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UNIFORM SUPERCONVERGENCE OF GALERKIN METHODS FOR SINGULARLY PERTURBED PROBLEMS*

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Abstract

In this paper, we are concerned with uniform superconvergence of Galerkin methods for singularly perturbed reaction-diffusion problems by using two Shishkin-type meshes. Based on an estimate of the error between spline interpolation of the exact solution and its numerical approximation, an interpolation post-processing technique is applied to the original numerical solution. This results in approximation exhibit superconvergence which is uniform in the weighted energy norm. Numerical examples are presented to demonstrate the effectiveness of the interpolation post-processing technique and to verify the theoretical results obtained in this paper.

Mathematics subject classification: 65L10, 65L60.
Key words: singularly perturbed, Hermite splines, Shishkin-type meshes, Interpolation post-processing, Uniform superconvergence.

1. Introduction

In this paper, we consider the singularly perturbed two-point boundary value problem of reaction-diffusion type. It is well-known that the solution of this problem exhibits singularities at the boundary layers where singularities depend upon perturbation parameters. When the problem is solved numerically, we must take this boundary layer behavior of the solution into account in order to produce an approximate solution with high-order convergence. Shishkin meshes are most commonly used meshes in numerical methods which include finite difference methods and finite element methods (see, e.g., [9, 10, 12, 13] and references cited therein). In [12,13], the finite element method on the standard Shishkin mesh (S-mesh) provided a numerical solution with convergence rate which is almost optimal uniformly in the weighted energy norm. Based on another Shishkin-type mesh, namely, the anisotropic mesh (A-mesh), Li [3] proved an optimal order of uniform convergence for high-order reaction-diffusion problems. One of the advantages of Shishkin-type meshes, broadly defined, is that they are piecewise equidistant meshes. This structure of Shishkin-type meshes can be exploited and we show in this paper that, when it is combined with the interpolation post-processing technique, uniform superconvergence of numerical solution can be obtained.

Therefore, the main purpose of this paper is to obtain uniform superconvergence in the weighted energy norm of the Galerkin method on S-mesh as well as on A-mesh for singularly

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perturbed reaction-diffusion problems. Previously, Zhang [14] investigated a superconvergence phenomena of a finite element solution obtained on a modified Shiskin mesh for the second-order singularly perturbed problems. The superconvergence estimate was given in a *discrete* weighted energy norm in which the L^2 norm of $\nabla(u - u_h)$ is replaced by a p-point Gaussian quadrature rule. In this paper, motivated by an estimate of the error between spline interpolation of the exact solution and its numerical approximation, which is commonly known as *superclose*, (see, e.g., [2, 4, 5]), we will apply the interpolation post-processing technique to the numerical solution obtained by the Galerkin method. This generates a higher order approximation which gives rise to uniform superconvergence in the weighted energy norm. It will be shown that we may gain improvements in the order of convergence in the following way. For S-mesh, the rate of convergence in the weighted energy norm is enhanced from the optimal order of convergence *up to a logarithmic factor* and for A-mesh, the order of convergence improves by one from the optimal. We point out that the superclose property mentioned above plays a key role in establishing the superconvergence result presented in this paper. The idea of superclose property was used to obtain numerical solution of different operator equations. For one dimensional problem, we refer the reader to [3,8,12] and for two dimensional problems [6,7]. Also, we point out that fact that the idea of the interpolation post-processing technique has been successfully used by several authors (see, [2, 4, 5]). Finally, we note that our current work can be extended to two-dimensional reaction-diffusion problems as well as to other singularly perturbed problems. These topics will be discussed in the forthcoming papers.

This paper is organized as follows. In section 2, we describe the Galerkin method on S-mesh as well as on A-mesh for solving high-order reaction-diffusion problems. Section 3 is devoted to a study of application of the interpolation post-processing technique. The post-processing is applied to the original numerical solution of high-order reaction-diffusion problems. The uniform superconvergence of the post-processed solution on S-mesh as well as on A-mesh is subsequently obtained in the weighted energy norm. In section 4, we consider the second-order reaction-diffusion problem. The superclose property of the approximate solution in the weighted energy norm is derived by constructing a special interpolant. Based on this estimate, we use the interpolation post-processing technique to achieve the uniform superconvergence property of the numerical solution. Finally in section 5, two numerical examples are presented to confirm the theoretical results obtained in the previous sections.

