, $\langle v \rangle_e$ and $\lfloor v \rfloor_K$.

Our ansatz space will be

$$V := \left\{ v \in H^1(\Omega, \mathcal{T}(\Omega)) : H^1(\Omega_c, \mathcal{T}(\Omega_c)), v|_{\Omega_f} \in C(\Omega_f), v|_{\Gamma \cap \partial \Omega_f} = 0 \right\}.$$

This space is slightly different than the one used in [12], where $V = H^1(\Omega, \mathcal{T}(\Omega))$ with zero boundary conditions on the whole boundary Γ. The local interpolation in Ω_c will be the local L_2-projection into the discontinuous polynomial space without restrictions to the boundary values. Therefore, we include the boundary conditions weakly in the bilinear form and not in the ansatz space.

Let $\Gamma_T = \partial\Omega_c \cap \partial\Omega_f$. Then the bilinear form reads

$$\begin{aligned} a(w,v) := &\sum_{K\subset\Omega} \Big((\varepsilon\nabla w, \nabla v)_K + (\mathbf{b}\cdot\nabla w + cw, v)_K \Big) \\ &- \sum_{K\subset\Omega_c} \left(\int_{\partial_- K\cap\partial\Omega_c} (\mathbf{b}\cdot\mu_K) w^+ v^+ \,\mathrm{d}s + \int_{\partial_- K\setminus\partial\Omega_c} (\mathbf{b}\cdot\mu_K) \lfloor w \rfloor v^+ \,\mathrm{d}s \right) \\ &- \sum_{K\subset\Omega_f} \int_{\partial_- K\cap\Gamma_T} (\mathbf{b}\cdot\mu_K) \lfloor w \rfloor v^+ \,\mathrm{d}s \\ &+ \sum_{e\subset\partial\Omega_c} \varepsilon \int_e \Big([w](\nabla v\cdot\nu) - (\nabla w\cdot\nu)[v] \Big) \,\mathrm{d}s + \sum_{e\subset\Omega_c} \int_e \sigma_e [w][v] \,\mathrm{d}s \\ &+ \sum_{e\subset\Omega_{c,int}} \varepsilon \int_e \Big([w]\langle\nabla v\cdot\nu\rangle - \langle\nabla w\cdot\nu\rangle[v] \Big) \,\mathrm{d}s. \end{aligned}$$

Herein $(w,v)_D$ denotes the usual L_2-scalar product over $D\subset\Omega$ and $\Omega_{c,int}$ the set containing all inner edges $e\in\mathcal{E}_{int}$ that belong to $\Omega_c\setminus\partial\Omega_c$. The penalization parameter σ_e is used for interior jumps and violation of boundary conditions.

Thus, in Ω_f the standard Galerkin bilinear form is used and in Ω_c the discontinuous Galerkin bilinear form in the non-symmetric version.

The conforming piecewise bilinear finite element space $V^N\subset V$ is

$$V^N := \big\{ v\in L_2(\Omega)\cap C(\Omega_f) : v|_{\Gamma\cap\partial\Omega_f} = 0,\ v|_K\in Q_1(K),\ \forall K\in\mathcal{T} \big\}$$

and the discrete problem reads: Find $u^N\in V^N$ such that

$$a(u^N, v^N) = \sum_{K\in\mathcal{T}} (f, v^N)_K, \quad \text{for all } v^N\in V^N. \tag{4.4}$$

