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STABILITY ANALYSIS OF YEE SCHEMES TO PML AND UPML FOR MAXWELL EQUATIONS*

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Abstract

We utilize Fourier methods to analyze the stability of the Yee difference schemes for Bérenger PML (perfectly matched layer) as well as the UPML (uniaxial perfectly matched layer) systems of two-dimensional Maxwell equations. Using a practical spectrum stability concept, we find that the two schemes are spectrum stable under the same conditions for mesh sizes. Besides, we prove that the UPML schemes with the same damping in both directions are stable. Numerical examples are given to confirm the stability analysis for the PML method.

Mathematics subject classification: 65M12.
Key words: Computational electromagnetic, Fourier method, Yee scheme, PML, UPML, Stability.

1. Introduction

A general approach in computational electromagnetic of finding infinite space solutions is to introduce an absorbing boundary condition in the outer lattices boundary to simulate the extension of the lattice to infinity. While an alternative approach is to terminate the outer boundary of the space lattice in an absorbing material medium, the difficulty is that such an absorbing layer is matched only to normally incident plane wave [1].

In 1994, Bérenger published a pioneer paper about the so-called perfectly matched layer (PML) method. By splitting the field, plane waves of arbitrary incidence, polarization, and frequency are matched at the boundary [2]. Since then, the PML has been very popular, and many works from both engineering and mathematical points have been carried out in the fields.

According to Chew and Weedon's observation, the system of the PML medium can be obtained by a complex change of independent variables, which is the famous UPML [5]. Moreover, Sack *et al.* imposed a physical model with perfectly matched medium on an anisotropic parameters without splitting the fields [6]. These two formulations are also mathematically identical, and the fact was proved by Zhao and Cangellaris, provided that the electric and magnetic fields presented in the Chew-Weedon stretched-coordinate formulation are properly defined [9]. Bramble and Pasciak have shown the existence and uniqueness of solutions to the truncated time harmonic PML problem provided that the truncated domain is sufficiently large [10]. Bao

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and Wu have given the convergence analysis in spherical coordinates for a three-dimensional electromagnetic scattering problem and established an explicit error estimate between the solution of the scattering problem and that of the truncated PML problem [11]. Chen and Liu have developed an adaptive PML technique for solving the time harmonic scattering problems. Numerical experiments are included to illustrate the competitive behavior of the proposed adaptive method [12].

Although too many works have been reported about the PML and its successful applications, still many questions are deserved to investigate. The stability of PML is one among them in spite of some known results.

It is well known that the Maxwell's equations are hyperbolic and symmetric; as a result, the initial value problem is well-posed. On the other hand, the Bérenger's PML is not symmetric which may causes instability. However, the damping term is a "good" ingredient to improve stability. The first analysis of the stability is due to the work of Abarbanel and Gottlieb [3]. They studied the split system of equations and proved that if damping parameter $\sigma = 0$ the initial value problem of the system is weakly stable, namely, the L^2-norm of the solutions depend not only on the L^2-norm of the initial data but also on the L^2-norm of their derivatives. They also studied the Yee's scheme to this problem and showed that the numerical solutions to the split TE model with $\sigma = 0$ grows linearly with the time step n, therefore the scheme is unstable. It has been proved that the PML is weakly stable for $\sigma > 0$ as well, for example see [13]. In order to overcome the weakly well-posedness, a lot of authors have designed new modified PMLs, such as, Lions *et al* have imposed a new type of absorbing layer for Maxwell equations and the linearized Euler equations, which is also valid for several classes of first order hyperbolic system, and the associated Cauchy problems are proved well-posed [8]. Besides, Zhao and Cangellaris in [9] proposed a modified PML by restoring the usual operator with a new introduced unknown without splitting fields. Becache and Joly [7] made a thorough investigation for the problem of stability. They proved the weak stability, and the equivalence between the PML and the system in [9]. Moreover, they proved the stability of the initial value problem of the system in [9] for all $\sigma \geq 0$, and the stability of the Yee's scheme to this problem for all $\sigma \geq 0$, too.

It looks like a contradiction that different works generate stable, unstable, or weakly stable results to the same model. In fact, the properties of stability relate to different unknowns in different formulations. For example, the Bérenger's PML is a 4×4 system for the TE mode with unknowns E_x, E_y, H_x and H_y, and the formulation by Zhao and Cangellaris is a 4×4 system with unknowns $E_x, E_y, \tilde{E}_x$ and H_z. Consequently the L^2-norm stability proved in [7] does not apply to the Bérenger's PML directly. Particularly if $\sigma = 0$, the Yee's scheme to the PML is unstable, while the scheme to the formulation by Zhao and Cangellaris is stable.

In this paper, we are interested in the stability property of the Yee's scheme to the PMLs for the case of $\sigma \geq 0$. We will show that the damping parameter $\sigma > 0$ can improve the behavior of stability. The scheme is no longer unstable but stable in a weaker sense which will be called spectrum stability. For the UPML, it may be stable in some cases.

Regarding the stability analysis of PML methods, there are some other related works need to be mentioned. Some more general formulations of PML have been derived by Appelo, Hagstrom and Kreiss, and their stability is analyzed by using Schur criterion in the continuous setting [15]. Ying has considered an exterior initial-boundary value problem of TM mode by truncating the domain with the UPML, and obtained the existence and uniqueness of the weak formulation [14]. Later on, Ying and Fang have analyzed the corresponding FDTD initial-boundary value problem on the truncated domain and proved the stability [16].

This following contents are organized as follows: Firstly in section 2, we recall the Bérenger PML as well as the UPML for two-dimensional Maxwell equations and some energy norm estimate results. Then in section 3, the Yee algorithms are applied to the PML system as well as the UPML system. Via Fourier transform, we get the amplification matrices of the difference equations. By analyzing the location of the latent roots of the amplification matrices, we find that the two discrete systems are spectrum stable under the same mesh sizes conditions. Moreover, we prove that the UPML scheme is stable with the same damping in both directions. We at last in section 4 give some numerical examples, and the results show that the Yee difference schemes are stable for the PML systems with proper initial boundary conditions.

2. PML and UPML Systems

In the following discussions, we will set the equations of a PML medium for the two-dimensional TE (transverse electric) mode. The results for the TM (transverse magnetic) mode are similar.

Consider a problem in Cartesian coordinates without variation along z, with the electric field lying in the (x, y) plane. The electromagnetic field involves three components E_x,E_y,H_z, and the simplified dimensionless Maxwell equations in a medium reduce to a set of three equations:

$$\frac{\partial E_y}{\partial t} + \sigma_1 E_y = \frac{\partial H_z}{\partial x}, \tag{2.1}$$

$$\frac{\partial E_x}{\partial t} + \sigma_2 E_x = \frac{\partial H_z}{\partial y}, \tag{2.2}$$

$$\frac{\partial H_z}{\partial t} + \rho^* H_z = \frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}. \tag{2.3}$$

Following Bérenger's idea [2], the orthogonal magnetic field H_z is split into nonphysical components H_x and H_y, i.e:, $H_z = H_x + H_y$. The system (2.1)-(2.3) becomes the following 4×4 PML system,

$$\frac{\partial E_y}{\partial t} + \sigma_1 E_y = \frac{\partial (H_x + H_y)}{\partial x}, \qquad \frac{\partial E_x}{\partial t} + \sigma_2 E_x = \frac{\partial (H_x + H_y)}{\partial y}, \tag{2.4}$$

$$\frac{\partial H_x}{\partial t} + \sigma_1 H_x = \frac{\partial E_y}{\partial x}, \qquad \frac{\partial H_y}{\partial t} + \sigma_2 H_y = \frac{\partial E_x}{\partial y}. \tag{2.5}$$

Let $U = (H_z, E_y, E_x)^T$, and

$$E(t) = \int_{\mathbb{R}^2} \left(\left|\frac{\partial U}{\partial t}\right|^2 + \left|\frac{\partial U}{\partial x}\right|^2 + \left|\frac{\partial U}{\partial y}\right|^2 \right) dxdy, \quad \|H_x(t)\|_0^2 = \int_{\mathbb{R}^2} H_x^2 \, dxdy.$$

The following results have been proved in or extended from Chapter 5 of [13] by energy methods.

Theorem 2.1. *If the solution (U, H_x) to the system (2.4)-(2.5) with $\sigma_1 = \sigma_2 = 0$ belongs to the Sobolev space $H^1([0,\infty); H^1(\mathbb{R}^2))$, then the following properties are valid:*

$$E(t) = E(0), \tag{2.6}$$

$$\|H_x(t)\|_0 \le \|H_x(0)\|_0 + \int_0^t E(\tau)^{\frac{1}{2}} \, d\tau. \tag{2.7}$$

Theorem 2.2. *If the solution (U, H_x) to the system (2.4)-(2.5) belongs to the space $H^1([0,\infty); H^1(\mathbb{R}^2))$, then the following properties are valid:*

$$\begin{aligned} &E(t) \le C(\sigma_1,\sigma_2)e^{C\,(\sigma_1+\sigma_2)t}\left(E(0)+\|H_x(0)\|_0^2+\|H_y(0)\|_0^2\right),\\ &\|H_x(t)\|_0 \le e^{-\sigma_1 t}\|H_x(0)\|_0 + \int_0^t E(\tau)^{\frac{1}{2}}\,d\tau, \\ &\|H_y(t)\|_0 \le e^{-\sigma_2 t}\|H_y(0)\|_0 + \int_0^t E(\tau)^{\frac{1}{2}}\,d\tau, \end{aligned} \tag{2.8}$$

where $C(\sigma_1,\sigma_2)$ depends on σ_1 and σ_2.

Remark 1. The estimates (2.7)-(2.8) are classified as a "weak stability", since the L^2-norms of H_x and H_y are bounded not only by the L^2-norms of the initial data, but also by the L^2-norms of the derivatives of the initial data. Similar results have also been obtained by using the energy methods in [7]. This property may cause instabilities in numerical computations, see Example 1 in Section 4.