2. High-order Reaction-Diffusion Problems

We introduce in this section the Galerkin method on S-mesh as well as on A-mesh for solving reaction-diffusion problems using Hermite splines. We begin with some notations. Set $\mathbb{N} := \{1, 2, ...\}$, $\mathbb{N}_0 := \{0, 1, ...\}$ and $\mathbb{Z}_n := \{0, 1, ..., n-1\}$. Let $I := [0, 1]$ and T be a subinterval of I. We denote by $(\cdot, \cdot)_T$ the inner product in $L^2(T)$ and by $\|\cdot\|_{0,T}$ for the associated norm on $L^2(I)$. Let $H^k(T), k \in \mathbb{N}$, be the Sobolev spaces on T with the norm $\|\cdot\|_{k,T}$ defined by

$$\|v\|_{k,T} = \left\{ \sum_{i=0}^{k} \int_T |v^{(i)}(x)|^2 dx \right\}^{1/2},$$

and the semi-norm $|\cdot|_{k,T}$ defined by

$$|v|_{k,T} = \left\{ \int_T |v^{(k)}(x)|^2 dx \right\}^{1/2}.$$

For notational convenience, we let $|\cdot|_{0,T} := \|\cdot\|_{0,T}$ and simply use $(\cdot,\cdot)$, $\|\cdot\|_k$ and $|\cdot|_k$ for $(\cdot,\cdot)_I$, $\|\cdot\|_{k,I}$ and $|\cdot|_{k,I}, k \in \mathbb{N}_0$, respectively. Also, we denote by $\|\cdot\|_\infty$ the essential maximum norm on $L^\infty(I)$ and by $\|\cdot\|_{k,\infty}$ the maximum norm on $C^k(I)$ for $k \in \mathbb{N}_0$. Let $H_0^k := H_0^k(I)$ be the closure of the set $\{v \in C^k(I):\ v^{(i)}(0) = v^{(i)}(1) = 0,\ i \in \mathbb{Z}_k\}$ in the Sobolev norm $\|\cdot\|_k$.

To describe the problem, we let m be a positive integer, let $\epsilon \in (0,1]$ be a singular perturbation parameter, and let $a_j,\ j \in \mathbb{Z}_{2(m-1)+1}$ and f be sufficiently smooth functions defined on I. We introduce the differential operator $\mathcal{L}_\epsilon$ by

$$\begin{aligned}\mathcal{L}_\epsilon u :=& (-1)^m \epsilon^2 u^{(2m)} + (-1)^{m-1}\big(a_{2(m-1)} u^{(m-1)}\big)^{(m-1)} \\ &+ \sum_{i=2}^{m} (-1)^{m-i}\big(a_{2(m-i)+1} u^{(m-i+1)} + a_{2(m-i)} u^{(m-i)}\big)^{(m-i)}\end{aligned}$$

and consider the singularly perturbed two-point boundary value problem of reaction-diffusion type

$$\begin{aligned}&(\mathcal{L}_\epsilon u)(x) = f(x), \quad x \in (0,1),\\ &u^{(j)}(0) = u^{(j)}(1) = 0, \quad j \in \mathbb{Z}_m.\end{aligned} \tag{2.1}$$

We define the bilinear form $A_\epsilon(\cdot,\cdot)$ by

$$\begin{aligned}A_\epsilon(u,v) :=& \epsilon^2\big(u^{(m)}, v^{(m)}\big) + \big(a_{2(m-1)} u^{(m-1)}, v^{(m-1)}\big)\\ &+ \sum_{i=2}^{m} \big(a_{2(m-i)+1} u^{(m-i+1)} + a_{2(m-i)} u^{(m-i)}, v^{(m-i)}\big).\end{aligned}$$

and the weighted energy norm $\|\cdot\|_\epsilon$ by

$$\|v\|_\epsilon := \big(\epsilon^2 |v|_m^2 + \|v\|_{m-1}^2\big)^{1/2}, \quad \text{for } v \in H_0^m.$$

By integration by parts, problem (2.1) may be written in a variational form in which we seek $u \in H_0^m$ such that

$$A_\epsilon(u,v) = (f,v), \quad \text{for all } v \in H_0^m. \tag{2.2}$$

The solution of Eq. (2.2) is a weak solution of problem (2.1). To guarantee the coercivity of the bilinear form in the weighted energy norm, we need to make more assumptions on the coefficient functions. Specifically, we let m constants $\alpha_i,\ i \in \mathbb{Z}_m$ satisfy for all $x \in I$,

$$a_{2(m-1)}(x) \ge \alpha_{m-1}, \qquad a_{2(m-i)}(x) - \frac{1}{2} a'_{2(m-i)+1} \ge \alpha_{m-i},\ i = 2, ..., m,$$

and define the index set

$$\mathbb{J}^+ := \{j:\ \alpha_j \ge 0, j \in \mathbb{Z}_m\} \quad \text{and} \quad \mathbb{J}^- := \mathbb{Z}_m \setminus \mathbb{J}^+.$$

We suppose that for any $\alpha_j,\ j \in \mathbb{J}^-$ there is a composition

$$\alpha_j = \sum_{k \in \mathbb{J}^+ \cap \{j+1, j+2, ..., m-1\}} \alpha_{j,k} \tag{2.3}$$

satisfying $\eta_{m-1} > 0$ and $\eta_k \ge 0$, for $k \in \mathbb{J}^+ \setminus \{m-1\}$, where

$$\eta_k := \alpha_k + \sum_{j \in \mathbb{J}^-, j<k} \alpha_{j,k} 2^{-(k-j)}, \qquad k \in \mathbb{J}^+.$$