Using

$$\|w\|_e^2 := \int_e |\mathbf{b}\cdot\mu_K| w^2 \,\mathrm{d}s, \text{ for all } e\subset\partial K$$

we define the dG-norm

$$\begin{aligned} |||w|||_{dG}^2 := a(w,w) = &\sum_{K\subset\Omega} \left(\varepsilon\|\nabla w\|_{0,K}^2 + \|c_0 w\|_{0,K}^2 \right) + \sum_{e\subset\Omega_c} \int_e \sigma_e [w]^2 \,\mathrm{d}s \\ &+ \frac{1}{2} \sum_{K\subset\Omega_c} \left(\|w^+\|_{\partial_- K\cap\partial\Omega_c}^2 + \|w^+ - w^-\|_{\partial_- K\setminus\partial\Omega_c}^2 \right) \\ &+ \frac{1}{2} \sum_{K\subset\Omega_f} \|w^+ - w^-\|_{\partial_- K\cap\Gamma_T}^2 \end{aligned}$$

and the energy norm

$$|||w|||_\varepsilon^2 := \sum_{K\subset\Omega} \left(\varepsilon\|\nabla w\|_{0,K}^2 + \|c_0 w\|_{0,K}^2 \right).$$

Because of the different discretizations on Ω_c and Ω_f, respectively, we use a mixed local interpolation operator Πu, defined by

$$(\Pi u)|_K = \begin{cases} \pi u, & K\subset\Omega_c, \\ u^I, & K\subset\Omega_f, \end{cases} \tag{4.5}$$

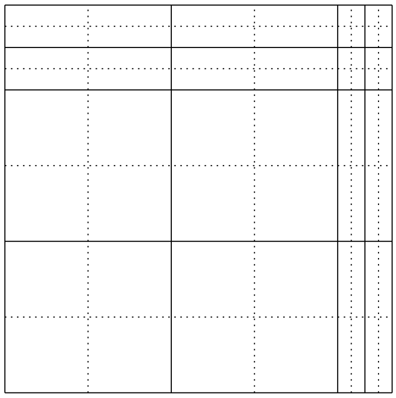


Fig. 4.1. Macro mesh $\mathcal{M}$ constructed from $\mathcal{T}$

with the local L_2-projection πu and the nodal bilinear interpolation u^I into V^N.

With $\Omega^*_{c,int} := \Omega_{c,int} \cap ([0,(1-\lambda_x)(1-2/N)] \times [0,(1-\lambda_y)(1-2/N)])$ denoting the set of all inner edges having no points on $\partial\Omega_c \cap \partial\Omega_f$, the main Theorem of [12] now states

Theorem 4.3. *Let u and u^N be the solutions of the continuous problem* (4.1) *and the discrete problem* (4.4)*, respectively.*

With Assumption 4.1*, $\varepsilon^{1/2}\ln^2 N \le C$, $\alpha \ge 5/2$ and*

$$\sigma_e = \begin{cases} \varepsilon N, & e \subset \Omega^*_{c,int} \\ 1, & e \subset \Omega_{c,int} \setminus \Omega^*_{c,int} \end{cases}$$

we have

$$|||\Pi u - u^N|||_{dG} \le C(\varepsilon^{1/2}N^{-1} + N^{-3/2}). \tag{4.6}$$

Remark 4.4. The minor changes in applying weak enforcement of the boundary conditions in the coarse region give rise to changes in the estimates on the convective part [12, Eqs.(16),(17)], but resulting in the same bounds.

The third sum in the bilinear form allows us to use differences over the transition line in the dG-norm instead of sums. In the supercloseness analysis this additional term can be estimated with second order by Cauchy-Schwarz inequalities and estimates of the interpolation/projection error.

4.2. Recovery

For recovery let N be divisible by 4. We build a macro mesh based on disjoint macrorectangles M, each consisting of 2 by 2 neighbouring rectangles of $\mathcal{T}$. Moreover, the transition lines $(1-\lambda_x) \times [0,1]$ and $[0,1] \times (1-\lambda_y)$ shall not be crossed, see Figure 4.1.

Similarly to the local interpolation operator (4.5), the recovery operator $\mathbf{P}$ is defined locally by

$$(\mathbf{P}u)|_M := \begin{cases} T_M^{-1} \circ \mathbf{P}^2(T_M(u|_M)), & M \subset \Omega_c, \\ P_c(u|_M), & M \subset \Omega_f \end{cases}$$

with the recovery operator $\mathbf{P}^2$ from (3.3), T_M the mapping from M to the reference macro and the nodal biquadratic interpolation $P_c(u|_M)$ used in [13].