Bérenger's PML method yields only a weakly stable system [3]. It is desirable to replace it to a form without splitting the field, that is the uniaxial perfectly matched layer [5].

Consider, for example, a UPML in a square domain $\{a < |\alpha| < a+d\}$ which surrounds the original Maxwell equations in $\{|\alpha| < a\}$, where α can be either x or y. d is the thickness of medium, and a is the length of truncated domain. Now we introduce a complex change of independent variables which are continuous on the interface,

$$\alpha = \begin{cases} \alpha, & |\alpha| < a, \\ a + (1+i\frac{\sigma}{\omega})(\alpha - a), & \alpha > a, \\ -a + (1+i\frac{\sigma}{\omega})(\alpha + a), & \alpha < -a. \end{cases} \tag{2.9}$$

By use of Fourier transform, it is easy to get the UPML system in different domains [14]. For $\sigma_2 = 0$, the UPML system is

$$\frac{\partial H_z}{\partial t} = \frac{\partial e_y}{\partial x} + \frac{\partial E_x}{\partial y}, \quad \frac{\partial E_y}{\partial t} = \frac{\partial h_z}{\partial x}, \tag{2.10a}$$

$$\frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y}, \quad \frac{\partial H_z}{\partial t} = \frac{\partial h_z}{\partial t} + \sigma_1 h_z, \quad \frac{\partial E_y}{\partial t} = \frac{\partial e_y}{\partial t} + \sigma_1 e_y. \tag{2.10b}$$

For $\sigma_1 = \sigma_2 = \sigma$, the UPML system is

$$\frac{\partial E_x}{\partial t} = \frac{\partial h_z}{\partial y}, \quad \frac{\partial E_y}{\partial t} = \frac{\partial h_z}{\partial x}, \quad \frac{\partial H_z}{\partial t} = \frac{\partial e_y}{\partial x} + \frac{\partial e_x}{\partial y}, \tag{2.11a}$$

$$\frac{\partial E_x}{\partial t} = \frac{\partial e_x}{\partial t} + \sigma e_x, \quad \frac{\partial E_y}{\partial t} = \frac{\partial e_y}{\partial t} + \sigma e_y, \quad \frac{\partial H_z}{\partial t} = \frac{\partial h_z}{\partial t} + \sigma h_z. \tag{2.11b}$$

The variables E_x, E_y, and H_z in (2.10) and (2.11) are the original ones in (2.2)-(2.3). They are linked with e_x, e_y, h_z by relationships

$$\hat{H}_z = \left(1 + i\frac{\sigma}{\omega}\right)\hat{h}_z, \quad \hat{E}_y = \left(1 + i\frac{\sigma}{\omega}\right)\hat{e}_y, \quad \hat{E}_x = \left(1 + i\frac{\sigma}{\omega}\right)\hat{e}_x,$$

and symbol " ˆ " represents the corresponding Fourier transform about the time and the spatial variables [14].

3. Fourier Analysis to the Yee Schemes

3.1. Yee Scheme to the PML System

A very practical difference scheme in computational electromagnetics is due to Yee. The algorithm solves both electric and magnetic fields in time and space using the coupled Maxwell's curl equations [1]. We now apply the Yee algorithm to the PML equations (2.4)-(2.5). In order to represent different systems, we introduce a parameter θ $(0 \le \theta \le 1)$ for the damping terms, then get the discrete system as (3.1)-(3.4). Since the Yee algorithm is a leapfrog scheme in time and central-difference in space, the scheme is of second-order accurate if $\theta = \frac{1}{2}$:

$$\frac{E_y|_{i,j+\frac{1}{2}}^{n+1} - E_y|_{i,j+\frac{1}{2}}^{n}}{\tau} + \sigma_1 \left(\theta E_y|_{i,j+\frac{1}{2}}^{n} + (1-\theta) E_y|_{i,j+\frac{1}{2}}^{n+1}\right) = \delta_x \left(H_x|_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} + H_y|_{i,j+\frac{1}{2}}^{n+\frac{1}{2}}\right), \quad (3.1)$$

$$\frac{E_x|_{i+\frac{1}{2},j}^{n+1} - E_x|_{i+\frac{1}{2},j}^{n}}{\tau} + \sigma_2 \left(\theta E_x|_{i+\frac{1}{2},j}^{n} + (1-\theta) E_x|_{i+\frac{1}{2},j}^{n+1}\right) = \delta_y \left(H_x|_{i+\frac{1}{2},j}^{n+\frac{1}{2}} + H_y|_{i+\frac{1}{2},j}^{n+\frac{1}{2}}\right), \quad (3.2)$$

$$\frac{H_x|_{i+\frac{1}{2},j+\frac{1}{2}}^{n+\frac{1}{2}} - H_x|_{i+\frac{1}{2},j+\frac{1}{2}}^{n-\frac{1}{2}}}{\tau} + \sigma_1 \left(\theta H_x|_{i+\frac{1}{2},j+\frac{1}{2}}^{n-\frac{1}{2}} + (1-\theta) H_x|_{i+\frac{1}{2},j+\frac{1}{2}}^{n+\frac{1}{2}}\right) = \delta_x E_y|_{i+\frac{1}{2},j+\frac{1}{2}}^{n}, \quad (3.3)$$

$$\frac{H_y|_{i+\frac{1}{2},j+\frac{1}{2}}^{n+\frac{1}{2}} - H_y|_{i+\frac{1}{2},j+\frac{1}{2}}^{n-\frac{1}{2}}}{\tau} + \sigma_2 \left(\theta H_y|_{i+\frac{1}{2},j+\frac{1}{2}}^{n-\frac{1}{2}} + (1-\theta) H_y|_{i+\frac{1}{2},j+\frac{1}{2}}^{n+\frac{1}{2}}\right) = \delta_y E_x|_{i+\frac{1}{2},j+\frac{1}{2}}^{n}, \quad (3.4)$$

where

$$E_\alpha|_{i,j}^{n} = E_\alpha|_{x=ih_1,y=jh_2}^{t=n\tau}, \quad H_\alpha|_{i,j}^{n+\frac{1}{2}} = H_\alpha|_{x=ih_1,y=jh_2}^{t=(n+\frac{1}{2})\tau},$$

τ is the time step length, h_1, h_2 are the mesh sizes, α represents x, y; δ_α is the finite difference operator, e.g.,

$$\delta_x H_x|_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{h_1}\left(H_x|_{i+\frac{1}{2},j+\frac{1}{2}}^{n+\frac{1}{2}} - H_x|_{i-\frac{1}{2},j+\frac{1}{2}}^{n+\frac{1}{2}}\right),$$

(i,j) can be either the integer grids or the shifted grids. The Fourier symbols δ_x and δ_y are, respectively

$$\delta_x \to \frac{2i\sin\theta_1}{h_1} = i\frac{k_1}{\tau}, \qquad \delta_y \to \frac{2i\sin\theta_2}{h_2} = i\frac{k_2}{\tau},$$

where $-\pi/2 \le \theta_1, \theta_2 \le \pi/2$.

The system (3.1)-(3.4) expressed in matrix form is $\mathbf{A}\, U_{n+1} = \mathbf{B} U_n$, where

$$\mathbf{A} = \begin{pmatrix} 1+\sigma_1\tau(1-\theta) & 0 & -ik_1 & -ik_1 \\ 0 & 1+\sigma_2\tau(1-\theta) & -ik_2 & -ik_2 \\ 0 & 0 & 1+\sigma_1\tau(1-\theta) & 0 \\ 0 & 0 & 0 & 1+\sigma_2\tau(1-\theta) \end{pmatrix},$$

$$U_{n+1} = \begin{pmatrix} E_y|_{i,j+\frac{1}{2}}^{n+1} & E_x|_{i+\frac{1}{2},j}^{n+1} & H_x|_{i+\frac{1}{2},j}^{n+\frac{1}{2}} & H_y|_{i+\frac{1}{2},j}^{n+\frac{1}{2}} \end{pmatrix}^T,$$

$$\mathbf{B} = \begin{pmatrix} 1-\sigma_1\tau\theta & 0 & 0 & 0 \\ 0 & 1-\sigma_2\tau\theta & 0 & 0 \\ ik_1 & 0 & 1-\sigma_1\tau\theta & 0 \\ 0 & ik_2 & 0 & 1-\sigma_2\tau\theta \end{pmatrix},$$

$$U_n = \begin{pmatrix} E_y|_{i,j+\frac{1}{2}}^{n} & E_x|_{i+\frac{1}{2},j}^{n} & H_x|_{i+\frac{1}{2},j}^{n-\frac{1}{2}} & H_y|_{i+\frac{1}{2},j}^{n-\frac{1}{2}} \end{pmatrix}^T.$$

A natural way to show the stability of the schemes is to judge the uniform bound of the amplification matrices. Unfortunately, we have found that the Kreiss theorem, J-conditions [17] and Buchanan theorem [18] are all too difficult to be applied here.

As is well-known, the Von Neumann condition that the spectrum radius is less than or equal to $1 + \mathcal{O}(\tau)$ is only a necessary condition for the stability of a difference scheme

$$U_{n+1} = GU_n, \quad n = 0, 1, \cdots. \quad (3.5)$$

The amplification matrix G can be decomposed as $G = T^{-1}JT$, where J is the Jordan Matrix of G, and T is the matrix which are made up of the eigenvectors as the corresponding rows. Setting up $V_n = TU_n$, the difference scheme (3.5) turns to

$$V_{n+1} = JV_n, \quad n = 0, 1, \cdots . \tag{3.6}$$

If the maximum mode of the latent roots of G is less than 1, or equal to 1 with G having complete eigenvectors, then $V_n = J^n V_0$ and $\|V_n\|_{l_2} \le C$, where C is independent of n. Consequently, we can introduce a weak stability concept, namely spectrum stability.