Condition (2.3) originally presented in [8] is weaker than a condition in [12] which is assumed that

$$\alpha_{m-1} > 0 \quad \text{and} \quad \sum_{i=1}^{j} \alpha_{m-i} > 0, \quad j = 2, 3, ..., m.$$

Throughout the rest of this paper, condition (2.3) is always assumed. It is proved in [8] that there exist positive constants c_1 and c_2 such that for all $v, w \in H_0^m$,

$$|A_\epsilon(v, w)| \leq c_1 \|v\|_\epsilon \|w\|_\epsilon, \tag{2.4}$$

and

$$A_\epsilon(v, v) \geq c_2 \|v\|_\epsilon^2. \tag{2.5}$$

Hence, by the Lax-Milgram theorem, equation (2.2) has a unique solution. Moreover, the solution is sufficiently smooth and has the following decomposition (see, [9])

$$u = E + F + G, \tag{2.6}$$

where E, F and G satisfy for all $i \in \mathbb{N}_0$ and $x \in I$,

$$\begin{aligned} &|G^{(i)}(x)| \leq c, \quad |E^{(i)}(x)| \leq c\epsilon^{m-1-i} \exp(-\alpha x/\epsilon), \\ &|F^{(i)}(x)| \leq c\epsilon^{m-1-i} \exp(-\alpha(1-x)/\epsilon), \end{aligned} \tag{2.7}$$

with $\alpha := \alpha_{m-1}$. Note that G, E, F describe, respectively, the solution of the reduced problem and two boundary layers at endpoints $0, 1$. Since the solution of equation (2.2) is sufficiently smooth, it is identical to the classic solution of problem (2.1). For this reason, we will not distinguish the weak solution from the classic solution.

When $m = 1$, we refer to (2.1) as being the second-order reaction-diffusion problem, which will be discussed in section 4. When $m \geq 2$, problem (2.1) is regarded as the high-order reaction-diffusion problem. In the remainder of section 2 and throughout section 3, we always assume $m \geq 2$.

We now describe the ideas of *S*-mesh and *A*-mesh for problem (2.1). Let $N \in \mathbb{N}$. We divide the interval I into three subintervals

$$\Omega_0 := [0, \sigma], \quad \Omega_1 := [\sigma, 1-\sigma], \quad \Omega_2 := [1-\sigma, 1],$$

where

$$\sigma = \min\{1/4, \ (m+1)\alpha^{-1}\epsilon\rho\}$$

and $\rho = \ln N$ in the case of *S*-mesh, and $\rho = |\ln \epsilon|$ in the case of *A*-mesh. Then, two Shishkin-type meshes $0 = x_0 < x_1 < \cdots < x_{4N} = 1$ are obtained by setting $x_{i+1} = x_i + h_i, \ i \in \mathbb{Z}_{4N}$, with

$$h_i = \begin{cases} (1/2 - \sigma)/N, & i = N, N+1, \ldots, 3N-1, \\ \sigma/N, & \text{otherwise.} \end{cases}$$

Note that the Shishkin-type meshes are piecewise equidistant meshes with the transition points $x_N = \sigma$ and $x_{3N} = 1 - \sigma$. In this paper, it is sufficient to assume that

$$\sigma = (m+1)\alpha^{-1}\epsilon\rho.$$

We next describe two finite-dimensional spaces of Hermite splines corresponding to the Shishkin-type meshes. For a positive integer q and a subinterval T of I, we denote by $P_q(T)$

the space of polynomials of degree $q-1$ on T. Associated with the Shishkin-type meshes, the spaces V_h and $V_{h,0}$ are defined, respectively, by

$$V_h := \left\{ v \in H^m(I) : \ v|_{I_i} \in P_{2m}(I_i), \ i \in \mathbb{Z}_{4N} \right\} \tag{2.8}$$

and

$$V_{h,0} := \left\{ v \in H_0^m : \ v|_{I_i} \in P_{2m}(I_i), \ i \in \mathbb{Z}_{4N} \right\}, \tag{2.9}$$

where $I_i := [x_i, x_{i+1}], \ i \in \mathbb{Z}_{4N}$. It is clear that $V_h \subset C^{m-1}(I)$ and $V_{h,0} \subset C_0^{m-1}(I)$ by the Sobolev embedding theorem and the dimensions of V_h and $V_{h,0}$ are $m(4N+1)$ and $m(4N-1)$, respectively. We now define the interpolation operator $\mathcal{Q}_h$ from $C^{m-1}(I)$ to V_h by

$$(\mathcal{Q}_h v)^{(j)}(x_i) = v^{(j)}(x_i), \quad j \in \mathbb{Z}_m, \quad i \in \mathbb{Z}_{4N+1}. \tag{2.10}$$

It is easily verified that the interpolation operator $\mathcal{Q}_h$ is well defined. Notice also that the interpolation operator $\mathcal{Q}_h$ maps H_0^m to $V_{h,0}$. Let $m \le l \le 2m$ be an integer. We denote by $|v|_{k,\infty,T}$ the maximum semi-norm of $v^{(k)}$ on the interval $T \subseteq I$. It is known from [11] that there exists a positive constant c such that for $v \in C^l(I_i)$,

$$|v - \mathcal{Q}_h v|_{j,\infty,I_i} \le c h_i^{l-j} |v|_{l,\infty,I_i}, \quad j \in \mathbb{Z}_{l+1}, \ i \in \mathbb{Z}_{4N}. \tag{2.11}$$

Here and in what follows, constant c is used to denote the generic positive constant independent of ϵ and the mesh.