Lemma 4.5 (Interpolation error) *Under Assumption* 4.1 *and* $\alpha \geq 5/2$ *we have*

$$|||\mathbf{P}u - u|||_{\varepsilon} \leq C(\varepsilon N^{-3/2} + N^{-2}\ln^2 N). \tag{4.7}$$

Proof. The recovery operator on Ω_f is the one used in [13]. Thus [13, Lemma 5.5] can be applied and to obtain

$$|||P_c u - u|||_{\varepsilon,\Omega_f} \leq CN^{-2}\ln^2 N \tag{4.8}$$

for the layer region, although Assumption 4.1 is weaker then [13, Assumption 5.1]. A closer look at the proof of [13, Lemma 5.5] reveals Assumption 4.1 to be sufficient.

In Ω_c we bound the different parts of u separately. Application of Theorem 3.3, the transformation T_K and the fact that the maximal mesh size is smaller than $2N^{-1}$ gives

$$\left|\left|\left|\mathbf{P}^2 S - S\right|\right|\right|_{\varepsilon,\Omega_c} \leq C(\varepsilon^{1/2}N^{-2} + N^{-3}). \tag{4.9}$$

For the layers parts, let E be any of E_1, E_2 or E_3. We apply the triangle inequality and obtain

$$\left|\left|\left|\mathbf{P}^2 E - E\right|\right|\right|_{\varepsilon,\Omega_c} \leq \left|\left|\left|\mathbf{P}^2 E\right|\right|\right|_{\varepsilon,\Omega_c} + |||E|||_{\varepsilon,\Omega_c}.$$

The second term is bounded by Assumption 4.1 with

$$|||E|||_{\varepsilon,\Omega_c} \leq CN^{-\alpha}. \tag{4.10}$$

To estimate $\|\mathbf{P}^2 E_1\|_{0,\Omega_c}$, let $M = [x_{2i}, x_{2(i+1)}] \times [y_{2j}, y_{2(j+1)}]$ with $i,j = 0,\ldots,N/4-1$. Then

$$\begin{aligned}
&\int_M (\mathbf{P}^2 E_1)^2 \,\mathrm{d}x\,\mathrm{d}y \\
=&\int_M \Big(\sum_{r,s=0}^{2} \frac{N^2_{k(r,s)}(E_1)}{\text{meas }(M)}\zeta^2_{k(r,s)}\Big)^2 \mathrm{d}x\,\mathrm{d}y \leq C\int_M \Big(\sum_{r,s=0}^{2} \frac{N^2_{k(r,s)}(E_1)}{\text{meas }(M)}\Big)^2 \mathrm{d}x\,\mathrm{d}y \\
\leq& C\int_M \Big(\sum_{r,s=0}^{2} \|E_1\|_{L_\infty(M)}\Big)^2 \mathrm{d}x\,\mathrm{d}y \leq C\int_M e^{-2\beta_1(1-x_{2(i+1)})/\varepsilon}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\mathbf{P}^2 E_1\|^2_{0,\Omega_c} &= \int_0^{1-\lambda_y} \sum_{i=0}^{N/4-1} \int_{x_{2i}}^{x_{2(i+1)}} (\mathbf{P}^2 E_1)^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\leq C\int_0^{1-\lambda_y} \sum_{i=0}^{N/4-1} \int_{x_{2i}}^{x_{2(i+1)}} e^{-2\beta_1(1-x_{2(i+1)})/\varepsilon}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

We bound the integrals in the sum for $i < N/4-1$ and $i = N/4-1$ differently.