Spectrum Stability: The difference scheme

$$U_{n+1} = GU_n, \quad n = 0, 1, \cdots$$

is spectrum stable if the maximum mode of the latent roots of G is less than 1, or equal to 1 with G having complete eigenvectors.

Using the spectrum stability concept, we consider some particular cases of system (3.1)-(3.4) in this section, and numerical examples for more general cases will be shown in the next section.

First, let $\sigma_2 = 0$ and $\theta = 1$. That is a PML with explicit damping terms in one direction case. The amplification matrix is

$$\mathbf{G} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} 1 - {k_1}^2 - \tau\,\sigma_1 & -k_1k_2 & ik_1\,(1-\tau\sigma_1) & ik_1 \\ -k_1k_2 & 1 - {k_2}^2 & ik_2\,(1-\tau\sigma_1) & ik_2 \\ ik_1 & 0 & 1-\tau\sigma_1 & 0 \\ 0 & ik_2 & 0 & 1 \end{pmatrix}. \tag{3.7}$$

The characteristic polynomial of $\mathbf{G}$ is

$$\begin{aligned} \phi(x) =& x^4 + x^3\,\left(-4 + {k_1}^2 + {k_2}^2 + 2\,\tau\,\sigma_1\right) \\ &+ x^2\,\left(6 - 2\,{k_1}^2 - 2\,{k_2}^2 - 6\,\tau\,\sigma_1 + 2\,\tau\,{k_2}^2\,\sigma_1 + \tau^2\,{\sigma_1}^2\right) \\ &+ x\,\left(-4 + {k_1}^2 + {k_2}^2 + 6\,\tau\,\sigma_1 - 2\,\tau\,{k_2}^2\,\sigma_1 - 2\,\tau^2\,{\sigma_1}^2 + \tau^2\,{k_2}^2\,{\sigma_1}^2\right) \\ &+ 1 - 2\,\tau\,\sigma_1 + \tau^2\,{\sigma_1}^2. \end{aligned} \tag{3.8}$$

Since the latent roots can not be effectively expressed, we need the following criterion, which will be useful for the location of roots of polynomials.

Miller-Schur Criterion (see [4], p.77) A k-th degree polynomial

$$\phi(x) = c_k x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$$

with complex coefficients c_s, $s = 0, 1, \cdots, k$, where $c_k \neq 0, c_0 \neq 0$, is said to be a Schur polynomial if its roots x_s satisfy $|x_s| < 1$. Define the polynomial

$$\varphi(x) = c_0^* x^k + c_1^* x^{k-1} + \cdots + c_{k-1}^* x + c_k^*, \tag{3.9}$$

where c_s^* is the complex conjugate of c_s, and the polynomial

$$\phi_1(x) = \frac{1}{x}\bigg(\varphi(0)\phi(x) - \phi(0)\varphi(x)\bigg) \tag{3.10}$$

which has degree at most $k-1$. Then $\phi(x)$ is a Schur polynomial if and only if $|\varphi(0)| > |\phi(0)|$ and that $\phi_1(x)$ is a Schur polynomial.

According to the Miller-Schur Criterion, we start with $\phi(x)$ and construct the polynomial series $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ and $\varphi(x)$, $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$ step by step.

• First Step: From (3.8) and (3.9), we obtain

$$\begin{aligned}\varphi(x) =&1 + x^4(1 - 2\,\tau\,\sigma_1 + \tau^2\,\sigma_1^2)\\ &+ x\,\left(-4 + k_1^2 + k_2^2 + 2\,\tau\,\sigma_1\right)\\ &+ x^2\,\left(6 - 2\,k_1^2 - 2\,k_2^2 - 6\,\tau\,\sigma_1 + 2\,\tau\,k_2^2\,\sigma_1 + \tau^2\,\sigma_1^2\right)\\ &+ x^3\,\left(-4 + k_1^2 + k_2^2 + 6\,\tau\,\sigma_1 - 2\,\tau\,k_2^2\,\sigma_1 - 2\,\tau^2\,\sigma_1^2 + \tau^2\,k_2^2\,\sigma_1^2\right),\end{aligned} \tag{3.11}$$

and the characteristic polynomial (3.8) is a Schur polynomial if and only if $|\varphi(0)| > |\phi(0)|$, i.e:

$$1 > (1 - \tau\,\sigma_1)^2, \tag{3.12}$$

and $\phi_1(x)$ is a Schur polynomial. For $\phi_1(x)$ we go to

• Second Step: From (3.8)-(3.11), we get

$$\begin{aligned}\phi_1(x) =& -\tau\,\sigma_1\,\left(-2 + \tau\,\sigma_1\right)\,\left(-2 + k_1^2 + 2\,\tau\,\sigma_1\right)\\ &+ x^3\,\left(4\,\tau\,\sigma_1 - 6\,\tau^2\,\sigma_1^2 + 4\,\tau^3\,\sigma_1^3 - \tau^4\,\sigma_1^4\right)\\ &- x\,\tau\,\sigma_1\,\left(-2 + \tau\,\sigma_1\right)\,\left(6 - 2\,k_1^2 - 6\,\tau\,\sigma_1 + \tau^2\,\sigma_1^2 + 2\,k_2^2\,\left(-1 + \tau\,\sigma_1\right)\right)\\ &- x^2\,\tau\,\sigma_1\,\left(-2 + \tau\,\sigma_1\right)\,\left(-6 + k_1^2 + 6\,\tau\,\sigma_1 - 2\,\tau^2\,\sigma_1^2 + k_2^2\,\left(p(\tau,\sigma_1)\right)\right),\end{aligned} \tag{3.13}$$

$$\begin{aligned}\varphi_1(x) =& -x^3\,\tau\,\sigma_1\,\left(-2 + \tau\,\sigma_1\right)\,\left(-2 + k_1^2 + 2\,\tau\,\sigma_1\right)\\ &+ \tau\,\sigma_1\,\left(4 - 6\,\tau\,\sigma_1 + 4\,\tau^2\,\sigma_1^2 - \tau^3\,\sigma_1^3\right)\\ &- x^2\,\tau\,\sigma_1\,\left(-2 + \tau\,\sigma_1\right)\,\left(6 - 2\,k_1^2 - 6\,\tau\,\sigma_1 + \tau^2\,\sigma_1^2 + 2\,k_2^2\,\left(-1 + \tau\,\sigma_1\right)\right)\\ &- x\,\tau\,\sigma_1\,\left(-2 + \tau\,\sigma_1\right)\,\left(-6 + k_1^2 + 6\,\tau\,\sigma_1 - 2\,\tau^2\,\sigma_1^2 + k_2^2\,\left(p(\tau,\sigma_1)\right)\right),\end{aligned} \tag{3.14}$$

where

$$p(\tau,\sigma_1) = 2 - 2\tau\sigma_1 + \tau^2\sigma_1^2.$$

Moreover, $\phi_1(x)$ is a Schur polynomial if and only if $|\varphi_1(0)| > |\phi_1(0)|$, i.e:

$$\begin{aligned}&|\tau\,\sigma_1\,\left(-2 + \tau\,\sigma_1\right)\,\left(2 - 2\,\tau\,\sigma_1 + \tau^2\,{\sigma_1}^2\right)|\\ >&|\tau\,\sigma_1\,\left(-2 + \tau\,\sigma_1\right)\,\left(-2 + {k_1}^2 + 2\,\tau\,\sigma_1\right)|,\end{aligned} \tag{3.15}$$

and $\phi_2(x)$ is a Schur polynomial. For $\phi_2(x)$ we go to

• Third Step: The process goes on. From (3.10), (3.9), (3.14) and (3.16)), we have

$$\begin{aligned}\phi_2(x) =&x^2\,\tau^2\,\sigma_1^2\,(-2 + \tau\,\sigma_1)^2\,\left(-k_1^4 + k_1^2\,\left(4 - 4\,\tau\,\sigma_1\right) + \tau^2\,\sigma_1^2\,(-2 + \tau\,\sigma_1)^2\right)\\ &+ x\,\tau^2\,\sigma_1^2\,(-2 + \tau\,\sigma_1)^2\,\{2\,k_1^4 + \tau^2\,\left(-2 + k_2^2\right)\,\sigma_1^2\,(-2 + \tau\,\sigma_1)^2\\ &- 2\,k_1^2\,\left(-4 + k_2^2\right)\,\left(-1 + \tau\,\sigma_1\right)\} + q(\tau,\sigma_1,k_1,k_2),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\varphi_2(x) =&\tau^2\,\sigma_1^2\,(-2 + \tau\,\sigma_1)^2\,\left(-k_1^4 + k_1^2\,\left(4 - 4\,\tau\,\sigma_1\right) + \tau^2\,\sigma_1^2\,(-2 + \tau\,\sigma_1)^2\right)\\ &+ x\,\tau^2\,\sigma_1^2\,(-2 + \tau\,\sigma_1)^2\,\{2\,k_1^4 + \tau^2\,\left(-2 + k_2^2\right)\,\sigma_1^2\,(-2 + \tau\,\sigma_1)^2\\ &- 2\,k_1^2\,\left(-4 + k_2^2\right)\,\left(-1 + \tau\,\sigma_1\right)\} + x^2 q(\tau,\sigma_1,k_1,k_2)\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}q(\tau,\sigma_1,k_1,k_2) =& -\tau^2\,\sigma_1^2\,(-2 + \tau\,\sigma_1)^2\,\{k_1^4 - \tau^2\,\sigma_1^2\,(-2 + \tau\,\sigma_1)^2\\ &+ k_1^2\,\left(-4 + 4\,\tau\,\sigma_1 + k_2^2\,\left(p(\tau,\sigma_1)\right)\right)\},\end{aligned}$$

and $\phi_2(x)$ is a Schur polynomial if and only if $|\varphi_2(0)| > |\phi_2(0)|$, i.e:

$$\begin{aligned}&|\tau^2\,{\sigma_1}^2\,(-2+\tau\,\sigma_1)^2\,\left(-2-k_1+\tau\,\sigma_1\right)\,\left(-2+k_1+\tau\,\sigma_1\right)\,\left({k_1}^2+\tau^2\,{\sigma_1}^2\right)|\\ &>|q(\tau,\sigma_1,k_1,k_2)|,\end{aligned} \tag{3.18}$$