The Galerkin method on S-mesh or A-mesh for solving Eq. (2.2) is to seek $u_h \in V_{h,0}$ such that

$$A_\epsilon(u_h, v_h) = (f, v_h), \quad \text{for all} \ \ v_h \in V_{h,0}. \tag{2.12}$$

Again, by the Lax-Milgram theorem, Eq. (2.12) has a unique solution $u_h \in V_{h,0}$. For notational convenience, we let $\lambda = \ln N$ in the case of S-mesh, or $\lambda = 1$ in the case of A-mesh throughout the rest of the paper. It is established in [3, 12] that there exist a positive integer N_0 and a positive constant c such that for all $N > N_0$,

$$\|u - u_h\|_\epsilon \le c(N^{-1}\lambda)^m \tag{2.13}$$

and

$$\|\mathcal{Q}_h u - u_h\|_\epsilon \le c(N^{-1}\lambda)^{m+1}, \tag{2.14}$$

where u is the exact solution of Eq. (2.2). The estimate (2.13) shows that the Galerkin method for Eq. (2.2) provides an almost optimal order of uniform convergence on S-mesh, and an optimal order of uniform convergence on A-mesh, in terms of the singular perturbation parameter ϵ.

3. Interpolation Post-Processing

In this section, we apply interpolation post-processing technique to the approximate solution u_h and subsequently obtain an approximation which exhibits superconvergence uniformly in the weight energy norm. To prepare for the use of the technique, we first introduce an interpolation operator and study the property of this interpolation operator.

We first define the other two finite-dimensional spaces of Hermite splines corresponding to the Shishkin-type meshes described in the last section. For convenience, we shall assume that $N > 0$ is an even number, i.e., $N = 2M$ with M being some positive integer. Let $z_i = x_{2i}$ and $J_i := [z_i, z_{i+1}],\ i \in \mathbb{Z}_{2N}$, so that

$$J_i = I_{2i} \cup I_{2i+1} = [x_{2i}, x_{2i+2}], \quad i \in \mathbb{Z}_{2N}.$$

Thus, for each of two meshes, the points $0 = z_0 < z_1 < \cdots < z_{2N} = 1$ form a new mesh of the interval I with $\bar{h}_i := z_{i+1} - z_i$ which is still a Shishkin-type mesh. Based on the new meshes, the space $U_{\bar{h}}$ is defined by

$$U_{\bar{h}} := \left\{ v \in H^m(I) : \ v|_{J_i} \in P_{2m+1}(J_i),\ i \in \mathbb{Z}_{2N} \right\}. \tag{3.1}$$

By the Sobolev embedding theorem, $U_{\bar{h}} \subset C^{m-1}(I)$ and the dimension of $U_{\bar{h}}$ is $m(2N+1)+2N$. The interpolation operator $\mathcal{P}_{\bar{h}}$ from $C^{m-1}(I)$ to $U_{\bar{h}}$ is defined by

$$(\mathcal{P}_{\bar{h}} v)(x_{2i+1}) = v(x_{2i+1}),\ i \in \mathbb{Z}_{2N} \quad \text{and} \tag{3.2a}$$

$$(\mathcal{P}_{\bar{h}} v)^{(j)}(z_k) = v^{(j)}(z_k),\ j \in \mathbb{Z}_m,\ k \in \mathbb{Z}_{2N+1}. \tag{3.2b}$$

It is easy to see that the interpolation operator $\mathcal{P}_{\bar{h}}$ is well defined.

The following lemma shows the boundedness of the operator $\mathcal{P}_{\bar{h}}$ on V_h in the weighted energy norm.