$$\begin{aligned}
&i < N/4-1 : e^{-2\beta_1(1-x_{2(i+1)})/\varepsilon} \leq e^{2\beta_1(x_{N/2}-x_{N/2-2})/\varepsilon} e^{-2\beta_1(1-x)/\varepsilon}, \text{ with } x \in [x_{2i}, x_{2(i+1)}] \\
&i = N/4-1 : e^{-2\beta_1(1-x_{2(i+1)})/\varepsilon} = e^{-2\beta_1\lambda_x/\varepsilon} = N^{-2\alpha}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|\mathbf{P}^2 E_1\|^2_{0,\Omega_c} &\leq C\left(e^{2\beta_1(x_{N/2}-x_{N/2-2})/\varepsilon}\int_0^{x_{N/2-2}} e^{-2\beta_1(1-x)/\varepsilon}\,\mathrm{d}x + N^{-1}N^{-2\alpha}\right) \\
&\leq C(\varepsilon N^{-2\alpha} + N^{-(2\alpha+1)}).
\end{aligned} \tag{4.11}$$

Similar results are obtained for $\|\mathbf{P}^2 E_2\|^2_{0,\Omega_c}$ and $\|\mathbf{P}^2 E_3\|^2_{0,\Omega_c}$. Combining them with (4.9) and (4.10) we get

$$\|\mathbf{P}^2 u - u\|_{0,\Omega_c} \leq C(\varepsilon^{1/2} N^{-2} + N^{-3} + N^{-\alpha}). \tag{4.12}$$

For the derivatives of $\mathbf{P}^2 E$ we use locally an inverse inequality

$$\|\nabla(\mathbf{P}^2 E)\|_{0,M} \leq CN\|\mathbf{P}^2 E\|_{0,M}$$

and apply (4.11) to get

$$\varepsilon^{1/2}\|\nabla(\mathbf{P}^2 E)\|_{0,\Omega_c} \leq C\varepsilon^{1/2} N\|\mathbf{P}^2 E\|_{0,\Omega_c} \leq C(\varepsilon N^{-\alpha+1} + \varepsilon^{1/2} N^{-\alpha+1/2}). \tag{4.13}$$

Combining (4.8)–(4.10), (4.12), and (4.13) with $\alpha \geq 5/2$ we are done. $\square$

Theorem 4.6 (Superconvergence) *Under the assumptions of Theorem* 4.3 *and Lemma* 4.5*, the postprocessed numerical solution and the solution of* (4.1) *satisfy*

$$\left|\left|\left|u - \mathbf{P}u^N\right|\right|\right|_\varepsilon \leq C(\varepsilon^{1/2} N^{-1} + N^{-3/2}).$$

Proof. With the triangle inequality and Lemma 4.2 we get

$$\left|\left|\left|u - \mathbf{P}u^N\right|\right|\right|_\varepsilon \leq |||u - \mathbf{P}u|||_\varepsilon + C\left|\left|\left|\Pi u - u^N\right|\right|\right|_\varepsilon.$$

The first term is bounded by the interpolation error of Lemma 4.5 and the second by the supercloseness property of Theorem 4.3 and the energy norm being part of the dG-norm. $\square$

4.3. Numerical example

We consider two test problems, given by

Problem I

$$\begin{aligned} -\varepsilon\Delta u + (3-x)u_x + (3+(1-y)^3)u_y + u = f &\quad \text{in } \Omega = (0,1)^2 \\ u = 0 &\quad \text{on } \partial\Omega \end{aligned}$$

with given right hand side f, such that

$$u = \cos((1-x)\pi/2)(1 - \exp(-2(1-x)/\varepsilon))y^3(1 - \exp(-3(1-y)/\varepsilon))$$

is the exact solution and

Problem II

$$\begin{aligned} -\varepsilon\Delta u + (1+x)u_x + (21-y)u_y + (1+xy)u = f &\quad \text{in } \Omega = (0,1)^2 \\ u = 0 &\quad \text{on } \partial\Omega \end{aligned}$$

with given right hand side f, such that

$$u = \sin^2 x(1 - \exp(-2(1-x)/\varepsilon))y(1 + \sin(4\pi y)/2)(1 - \exp(-20(1-y)/\varepsilon))$$