and $\phi_3(x)$ is a Schur polynomial. For $\phi_3(x)$, we need to go to

• Fourth Step: Now it turns to require the first-order polynomial $\phi_3(x)$ be a Schur polynomial, where

$$\begin{aligned}\phi_3(x) =&\tau^4\,k_1^2\,k_2^2\,\sigma_1^4\,(-2+\tau\,\sigma_1)^4\,\left(p(\tau,\sigma_1)\right)\\ &\cdot\left(2\,k_1^4+\tau^2\,\left(-2+k_2^2\right)\,\sigma_1^2\,(-2+\tau\,\sigma_1)^2-2\,k_1^2\,\left(-4+k_2^2\right)\,\left(-1+\tau\,\sigma_1\right)\right)\\ &+x\,\tau^4\,k_1^2\,k_2^2\,\sigma_1^4\,(-2+\tau\,\sigma_1)^4\,\left(p(\tau,\sigma_1)\right)\\ &\cdot\left(-2\,k_1^4+2\,\tau^2\,\sigma_1^2\,(-2+\tau\,\sigma_1)^2-k_1^2\,\left(-8+8\,\tau\,\sigma_1+k_2^2\,\left(p(\tau,\sigma_1)\right)\right)\right),\end{aligned} \tag{3.19}$$

whose root satisfies $|x| < 1$ which is equivalent to

$$\begin{aligned}&\Big|\tau^4\,k_1^2\,k_2^2\,\sigma_1^4\,(-2+\tau\,\sigma_1)^4\,\left(p(\tau,\sigma_1)\right)\\ &\cdot\left(-2\,k_1^4+2\,\tau^2\,\sigma_1^2\,(-2+\tau\,\sigma_1)^2-k_1^2\,\left(-8+8\,\tau\,\sigma_1+k_2^2\,\left(p(\tau,\sigma_1)\right)\right)\right)\Big|\\ >&\Big|\tau^4\,k_1^2\,k_2^2\,\sigma_1^4\,(-2+\tau\,\sigma_1)^4\,\left(p(\tau,\sigma_1)\right)\\ &\left(2\,k_1^4+\tau^2\,\left(-2+k_2^2\right)\,\sigma_1^2\,(-2+\tau\,\sigma_1)^2-2\,k_1^2\,\left(-4+k_2^2\right)\,\left(-1+\tau\,\sigma_1\right)\right)\Big|.\end{aligned} \tag{3.20}$$

To sum up, $\phi(x)$ is a Schur polynomial if and only if (3.12), (3.15), (3.18) and (3.20) are satisfied at the same time. Recurring to the software *Mathematica 5.0*, we solve these inequalities one by one, and then find the intersection to get the inequality solution sets as follows,

$$\begin{cases} 0<\tau<\dfrac{2}{\sigma_1},\\ k_1\in(-2+\sigma_1\tau,\,0)\cup(0,\,2-\sigma_1\tau),\\ k_2\in\left(-\dfrac{2\sqrt{-{k_1}^2+(2-\tau\,\sigma_1)^2}}{2-\tau\,\sigma_1},\,0\right)\cup\left(0,\,\dfrac{2\sqrt{-{k_1}^2+(2-\tau\,\sigma_1)^2}}{2-\tau\,\sigma_1}\right).\end{cases} \tag{3.21}$$

Notice that (3.21) excludes $k_i = 0$, $i = 1, 2$. In this case, the latent roots of the amplification matrix are $|x| = 1$. However, it does not mean the schemes are not spectrum stable.

In fact, for $k_1 = 0$, the latent roots of the amplification matrix (3.7) are

$$\left\{\frac{2-{k_2}^2-k_2\,\sqrt{-4+{k_2}^2}}{2},\frac{2-{k_2}^2+k_2\,\sqrt{-4+{k_2}^2}}{2},1-\tau\,\sigma_1,1-\tau\,\sigma_1\right\}, \tag{3.22}$$

and the corresponding eigenvectors are

$$\begin{aligned}&\left\{0,\frac{i}{2}\,\left(k_2+\sqrt{-4+k_2^2}\right),0,1\right\},\ \left\{0,\frac{i}{2}\,\left(k_2-\sqrt{-4+k_2^2}\right),0,1\right\},\\ &\left\{0,\frac{i\,\tau\,\sigma_1}{k_2},-\left(\frac{-{k_2}^2+\tau\,{k_2}^2\,\sigma_1-\tau^2\,{\sigma_1}^2}{{k_2}^2\,(-1+\tau\,\sigma_1)}\right),1\right\},\ \{1,0,0,0\},\end{aligned} \tag{3.23}$$

when $0 \le k_2^2 < 4$, the maximum mode of the latent roots (3.22) is 1 and the eigenvectors are complete.

For $k_2 = 0$, the latent roots of (3.7) are

$$\left\{1, 1, \frac{2 - {k_1}^2 - 2\,\tau\,\sigma_1 - k_1\sqrt{-4 + {k_1}^2 + 4\,\tau\,\sigma_1}}{2}, \frac{2 - {k_1}^2 - 2\,\tau\,\sigma_1 + k_1\sqrt{-4 + {k_1}^2 + 4\,\tau\,\sigma_1}}{2}\right\}, \tag{3.24}$$

and the corresponding eigenvectors are

$$\left\{\frac{i\,\tau\,k_1\,\sigma_1}{{k_1}^2 + \tau^2\,{\sigma_1}^2}, 0, -\frac{{k_1}^2}{{k_1}^2 + \tau^2\,{\sigma_1}^2}, 1\right\}, \quad \{0, 1, 0, 0\}, \tag{3.25}$$

$$\left\{\frac{i}{2}\left(k_1 + \sqrt{-4 + {k_1}^2 + 4\,\tau\,\sigma_1}\right), 0, 1, 0\right\}, \quad \left\{\frac{i}{2}\left(k_1 - \sqrt{-4 + {k_1}^2 + 4\,\tau\,\sigma_1}\right), 0, 1, 0\right\}.$$

If $\sigma_1\tau < 1$ and $0 \le k_1^2 < 4(1 - \sigma_1\tau)$, the modes of the latent roots of (3.24) satisfy

$$|x_s| \in \{1, 1, 1 - \sigma_1\tau, 1 - \sigma_1\} \le 1, \quad s = 1, 2, 3, 4,$$

and the eigenvectors are complete as well.

Theorem 3.1. *The Yee's scheme* (3.1)-(3.4) *with* $\sigma_2 = 0$, $\theta = 1$ *is spectrum stable under conditions:*

$$0 < \sigma_1\tau < 1, \quad k_1^2 < 4(1 - \sigma_1\tau), \quad k_2^2 < \frac{4((2 - \sigma_1\tau)^2 - k_1^2)}{(2 - \sigma_1\tau)^2}, \tag{3.26}$$

or written in the form of

$$\tau < \sqrt{(1 - \sigma_1\tau)}h_1, \quad (2 - \sigma_1\tau)^2\left(\frac{\tau}{h_2}\right)^2 + 4\left(\frac{\tau}{h_1}\right)^2 < (2 - \sigma_1\tau)^2. \tag{3.27}$$

Then, let $\sigma_2 = 0$ and $\theta = 0$. That is a PML with implicit damping terms in one direction case. Following the idea above we get the characteristic polynomial of the corresponding amplification matrix as

$$\phi^{im}(x) = \frac{1}{(1 + \tau\,\sigma_1)^2}\left((-1 + x)^2\,x\,{k_1}^2 + \{(-1 + x)^2 + x\,{k_2}^2\}\,(-1 + x + x\,\tau\,\sigma_1)^2\right), \tag{3.28}$$

which is a Schur polynomial if and only if

$$\begin{cases} \tau > 0,\ \sigma_1 > 0, \\ k_1 \in (-2 - 2\sigma_1\tau,\ 0) \cup (0,\ 2 + 2\sigma_1\tau), \\ k_2 \in (-\dfrac{2\sqrt{-{k_1}^2 + (2 + 2\tau\,\sigma_1)^2}}{2 + \tau\,\sigma_1},\ 0) \cup (0,\ \dfrac{2\sqrt{-{k_1}^2 + (2 + \tau\,\sigma_1)^2}}{2 + \tau\,\sigma_1}). \end{cases} \tag{3.29}$$

For $k_1 = 0$, the latent roots are

$$x_s = \left\{\frac{2 - {k_2}^2 - k_2\sqrt{-4 + {k_2}^2}}{2}, \frac{2 - {k_2}^2 + k_2\sqrt{-4 + {k_2}^2}}{2}, \frac{1}{1 + \tau\,\sigma_1}, \frac{1}{1 + \tau\,\sigma_1}\right\},$$

for s=1,2,3,4; correspondingly the eigenvectors are

$$\left\{0, \frac{i}{2}\left(k_2 + \sqrt{-4 + {k_2}^2}\right), 0, 1\right\}, \quad \left\{0, \frac{i}{2}\left(k_2 - \sqrt{-4 + {k_2}^2}\right), 0, 1\right\},$$

$$\left\{0, \frac{i\,\tau\,\sigma_1}{k_2\,(1 + \tau\,\sigma_1)}, -\left(\frac{{k_2}^2 + \tau\,{k_2}^2\,\sigma_1 + \tau^2\,{\sigma_1}^2}{{k_2}^2\,(1 + \tau\,\sigma_1)}\right), 1\right\}, \quad \{1, 0, 0, 0\}.$$

If $k_2^2 < 4$, then $\max\{|x_s|\} = 1$, and then it has complete eigenvectors.