Lemma 3.1. *Let $v \in V_h$. Then we have*

$$|\mathcal{P}_{\bar{h}} v\|_\epsilon \le c\|v\|_\epsilon. \tag{3.3}$$

Proof. By Theorem 4.4.4 and the estimate (4.4.23) of [1], we obtain that for any $w \in H^k(I)$ with $m \le k \le 2m+1$,

$$|w - \mathcal{P}_{\bar{h}} w|_{j,J_i} \le c\bar{h}_i^{k-j} |w|_{k,J_i}, \quad j \in \mathbb{Z}_{k+1},\ i \in \mathbb{Z}_{2N}. \tag{3.4}$$

Recall that the Shishkin-type meshes are piecewise equidistant. It follows from (3.4) that for $v \in V_h$,

$$\epsilon |v - \mathcal{P}_{\bar{h}} v|_m \le c\epsilon |v|_m$$

and for $i \in \mathbb{Z}_3$,

$$\|v - \mathcal{P}_{\bar{h}} v\|_{m-1,\Omega_i} \le c\bar{H}_i \|v\|_{m,\Omega_i},$$

where $\bar{H}_0 = \bar{H}_2 := \bar{h}_0$ and $\bar{H}_1 := \bar{h}_M$. Employing the inverse estimate (see, e.g., Theorem 4.5.11 of [1]), we obtain that for $i \in \mathbb{Z}_3$,

$$\bar{H}_i \|v\|_{m,\Omega_i} \le c\|v\|_{m-1,\Omega_i}. \tag{3.5}$$

This implies

$$\|v - \mathcal{P}_{\bar{h}} v\|_{m-1} \le c\|v\|_{m-1}.$$

Combining the results above, we conclude that

$$\|\mathcal{P}_{\bar{h}} v\|_\epsilon \le \|v\|_\epsilon + \|v - \mathcal{P}_{\bar{h}} v\|_\epsilon \le c\|v\|_\epsilon,$$

which yields the desired result. □

In the next lemma, we establish an uniform convergence result for the interpolation projection $\mathcal{P}_{\bar{h}}$ in the weighted energy norm.

Lemma 3.2. *Let u be the exact solution of problem* (2.1)*, and let $\mathcal{P}_{\bar{h}}$ be the interpolation operator defined by* (3.2)*. Then there exists a positive constant c independent of ϵ and N such that*

$$\|u - \mathcal{P}_{\bar{h}}u\|_\epsilon \le c(N^{-1}\lambda)^{m+1}, \tag{3.6}$$

Proof. Recall that the solution u of problem (2.1) can be written as

$$u = E + F + G$$

where E, F and G satisfy the condition (2.7). Hence, we obtain

$$\|u - \mathcal{P}_{\bar{h}}u\|_\epsilon \le \|E - \mathcal{P}_{\bar{h}}E\|_\epsilon + \|F - \mathcal{P}_{\bar{h}}F\|_\epsilon + \|G - \mathcal{P}_{\bar{h}}G\|_\epsilon. \tag{3.7}$$

It is known from [12] that for $v \in C^k(J_i)$ with $m-1 \le k \le 2m+1$,

$$|v - \mathcal{P}_{\bar{h}}v|_{j,\infty,J_i} \le c\bar{h}_i^{k-j}|v|_{k,\infty,J_i}, \quad j \in \mathbb{Z}_{k+1},\ i \in \mathbb{Z}_{2N}. \tag{3.8}$$

Using (2.7) and following a standard argument yields that (see, [1,12])

$$\|G - \mathcal{P}_{\bar{h}}G\|_\epsilon \le cN^{-m-1}. \tag{3.9}$$

Next, we estimate $\|E - \mathcal{P}_{\bar{h}}E\|_\epsilon$. By (2.7), a direct computation shows that for $j \in \mathbb{N}_0$,

$$|E|_{j,\Omega_0} \le c\left(\int_0^\sigma \epsilon^{2(m-j-1)}\exp(-2\alpha x/\epsilon)dx\right)^{1/2} \le c\epsilon^{m-j-1/2}. \tag{3.10}$$

Combining this and (3.4), we conclude that for $m \le k \le 2m+1$,

$$|E - \mathcal{P}_{\bar{h}}E|_{j,\Omega_0} \le c\bar{h}_0^{k-j}|E|_{k,\Omega_0} \le c(N^{-1}\sigma)^{k-j}\epsilon^{m-k-1/2}, \quad j \in \mathbb{Z}_{k+1}. \tag{3.11}$$

Since $\bar{h}_i \le 1/N$ for all $i \in \mathbb{Z}_{2N}$, it follows from (2.7) and (3.8) that for $m-1 \le k \le 2m+1$,

$$\begin{aligned}|E - \mathcal{P}_{\bar{h}}E|_{j,\infty,\Omega_1\cup\Omega_2} &\le cN^{j-k}|E|_{k,\infty,\Omega_1\cup\Omega_2}\\ &\le cN^{j-k}\epsilon^{m-k-1}\exp(-\alpha\sigma/\epsilon), \quad j \in \mathbb{Z}_{k+1}.\end{aligned} \tag{3.12}$$

Consequently, we have that for $j \in \mathbb{Z}_{k+1}$,

$$\begin{aligned}|E - \mathcal{P}_{\bar{h}}E|_{j,\Omega_1\cup\Omega_2} &\le \left(\sum_{i=M}^{4M-1} |E - \mathcal{P}_{\bar{h}}E|^2_{j,\infty,J_i}|J_i|\right)^{1/2}\\ &\le cN^{j-k}\epsilon^{m-k-1}\exp(-\alpha\sigma/\epsilon).\end{aligned} \tag{3.13}$$