Table 4.1: Errors for test problem I, $\varepsilon = 10^{-8}$

N	$\vert\vert\vert u-u^N\vert\vert\vert_{dG}$	EOC	$\vert\vert\vert \Pi u-u^N\vert\vert\vert_{dG}$	EOC	$dGt5(\Pi u-u^N)$	EOC	$\vert\vert\vert u-\mathbf{P}u^N\vert\vert\vert_{\varepsilon}$	EOC
32	1.739e-01	0.72	2.426e-02	1.45	7.334e-03	1.44	2.740e-02	1.58
64	1.056e-01	0.77	8.907e-03	1.53	2.707e-03	1.47	9.172e-03	1.61
128	6.189e-02	0.81	3.077e-03	1.59	9.778e-04	1.48	3.004e-03	1.63
256	3.542e-02	0.83	1.023e-03	1.62	3.494e-04	1.49	9.705e-04	1.65
512	1.994e-02	0.85	3.321e-04	1.64	1.242e-04	1.50	3.089e-04	1.67
1024	1.108e-02	0.86	1.067e-04	1.64	4.403e-05	1.50	9.724e-05	1.67
2048	6.093e-03		3.430e-05		1.559e-05		3.057e-05	

Table 4.2: Errors for test problem II, $\varepsilon = 10^{-8}$

N	$\vert\vert\vert u-u^N\vert\vert\vert_{dG}$	EOC	$\vert\vert\vert \Pi u-u^N\vert\vert\vert_{dG}$	EOC	$dGt5(\Pi u-u^N)$	EOC	$\vert\vert\vert u-\mathbf{P}u^N\vert\vert\vert_{\varepsilon}$	EOC
32	2.374e-01	0.80	9.571e-02	1.42	8.953e-02	1.41	3.546e-02	1.60
64	1.361e-01	0.80	3.572e-02	1.48	3.380e-02	1.47	1.168e-02	1.63
128	7.814e-02	0.82	1.278e-02	1.50	1.220e-02	1.49	3.787e-03	1.64
256	4.438e-02	0.83	4.515e-03	1.51	4.344e-03	1.50	1.215e-03	1.66
512	2.491e-02	0.85	1.588e-03	1.51	1.540e-03	1.50	3.845e-04	1.68
1024	1.382e-02	0.86	5.584e-04	1.51	5.451e-04	1.50	1.203e-04	1.68
2048	7.600e-03		1.964e-04		1.928e-04		3.760e-05	

is the exact solution.

The parameter ε is chosen to be 10^{-8}—sufficiently small to bring out the singularly perturbed nature of the test problem. The Shishkin parameter is $\alpha = 3$.

Tables 4.1 and 4.2 show the errors for the numerical test problems. For each column with errors the estimated orders of convergence corresponding to

$$E_N = CN^{-EOC}$$

are given. We clearly see convergence in the dG-norm of order $N^{-1}\ln N$ in the first column of both tables. Theorem 4.3 predicts the rates in the second column for $|||\Pi u - u^N|||_{dG}$ to be $3/2$. In Table 4.1 we see a better rate, but a closer look at the components of the dG-norm reveals the fifth part

$$dGt5(w) := \left(\sum_{K\subset\Omega_c} \|w^+ - w^-\|^2_{\partial_{-K}\setminus\partial\Omega_c} \right)^{1/2}$$

to be only of order $3/2$, as can be seen in the third column. The second test problem was selected, such that this part is dominating the energy-norm. Thus, in Table 4.2 the predicted rate $3/2$ in the second column can be seen.

Nevertheless, the errors in the energy norm of the postprocessed solution given in the last column indicate superconvergence of order $N^{-2}\ln^2 N$, although Theorem 4.6 predicts only order $3/2$ due to the supercloseness of Theorem 4.3.

Acknowledgment. This research was supported by the Ministry of Science and Technological Development of the Republic of Serbia, grant 144006.

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