For $k_2 = 0$, the latent roots are

$$\left\{1, 1, \frac{-2 - k_1^2 + 2\,\tau\,\sigma_1}{-}\sqrt{k_1^2\,(-4 + k_1^2 - 4\,\tau\,\sigma_1)}2\,(1 + \tau\,\sigma_1)^2,\right.$$

$$\left.\frac{2 - k_1^2 + 2\,\tau\,\sigma_1 + \sqrt{k_1^2\,(-4 + k_1^2 - 4\,\tau\,\sigma_1)}}{2\,(1 + \tau\,\sigma_1)^2}\right\},$$

correspondingly the eigenvectors are

$$\left\{\frac{i\,\tau\,k_1\,\sigma_1}{{k_1}^2+\tau^2\,{\sigma_1}^2},0,-\frac{{k_1}^2}{{k_1}^2+\tau^2\,{\sigma_1}^2},1\right\},\quad \{0,1,0,0\},$$
$$\left\{\frac{i}{k_1}(1+\tau\,\sigma_1)\left(\frac{1}{1+\tau\,\sigma_1}-\frac{2-{k_1}^2+2\,\tau\,\sigma_1-\sqrt{-4\,{k_1}^2+{k_1}^4-4\,\tau\,{k_1}^2\,\sigma_1}}{2\,(1+\tau\,\sigma_1)^2}\right),0,1,0\right\},$$
$$\left\{\frac{i}{k_1}(1+\tau\,\sigma_1)\left(\frac{1}{1+\tau\,\sigma_1}-\frac{2-{k_1}^2+2\,\tau\,\sigma_1+\sqrt{-4\,{k_1}^2+{k_1}^4-4\,\tau\,{k_1}^2\,\sigma_1}}{2\,(1+\tau\,\sigma_1)^2}\right),0,1,0\right\},$$

If ${k_1}^2 < 4\,(1+\tau\,\sigma_1)$, then $\max\{|x_s|\}=1$, and then it has complete eigenvectors as well.

Theorem 3.2. *The Yee's scheme* (3.1)-(3.4) *with* $\sigma_2 = 0, \theta = 0$ *is spectrum stable under conditions:*

$$\sigma_1 > 0, \quad k_1^2 < 4(1+\tau\sigma_1), \quad k_2^2 < \frac{4\{(2+\sigma_1\tau)^2 - k_1^2\}}{(2+\sigma_1\tau)^2}, \tag{3.30}$$

or written in the form of

$$\tau < \sqrt{1+\tau\sigma_1}h_1, \quad (2+\sigma_1\tau)^2\left(\frac{\tau}{h_2}\right)^2 + 4\left(\frac{\tau}{h_1}\right)^2 < (2+\sigma_1\tau)^2. \tag{3.31}$$

We list below some more results with different parameters to the Yee schemes.

Theorem 3.3. *For* $\sigma_2 = 0$ *and* $\theta = 1/2$*, the Yee's scheme* (3.1)-(3.4) *is spectrum stable under conditions:*

$$\sigma_1 > 0, \quad \tau > 0, \quad k_1^2 + k_2^2 < 4, \tag{3.32}$$

or written in an equivalent form of

$$\sigma_1 > 0, \quad \frac{\tau^2}{h_1^2} + \frac{\tau^2}{h_2^2} < 1. \tag{3.33}$$

Theorem 3.4. *Consider the case* $\sigma_1 = \sigma_2 = \sigma$.

1. *If* $\theta = 1$*, the Yee's scheme* (3.1)-(3.4)*, which has characteristic polynomial*

$$\phi^1(x) = (-1+x+\sigma\,\tau)^2\left(1-2\,x+x^2-2\,\sigma\,\tau+2\,x\,\sigma\,\tau+\sigma^2\,\tau^2+x\,{k_1}^2+x\,{k_2}^2\right), \tag{3.34}$$

is spectrum stable under conditions:

$$0 < \sigma\tau < 2, k_1^2 + k_2^2 < (2-\sigma\tau)^2, \tag{3.35}$$

or written in an equivalent form of

$$\sigma > 0, \quad \tau < \frac{2}{\sigma} \quad \text{and} \quad \left(\frac{\tau}{h_1}\right)^2 + \left(\frac{\tau}{h_2}\right)^2 < \frac{(2-\sigma\tau)^2}{4}. \tag{3.36}$$

2. *If* $\theta = 0$*, the Yee's scheme* (3.1)-(3.4)*, whose characteristic polynomial expresses as*

$$\begin{aligned}\phi^2(x) =& (1+\sigma\,\tau)^{-4}(-1+x+x\,\sigma\,\tau)^2\\ &\cdot\left(1-2\,x+x^2-2\,x\,\sigma\,\tau+2\,x^2\,\sigma\,\tau+x^2\,\sigma^2\,\tau^2+x\,{k_1}^2+x\,{k_2}^2\right),\end{aligned} \tag{3.37}$$

is spectrum stable under condition:

$$k_1^2 + k_2^2 < (2+\tau\sigma)^2, \tag{3.38}$$

or written in an equivalent form of

$$\left(\frac{\tau}{h_1}\right)^2 + \left(\frac{\tau}{h_2}\right)^2 < \frac{(2+\sigma\tau)^2}{4}. \tag{3.39}$$

Remark 3. The stablility results from Theorem 3.1 to Theorem 3.4 all naturally exclude the critical condition $\sigma_1 = \sigma_2 = 0$ which has been proved unstable in [3]. We observed that the case $\sigma_1 = \sigma_2 = 0$ happens to correspond to multiple latent roots $|x| = 1$ but without complete eigenvectors. If $h_1 = h_2 = h$ in Theorem 3.3, the spectrum stable condition becomes $\sqrt{2}\tau/h < 1$, which is the same as the CFL condition proved stable in [7]. The difference is that, the spectrum stability is shown for the original variables E_x, E_y, H_x, H_y, while the stability in [7] is proved for the restored unsplit variables $E_x, E_y, \tilde{E}_x, H_z$. In fact, the spectrum stability of the PML schemes can not be improved over the stability for the original variables, because one can observe that for the amplification matrix, when $\tau = 0$ it degenerates to the case as the same as that of $\sigma_1 = \sigma_2 = 0$.

3.2. Yee Scheme to the UPML System

It comes to investigate the Yee difference scheme for the UPML system. First for system (2.10), its Yee difference scheme written in matrix form is

$$\mathbf{C}W_{n+1} = \mathbf{D}W_n, \tag{3.40}$$

where

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -i\tau k_1 \\ -i\tau k_2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1-(1-\theta)\sigma_1\tau \\ 0 & 1 & 0 & -1-(1-\theta)\sigma_1\tau & 0 \end{pmatrix},$$

$$W_{n+1} = \left(\; H_z|^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}, E_y|^{n+1}_{i,j+\frac{1}{2}}, E_x|^{n+1}_{i+\frac{1}{2},j}, e_y|^{n+1}_{i,j+\frac{1}{2}}, h_z|^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \; \right)^T,$$

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & ik_2 & ik_1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1+\theta\sigma_1\tau \\ 0 & 1 & 0 & -1+\theta\sigma_1\tau & 0 \end{pmatrix},$$

$$W_n = \left(\; H_z|^{n-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}, E_y|^{n}_{i,j+\frac{1}{2}}, E_x|^{n}_{i+\frac{1}{2},j}, e_y|^{n}_{i,j+\frac{1}{2}}, h_z|^{n-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \; \right)^T.$$

For $\theta = 1$, the characteristic polynomial of $\mathbf{C}^{-1}\mathbf{D}$ is simplified as

$$\begin{aligned} \tilde{\phi}(x) =& (x-1)\big\{x^4 + x^3\,\left(-4 + {k_1}^2 + {k_2}^2 + 2\,\tau\,\sigma_1\right) \\ &+ x^2\,\left(6 - 2\,{k_1}^2 - 2\,{k_2}^2 - 6\,\tau\,\sigma_1 + 2\,\tau\,{k_2}^2\,\sigma_1 + \tau^2\,{\sigma_1}^2\right) \\ &+ x\,\left(-4 + {k_1}^2 + {k_2}^2 + 6\,\tau\,\sigma_1 - 2\,\tau\,{k_2}^2\,\sigma_1 - 2\,\tau^2\,{\sigma_1}^2 + \tau^2\,{k_2}^2\,{\sigma_1}^2\right) \\ &+ 1 - 2\,\tau\,\sigma_1 + \tau^2\,{\sigma_1}^2\big\} \\ =& (x-1)\phi(x), \end{aligned} \tag{3.41}$$

where $\phi(x)$ is defined in (3.8). The former Miller-Schur criterion process has given the spectrum stability conditions (3.21). We verify critical cases: If $k_1 = 0$, $|x| = 1$ has only a single latent root, and the other latent roots all satisfy $|x| < 1$; if $k_2 = 0$, $|x| = 1$ are three-multiple roots with complete eigenvectors as $(0, 0, 1, 0, 0)^T$, $(0, 1, 0, 0, 0)^T$ and $(1, 0, 0, 0, 0)^T$.

Then for $\theta = 0$, by calculation, we have $\tilde{\phi}(x) = (x-1)\phi^{im}(x)$, where $\phi^{im}(x)$ has been defined in (3.28). Moreover, for the particular cases $k_1 = 0$ or $k_2 = 0$, we have checked that the latent roots satisfy $|x| = 1$ associating with complete eigenvectors. Thus we have

Theorem 3.5. *Consider the Yee scheme* (3.40) *to the UPML system* (2.10)*.*

1. *For* $\theta = 1$*, the scheme is spectrum stable under the conditions*
$$0 < \sigma_1\tau < 2, \quad k_1^2 < 4(1-\sigma_1\tau), \quad k_2^2 < \frac{4((2-\sigma_1\tau)^2 - k_1^2)}{(2-\sigma_1\tau)^2}, \tag{3.42}$$
or written in the equivalent form of
$$\tau < \sqrt{(1-\sigma_1\tau)}h_1, \quad (2-\sigma_1\tau)^2\left(\frac{\tau}{h_2}\right)^2 + 4\left(\frac{\tau}{h_1}\right)^2 < (2-\sigma_1\tau)^2. \tag{3.43}$$

2. *For* $\theta = 0$*, the scheme is spectrum stable under the conditions*
$$\sigma_1 > 0, \quad k_1^2 < 4(1+\tau\sigma_1), \quad k_2^2 < \frac{4\{(2+\sigma_1\tau)^2 - k_1^2\}}{(2+\sigma_1\tau)^2}, \tag{3.44}$$
or written in the form of
$$\tau < \sqrt{1+\tau\sigma_1}h_1, \quad (2+\sigma_1\tau)^2\left(\frac{\tau}{h_2}\right)^2 + 4\left(\frac{\tau}{h_1}\right)^2 < (2+\sigma_1\tau)^2. \tag{3.45}$$

Below we will demonstrate that the spectrum stability of (3.40) can not be improved to strong stability.