For S-mesh, we have $\sigma = (m+1)\alpha^{-1}\epsilon\ln N$. It follows from (3.11) and (3.13) that

$$\begin{aligned}\epsilon|(E - \mathcal{P}_{\bar{h}}E)|_m &\le c\epsilon\big(|E - \mathcal{P}_{\bar{h}}E|_{m,\Omega_0} + |E - \mathcal{P}_{\bar{h}}E|_{m,\Omega_1\cup\Omega_2}\big)\\ &\le c\epsilon\Big((N^{-1}\sigma)^{m+1}\epsilon^{-m-3/2} + \epsilon^{-1}\exp(-\alpha\sigma/\epsilon)\Big)\\ &\le c\Big(\epsilon^{1/2}(\ln N)^{m+1} + 1\Big)N^{-m-1} \le c(N^{-1}\ln N)^{m+1}\end{aligned} \tag{3.14}$$

and for $j \in \mathbb{Z}_m$,

$$\begin{aligned}|(E-\mathcal{P}_{\bar h}E)|_j &\le c\big(|E-\mathcal{P}_{\bar h}E|_{j,\Omega_0}+|E-\mathcal{P}_{\bar h}E|_{j,\Omega_1\cup\Omega_2}\big)\\ &\le c\bigg((N^{-1}\sigma)^{2m+1-j}\epsilon^{-m-3/2}+N^{j-m+1}\exp(-\alpha\sigma/\epsilon)\bigg)\\ &\le c\bigg(\epsilon^{m-j-1/2}(N^{-1}\ln N)^{2m+1-j}+N^{j-2m}\bigg).\end{aligned} \tag{3.15}$$

Similarly, for A-mesh, we have $\sigma=(m+1)\alpha^{-1}\epsilon|\ln\epsilon|$ and thus obtain

$$\begin{aligned}\epsilon|(E-\mathcal{P}_{\bar h}E)|_m &\le c\epsilon\big(|E-\mathcal{P}_{\bar h}E|_{m,\Omega_0}+|E-\mathcal{P}_{\bar h}E|_{m,\Omega_1\cup\Omega_2}\big)\\ &\le c\epsilon\bigg((N^{-1}\sigma)^{m+1}\epsilon^{-m-3/2}+N^{-m-1}\epsilon^{-m-2}\exp(-\alpha\sigma/\epsilon)\bigg)\\ &\le c\bigg(\epsilon^{1/2}|\ln\epsilon|^{m+1}+1\bigg)N^{-m-1}\le cN^{-m-1}\end{aligned} \tag{3.16}$$

and for $j \in \mathbb{Z}_m$,

$$\begin{aligned}|(E-\mathcal{P}_{\bar h}E)|_j &\le c\big(|E-\mathcal{P}_{\bar h}E|_{j,\Omega_0}+|E-\mathcal{P}_{\bar h}E|_{j,\Omega_1\cup\Omega_2}\big)\\ &\le c\bigg((N^{-1}\sigma)^{2m+1-j}\epsilon^{-m-3/2}+N^{j-2m}\epsilon^{-m-1}\exp(-\alpha\sigma/\epsilon)\bigg)\\ &\le c\bigg(\epsilon^{m-j-1/2}|\ln\epsilon|^{2m+1-j}N^{-1}+1\bigg)N^{-2m+j}.\end{aligned} \tag{3.17}$$

The last inequality in (3.16) holds because $\epsilon^{1/2}|\ln\epsilon|^{m+1}$ is bounded on $(0,1]$. Noting that $\epsilon^{m-j-1/2}|\ln\epsilon|^{2m+1-j}$ in (3.17) is also bounded on $(0,1]$ for each $j\in\mathbb{Z}_m$, we conclude from (3.14)-(3.17) that

$$\|E-\mathcal{P}_{\bar h}E\|_\epsilon \le c(N^{-1}\lambda)^{m+1}. \tag{3.18}$$

Following a similar argument, we obtain

$$\|F-\mathcal{P}_{\bar h}F\|_\epsilon \le c(N^{-1}\lambda)^{m+1}. \tag{3.19}$$

Combining this with (3.7), (3.11) and (3.18) proves the lemma. □

Using the two lemmas above, we are now ready to prove the main result of this section, which gives the uniform superconvergence of the Galerkin method using Hermite splines for problem (2.1) in the weighted energy norm.