Theorem 3.6. (*Theorem* 4.1 *in* [19]) *If the amplification matrix* $G(k_i,\tau)$ *is Lipschitz continuous about* τ *near* $\tau = 0$*, then the difference scheme is stable if and only if the matrix family*
$$\{G_0 = G^n(k_i,0) : 0 < \tau \le \tau_0, 0 < n\tau \le T, i = 1,2\}$$
is uniformly bounded.

In fact, one can easily check that the amplification matrices of (3.40) are Lipschitz continuous about τ near $\tau = 0$. Hence Theorem 3.6 can be applied. When $\tau = 0$,
$$G(k_i,0) = \begin{pmatrix} 1 & 0 & \iota k_2 & -\iota k_1 & 0 \\ 0 & 1 & k_1k_2 & -{k_1}^2 & -\iota k_1 \\ \iota k_2 & 0 & 1-{k_2}^2 & k_1k_2 & 0 \\ 0 & 0 & k_1k_2 & 1-{k_1}^2 & -\iota k_1 \\ 0 & 0 & \iota k_2 & -\iota k_1 & 1 \end{pmatrix},$$
whose eigenvalues are
$$\Lambda_1 = \left\{1,1,1,\frac{2-{k_1}^2-{k_2}^2 \pm \sqrt{-4+\left(-2+{k_1}^2+{k_2}^2\right)^2}}{2}\right\} \triangleq \{1,1,1,v,u\},$$
which can not give complete eigenvectors. Consider the Jordan decomposition
$$G(k_i,0) = SJS^{-1} = S\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & v \end{pmatrix}S^{-1},$$
where
$$S = \begin{pmatrix} 0 & 0 & -\frac{{k_1}^2}{{k_2}^2} & 1 & 1 \\ 0 & -\iota k_1 & 0 & \frac{-\iota u k_1}{-1+u} & \frac{-\iota v k_1}{-1+v} \\ \frac{k_1}{k_2} & \frac{-\iota {k_1}^2}{k_2} & 0 & \frac{\iota u k_2}{-1+u} & \frac{\iota v k_2}{-1+v} \\ 1 & -\iota k_1 & 0 & \frac{-\iota u k_1}{-1+u} & \frac{-\iota v k_1}{-1+v} \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Noticing

$$G^n(k_i,0) = SJ^nS^{-1} = S\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & n & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & v \end{pmatrix} S^{-1},$$

and using

$$\|G^n(k_i,0)\| = \max_{\forall\alpha\neq 0} \frac{\|G^n(k_i,0)\alpha\|}{\|\alpha\|},$$

we have

$$\|G^n(k_i,0)\| \geq \frac{\|G^n(k_i,0)\alpha_0\|}{\|\alpha_0\|}, \quad \forall \alpha_0.$$

Especially, taking $\alpha_0 = S(1,0,0,0,0)^T = (0,0,k_1/k_2,1,0)$, we get

$$\|G^n(k_i,0)\| \geq \frac{\|G^n(k_i,0)\alpha_0\|}{\|\alpha_0\|} \geq \frac{\sqrt{k_1^4/k_2^2 + k_1^6/k_2^4}\; n}{\sqrt{1+(k_1/k_2)^2}}, \quad \forall\, k_1, k_2.$$

Thus $\|G^n(k_i,0)\| \sim n$, which is not uniformly bounded.

For the UPML system (2.11) with the same damping in two directions, the Yee difference scheme is

$$\mathbf{E}R_{n+1} = \mathbf{F}R_n, \tag{3.46}$$

where

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -ik_2 \\ 0 & 1 & 0 & 0 & 0 & -ik_1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1-(1-\theta)\sigma\tau & 0 & 0 \\ 0 & 1 & 0 & 0 & -1-(1-\theta)\sigma\tau & 0 \\ 0 & 0 & 1 & 0 & 0 & -1-(1-\theta)\sigma\tau \end{pmatrix},$$

$$R_{n+1} = \Big(E_x|^{n+1}_{i+\frac12,j}, E_y|^{n+1}_{i,j+\frac12}, H_z|^{n+\frac12}_{i+\frac12,j+\frac12}, e_x|^{n+1}_{i+\frac12,j}, e_y|^{n+1}_{i,j+\frac12}, h_z|^{n+\frac12}_{i+\frac12,j+\frac12} \Big)^T,$$

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & ik_2 & ik_1 & 0 \\ 1 & 0 & 0 & -1+\theta\sigma\tau & 0 & 0 \\ 0 & 1 & 0 & 0 & -1+\theta\sigma\tau & 0 \\ 0 & 0 & 1 & 0 & 0 & -1+\theta\sigma\tau \end{pmatrix},$$

$$R_n = \Big(E_x|^{n}_{i+\frac12,j}, E_y|^{n}_{i,j+\frac12}, H_z|^{n-\frac12}_{i+\frac12,j+\frac12}, e_x|^{n}_{i+\frac12,j}, e_y|^{n}_{i,j+\frac12}, h_z|^{n-\frac12}_{i+\frac12,j+\frac12} \Big)^T.$$

For $\theta = 1$, the characteristic polynomial of the amplification matrix $\mathbf{E}^{-1}\mathbf{F}$ is

$$\begin{aligned}\tilde{\phi}^1(x) =& (-1+x)^3\, (-1+x+\sigma\,\tau) \\ & \left(1 - 2\,x + x^2 - 2\,\sigma\,\tau + 2\,x\,\sigma\,\tau + \sigma^2\,\tau^2 + x\,{k_1}^2 + x\,{k_2}^2\right).\end{aligned} \tag{3.47}$$

While for $\theta = 0$, the characteristic polynomial is

$$\begin{aligned}\tilde{\phi}^2(x) =& (1+\sigma\,\tau)^{-3}(-1+x)^3\, (-1+x+x\,\sigma\,\tau) \\ & \cdot\left(1 - 2\,x + x^2 - 2\,x\,\sigma\,\tau + 2\,x^2\,\sigma\,\tau + x^2\,\sigma^2\,\tau^2 + x\,{k_1}^2 + x\,{k_2}^2\right).\end{aligned} \tag{3.48}$$

And for $\theta = 1/2$, the characteristic polynomial is

$$\begin{aligned}\tilde{\phi}^3(x) =&(2+\sigma\,\tau)^{-3}(-1+x)^3\,\left(-2+2\,x+\sigma\,\tau+x\,\sigma\,\tau\right)\left(4-8\,x+4\,x^2-4\,\sigma\,\tau\right.\\ &\left.+4\,x^2\,\sigma\,\tau+\sigma^2\,\tau^2+2\,x\,\sigma^2\,\tau^2+x^2\,\sigma^2\,\tau^2+4\,x\,{k_1}^2+4\,x\,{k_2}^2\right)\end{aligned} \tag{3.49}$$

It is noted that (3.47) has same factors as those of (3.34), and (3.48) has same factors as those of (3.37). With the Miller-Schur criterion, we get the spectrum stablility conditions. For the particular case $k_1 k_2 = 0$, we have checked carefully and found that it has multiple latent roots $|x| = 1$ with complete eigenvectors. Thus the spectrum stability conditions are the same as those given in Theorem 3.4.

However, by Theorem 3.6, we shall get a stability result . One can check the amplification matrix of (3.46) is Lipschitz continuous about τ near $\tau = 0$. It is easy to compute that the eigenvalues of $G(k_i, 0)$ are

$$\Lambda_2 = \left\{1,1,1,1,\frac{1}{2}(2-{k_1}^2-{k_2}^2 \pm \sqrt{-4+\left(-2+{k_1}^2+{k_2}^2\right)^2})\right\},$$

which has complete eigenvectors. Thus the matrix can be diagonalized. When ${k_1}^2 + {k_2}^2 \le 4$, all the mode of the eigenvalues are 1. $G^n(k_i, 0)$ are uniformly bounded about τ, and the scheme (3.46) is stable.