Theorem 3.1. *Let u be the exact solution of problem* (2.1), *and let u_h be the corresponding approximate solution determined by Eq.* (2.12). *Assume that $\mathcal{P}_{\bar h}$ is the interpolation operator defined by* (3.2). *Then, for sufficiently large N, there exists a positive constant c independent of ϵ and N such that*

$$\|u-\mathcal{P}_{\bar h}u_h\|_\epsilon \le c(N^{-1}\lambda)^{m+1}. \tag{3.20}$$

Proof. Suppose that $\mathcal{Q}_h$ is the interpolation operator defined in (2.10). We first show that for any $v\in C^{m-1}(I)$,

$$\mathcal{P}_{\bar h}v=\mathcal{P}_{\bar h}\mathcal{Q}_hv. \tag{3.21}$$

By the definition of the interpolation operator $\mathcal{Q}_h$, we have

$$(\mathcal{Q}_hv)^{(j)}(x_i)=v^{(j)}(x_i),\quad j\in\mathbb{Z}_m,\ i\in\mathbb{Z}_{4N+1}.$$

Recalling that $z_i = x_{2i}$ for $i \in \mathbb{Z}_{2N+1}$, we obtain

$$(\mathcal{Q}_h v)(x_{2i+1}) = v(x_{2i+1}),\ i \in Z_{2N} \quad \text{and} \tag{3.22a}$$
$$(\mathcal{Q}_h v)^{(j)}(z_k) = v^{(j)}(z_k),\ j \in \mathbb{Z}_m,\ k \in \mathbb{Z}_{2N+1}. \tag{3.22b}$$

Using this with the definition of the interpolation operator $\mathcal{P}_{\bar{h}}$ confirms (3.21). It follows from (3.21) that

$$u - \mathcal{P}_{\bar{h}} u_h = u - \mathcal{P}_{\bar{h}} u + \mathcal{P}_{\bar{h}} \mathcal{Q}_h u - \mathcal{P}_{\bar{h}} u_h. \tag{3.23}$$

Combining this with (2.14), (3.3) and (3.6), we conclude that

$$\begin{aligned}\|u - \mathcal{P}_{\bar{h}} u_h\|_\epsilon &\le \|u - \mathcal{P}_{\bar{h}} u\|_\epsilon + \|\mathcal{P}_{\bar{h}}(\mathcal{Q}_h u - u_h)\|_\epsilon \\ &\le \|u - \mathcal{P}_{\bar{h}} u\|_\epsilon + c\|\mathcal{Q}_h u - u_h\|_\epsilon \le c(N^{-1}\lambda)^{m+1},\end{aligned}$$

which completes the proof. □

4. Second-order Reaction-Diffusion Problems

The superclose property (2.14) plays a key role in obtaining uniform superconvergence of the numerical solution in Theorem 3.1. This property is valid only for the Hermite spline interpolation of degree $2m-1$ applied to higher-order problem (2.1) with $m > 1$ (see, [3,8,12]). We may not obtain the superclose property of the numerical solution for the spline interpolation of higher degree (see, [8]). There is, however, an exception for the second-order problem. For this reason, in this section we investigate the superclose property of numerical solution of problem (2.1) with $m = 1$, i.e., the second-order reaction-diffusion problem, based on the Galerkin method using piecewise polynomials of degree no less than one. It will be seen that we may choose particular interpolation points, namely Gauss-Lobatto points, to construct the spline interpolation of higher degree which satisfies the superclose property. This, together with the application of the interpolation post-processing technique, leads to an approximation of the second-order problem which superconverges uniformly in the weighted energy norm.

Let r be a positive integer. In constructing S-mesh and A-mesh in this section, we take $\sigma = (r+1)\alpha^{-1}\epsilon\rho$. Associated with the Shishkin-type meshes, we define the spaces S_h and $S_{h,0}$, respectively, by

$$S_h := \left\{ v \in H^1(I) :\ v|_{I_i} \in P_{r+1}(I_i),\ i \in \mathbb{Z}_{4N} \right\} \tag{4.1}$$

and

$$S_{h,0} := \left\{ v \in H_0^1 :\ v|_{I_i} \in P_{r+1}(I_i),\ i \in \mathbb{Z}_{4N} \right\}, \tag{4.2}$$

It is clear that $S_h \subset C(I)$ and $S_{h,0} \subset C_0(I)$ by the Sobolev embedding theorem and the dimensions of S_h and $S_{h,0}$ are $4rN + 1$ and $4rN - 1$, respectively. We now define a special interpolation operator $\mathcal{R}_h$ from $C(I)$ to S_h by appropriately choosing the interpolation points. Specifically, we let $x_k = x_{k0} < x_{k1} < \cdots < x_{kr} = x_{k+1}$ be the $r+1$ Gauss-Lobatto points in $I_k, k \in \mathbb{Z}_{4N}$, that is, $x_{kj}, j-1 \in \mathbb{Z}_{r-1}$ are the $r-1$ zeros of the first derivative of the Legendre polynomial of degree r on I_k. Let

$$\phi_{kr}(x) = (x - x_{k0})(x - x_{k1}) \cdots (x - x_{kr}), \quad k \in \mathbb{Z}_{4N}.$$

It is known that ϕ'_{kr} can be written as the Legendre polynomial of degree r multiplied by a constant on I_k. This implies for $k \in \mathbb{Z}_{4N}$,

$$(\phi'_{kr}, w')_{I_k} = 0, \qquad \text{for all } \ w \in P_{r+1}(I_k). \tag{4.3}$$