Theorem 3.7. *For all $0 \le \theta \le 1$ and a given $T > 0$, the scheme* (3.46) *is stable on* $[0, T]$ *if and only if*

$$\sigma > 0, \quad \tau > 0, \quad k_1^2 + k_2^2 < 4, \tag{3.50}$$

or written in the form

$$\sigma > 0, \quad \tau > 0, \quad \left(\frac{\tau}{h_1}\right)^2 + \left(\frac{\tau}{h_2}\right)^2 < 1. \tag{3.51}$$

In particular,

1. *for $\theta = 1$, the scheme* (3.46) *is spectrum stable on* $[0,\infty)$ *under the conditions:*

$$0 < \sigma\tau < 2, \;\; k_1^2 + k_2^2 < (2-\sigma\tau)^2, \tag{3.52}$$

or written in the equivalent form of

$$\sigma > 0, \quad \tau < \frac{2}{\sigma} \quad \textit{and} \quad \left(\frac{\tau}{h_1}\right)^2 + \left(\frac{\tau}{h_2}\right)^2 < \frac{(2-\sigma\tau)^2}{4}. \tag{3.53}$$

2. *for $\theta = 0$, the scheme* (3.46) *is spectrum stable on* $[0,\infty)$ *under the conditions:*

$$\sigma > 0, \quad k_1^2 + k_2^2 < (2+\tau\sigma)^2, \tag{3.54}$$

or written in the equivalent form of

$$\sigma > 0, \quad \left(\frac{\tau}{h_1}\right)^2 + \left(\frac{\tau}{h_2}\right)^2 < \frac{(2+\sigma\tau)^2}{4}. \tag{3.55}$$

4. Numerical Examples

We carry out some numerical experiments for the problem of Eqs. (2.4) and (2.5) by using the Yee difference scheme (3.1)-(3.4). The boundary conditions are set as total reflection boundary conditions [13]

$$\left.\frac{\partial H_x}{\partial n}\right|_{\partial\Omega} = 0, \quad \left.\frac{\partial H_y}{\partial n}\right|_{\partial\Omega} = 0. \tag{4.1}$$

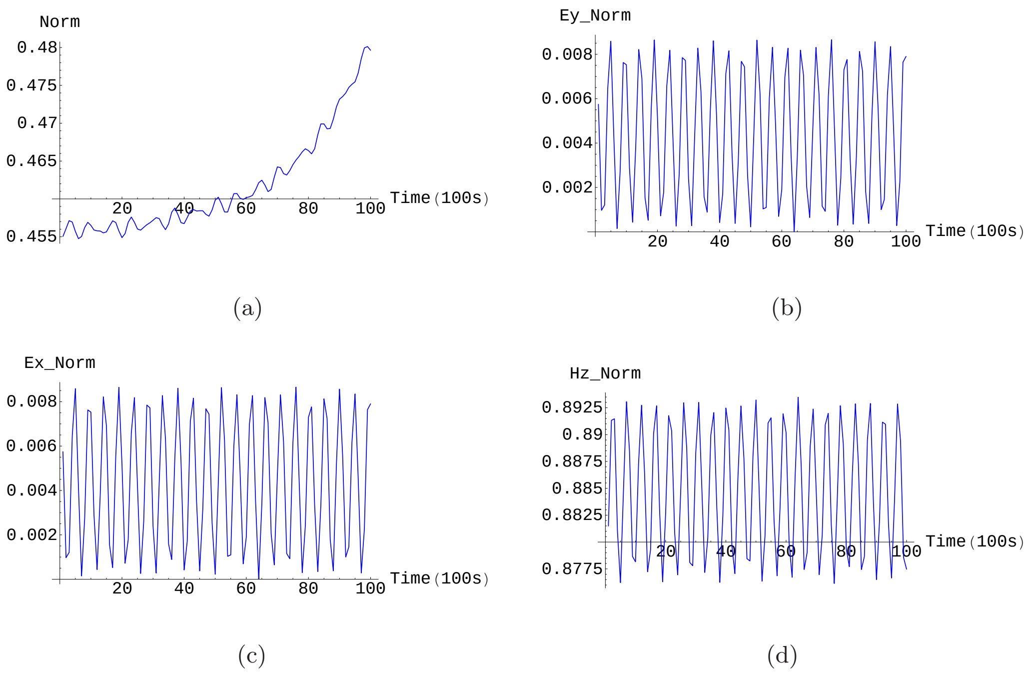
and the initial conditions can be arbitrary functions which are consistent to (4.1).

For simplicity, take a square domain $\Omega = [0,1] \times [0,1]$, and let $h_1 = h_2$. All the numerical results are illuminated in the figures, where the abscissa represents the time steps, and y-axis represents the norms of the corresponding fields. The discrete norms $\|\cdot\|_{l_2}$ are defined as, take E_x norm for example, $\|E_x\|_{l_2} = (h_1 h_2 \sum_{i,j} E_x^2(a_{i,j}))^{\frac{1}{2}}$, where $a_{i,j}$ are the nodes.

Example 1. Let $\sigma_1 = \sigma_2 = 0$, $\theta = 1$, $h_1 = h_2 = 0.1$, $\tau = 0.01$, *steps* $= 10^6$, terminal time=10000. The initial values

$$E_x^0 = y(y-1),\ E_y^0 = x(x-1),\ H_x^0 = H_y^0 = \mathtt{exp}(-x^2(x-1)^2 - y^2(y-1)^2).$$

The numerical results are shown in Fig. 4.1. The results indicate that the numerical method is unstable as the norm of (E_x, E_y, H_x, H_y) is increasing with step number n. This is caused by the norm of (H_x, H_y), i.e, $\big(h_1 h_2 \sum_{i,j}(H_x^2 + H_y^2)(a_{i,j})\big)^{\frac{1}{2}}$, which is increasing with the step number. On the other hand, the components norm for E_x and E_y oscillate in a strip domain but do not increase. So is the norm for $H_z = H_x + H_y$, i.e, $\big(h_1 h_2 \sum_{i,j}(H_x + H_y)^2(a_{i,j})\big)^{\frac{1}{2}}$, which oscillate but does not increase with time. All these unstable phenomenons are in agreement with the



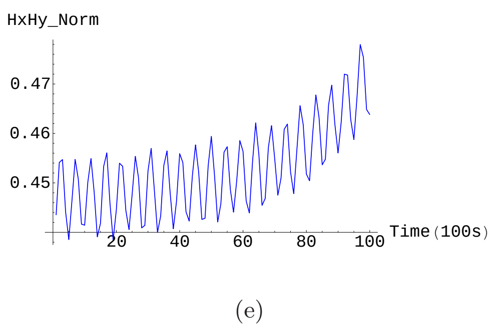


Fig. 4.1. Example 1: $\sigma_1 = \sigma_2 = 0$, $\theta = 1$, $h_1 = h_2 = 0.1$, *steps* $= 10^6$, $\tau = 0.01$. L_2-energy for (a) (E_x, E_y, H_x, H_y); (b) E_y; (c) E_x; (d) H_z; (e) (H_x, H_y).

theoretical results of [3] and our analysis.

Example 2. As pointed out in Remark 3 that $\sigma_1 = \sigma_2 = 0$ is just the critical values which cause the scheme unstable. If there exists even a little damping, the schemes can be stable. We test with a small positive $\sigma_1 = 0.001$, and $\sigma_2 = 0$, $\theta = 1$, $h_1 = h_2 = 0.1$, $\tau = 0.01$, the terminal time is 2000, the initial values are the same as that of Example 1. The numerical results are shown in Fig. 4.2, which indicates stability of the scheme. An interesting phenomenon is that only the E_y norm (whose direction adds damping) decays to 0, whereas E_x without adding damping does not decay to 0.

Example 3. Choosing proper parameters that (3.27) is not satisfied. Let $\sigma_1 = 10$, $\sigma_2 = 0$, $\theta = 1$, $h_1 = h_2 = 0.01$, $\tau = 0.01$, and the initial values be the same as that of example 1. The numerical results indicate that the norms blow up, and Fig. 4.3 shows the norms rising rapidly in only a few steps.

Example 4. Set $\theta = 0$, $\sigma_1 = 10$, $\sigma_2 = 0$, $h_1 = h_2 = 0.1$, $\tau = 0.01$ satisfying (3.31). The initial values are

$$E_x^0 = y(y-1),\ E_y^0 = x(x-1),\ H_x^0 = H_y^0 = \mathtt{exp}\Big(-x^2(x-1)^2 - y^2(y-1)^2\Big).$$

The numerical results show the stability in Fig. 4.4. The norms damp to 0 rapidly.

Example 5. Set $\sigma_1 = \sigma_2 = 10$, $\theta = 1$, $h_1 = h_2 = 0.1$, $\tau = 0.01$ satisfying (3.36). The norms damp to 0 at finite steps. The initial values are chosen as

$$E_x^0 = \sin \pi y, \quad E_y^0 = \sin \pi x, \quad H_x^0 = H_y^0 = \exp(-x^2(x-1)^2 - y^2(y-1)^2).$$

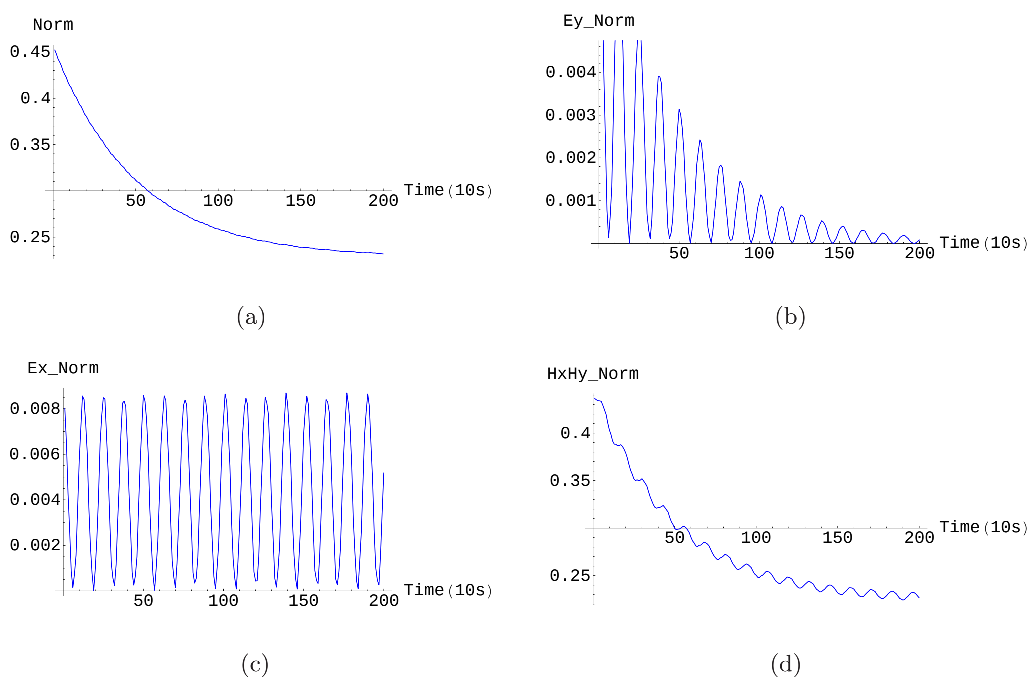


Fig. 4.2. Example 2: $\sigma_1 = 0.001$, $\sigma_2 = 0$, $\theta = 1$, $h_1 = h_2 = 0.1$, *steps* $= 100000$, $\tau = 0.01$. L_2-energy for (a) (E_x, E_y, H_x, H_y); (b) E_y; (c) E_x; (d) (H_x, H_y).

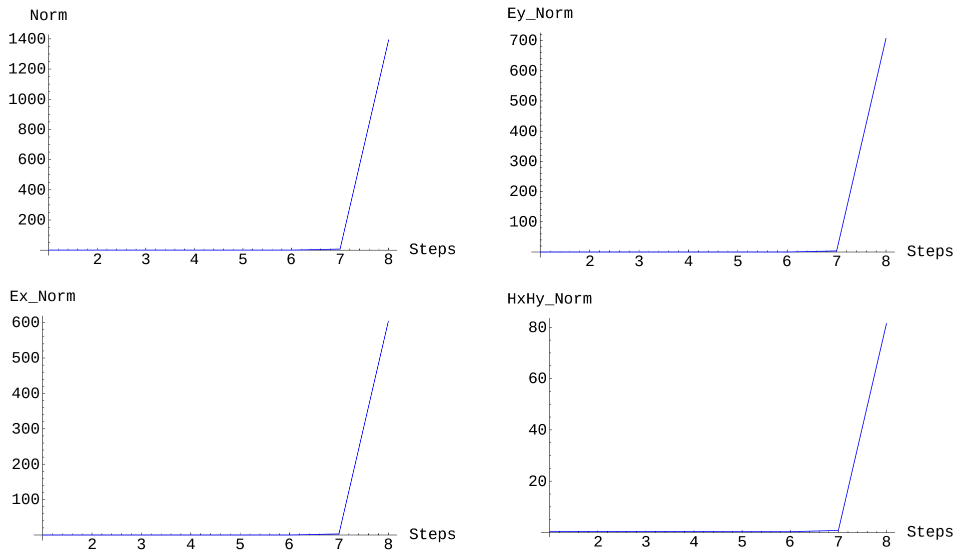


Fig. 4.3. Example 3: $\sigma_1 = 10, \sigma_2 = 0, \theta = 1, h_1 = h_2 = 0.01, \tau = 0.01$, steps=8, which do not satisfy (3.27). All energy norms are blowed-up after a few steps.

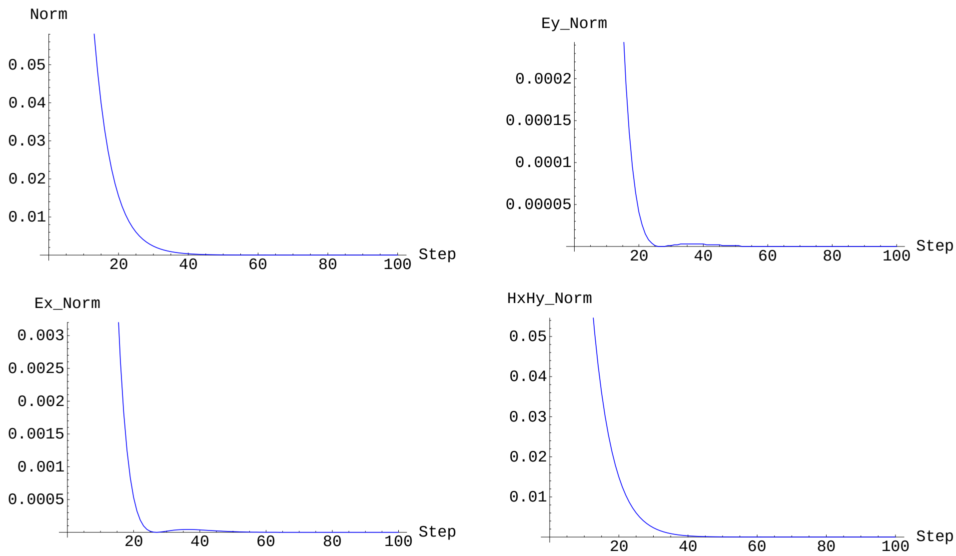
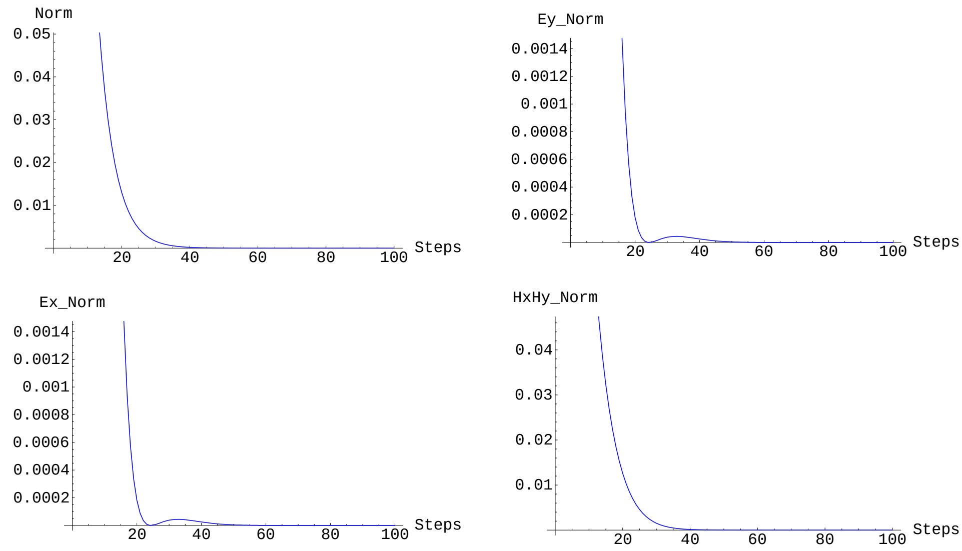


Fig. 4.4. Example 4: $\theta = 0$, $\sigma_1 = 10$, $\sigma_2 = 0$, $h_1 = h_2 = 0.1$, $\tau = 0.01$. The energy norms damp to 0 rapidly.

Example 6. For the explicit form $\theta = 1$, we set different damping constants $\sigma_1 = 2$, $\sigma_2 = 3$, $h_1 = h_2 = 0.1$, $\tau = 0.01$, and change the initial values as

$$E_x^0 = \mathtt{sin}\pi y,\ E_y^0 = x(x-1),\ H_x^0 = H_y^0 = \mathtt{exp}(-x^2(x-1)^2 - y^2(y-1)^2).$$

The numerical results are illuminated in Fig. 4.6. Similar as Example 5, the norms damp to 0 quickly at finite steps, which shows the stability for more general PML dampings.



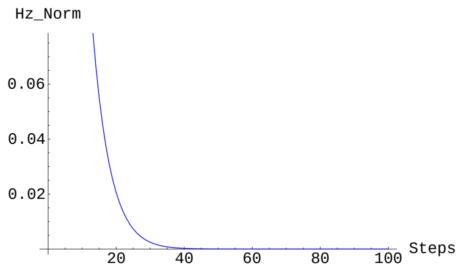


Fig. 4.5. Example 5: Both directions have damping, with parameters $\theta = 1$, $\sigma_1 = 10$, $\sigma_2 = 10$. $h_1 = h_2 = 0.1$, $\tau = 0.01$. All energy norms decay to 0

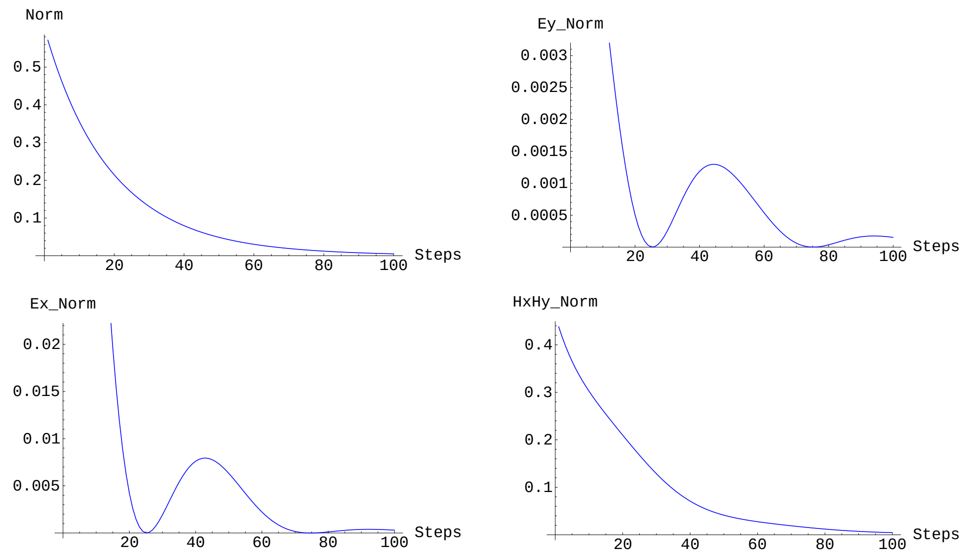


Fig. 4.6. Example 6: PML with two directions having different damping, $\sigma_1 = 2$, $\sigma_2 = 3$. $\theta = 1$, $h_1 = h_2 = 0.1$, $\tau = 0.01$, All energy norms decay to 0

5. Conclusions

Different Yee schemes of PMLs can yield stability, instability, and weak stability. Abarbanel and Gottlieb [3] give an unstable counterexample about PML for the case of $\sigma = 0$. Becache and Joly proved in [7] that the Yee schemes further Cauchy problem of PML are stable for the restored unsplit variables $E_x, E_y, \tilde{E}_x, H_z$ for $\sigma \geq 0$ by using the energy methods. We make clear that the Yee schemes of PML and UPML systems are spectrum stable under the same mesh conditions, and UPML schemes having the same damping in both directions are stable.

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