Associated with the Gauss-Lobatto points, the interpolation operator $\mathcal{R}_h : C(I) \to S_h$ is defined by

$$(\mathcal{R}_h v)(x_i) = v(x_i), \ i \in \mathbb{Z}_{4N+1}, \quad \text{and} \tag{4.4a}$$

$$(\mathcal{R}_h v)(x_{kj}) = v(x_{kj}), \ j-1 \in \mathbb{Z}_{r-1}, \ k \in \mathbb{Z}_{4N}. \tag{4.4b}$$

It is easy to verify that the interpolation operator $\mathcal{R}_h$ is well defined. For $v \in C^{r+2}(I_k)$ and $w \in P_{r+1}(I_k)$, it follows from (4.3) that (see, [14])

$$\left| \left((v - \mathcal{R}_h v)', w' \right)_{I_k} \right| \le c h_k^{r+1} |v|_{r+2,I_k} |w|_{1,I_k}, \quad k \in \mathbb{Z}_{4N}. \tag{4.5}$$

Also, it is known from [1,12] that for $v \in C^k(I_i)$ with $0 \le k \le r+1$,

$$|v - \mathcal{R}_h v|_{j,\infty,I_i} \le c h_i^{k-j} |v|_{k,\infty,I_i}, \quad j \in \mathbb{Z}_{k+1}, \ i \in \mathbb{Z}_{4N}, \tag{4.6}$$

and for for $v \in C^k(I_i)$ with $1 \le k \le r+1$,

$$|v - \mathcal{R}_h v|_{j,I_i} \le c h_i^{k-j} |v|_{k,I_i}, \quad j \in \mathbb{Z}_{k+1}, \ i \in \mathbb{Z}_{4N}. \tag{4.7}$$

The Galerkin method on S-mesh or A-mesh for solving Eq. (2.2) with $m = 1$ is to seek $u_h \in S_{h,0}$ such that

$$A_\epsilon(u_h, v_h) = (f, v_h), \quad \text{for all } \ v_h \in S_{h,0}. \tag{4.8}$$

By the Lax-Milgram theorem, Eq. (4.8) has a unique solution $u_h \in S_{h,0}$.

In the next theorem, an almost optimal (or optimal) order of uniform convergence is provided for the Galerkin method on S-mesh (or A-mesh) for solving equation (4.8). In addition, the superclose property of the numerical solution is presented.

Theorem 4.1. *Let u be the exact solution of problem* (2.1) *with $m = 1$, and let u_h be the solution of Eq.* (4.8). *Assume that $\mathcal{R}_h$ is the interpolation operator defined by* (4.4). *Then, for sufficiently large N, there exists a positive constant c independent of N such that*

$$\|u - u_h\|_\epsilon \le c(N^{-1}\lambda)^r \tag{4.9}$$

and

$$\|\mathcal{R}_h u - u_h\|_\epsilon \le c(N^{-1}\lambda)^{r+1}. \tag{4.10}$$

Proof. Using (4.6) and following similar arguments on the high-order problem as those given in [3,12], the estimate (4.9) is obtained. Recall that u has a decomposition (2.6). To prove (4.10), we first give an estimate on $A_\epsilon(E - \mathcal{R}_h E, v_h)$. By a similar argument as that in (3.14) and (3.15), it follows from (2.7), (3.10), (4.6) and (4.7) that for S-mesh,

$$\begin{aligned} \epsilon^2 \left| ((E - \mathcal{R}_h E)', v_h') \right| &\le c\epsilon^2 \left((N^{-1}\sigma)^{r+1} |E|_{r+2,\Omega_0} + |E|_{1,\infty,\Omega_1 \cup \Omega_2} \right) |v_h|_1 \\ &\le c\epsilon^2 \left((N^{-1}\epsilon \ln N)^{r+1} \epsilon^{-r-3/2} + \epsilon^{-1} \exp(-\alpha\sigma/\epsilon) \right) |v_h|_1 \\ &\le c\epsilon \left(\epsilon^{1/2} (N^{-1} \ln N)^{r+1} + N^{-r-1} \right) |v_h|_1 \\ &\le c\epsilon (N^{-1} \ln N)^{r+1} |v_h|_1 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\left|\left(a_0(E-\mathcal{R}_hE),v_h\right)\right| &\le c\left[(N^{-1}\sigma)^{r+1}|E|_{r+1,\Omega_0}+|E|_{0,\infty,\Omega_1\cup\Omega_2}\right]|v_h|_0\\ &\le c\left((N^{-1}\epsilon\ln N)^{r+1}\epsilon^{-r-1/2}+\exp\left(-\alpha\sigma/\epsilon\right)\right)|v_h|_0\\ &\le c\left(\epsilon^{1/2}(N^{-1}\ln N)^{r+1}+N^{-r-1}\right)|v_h|_0\\ &\le c(N^{-1}\ln N)^{r+1}|v_h|_0. \end{aligned}\tag{4.12}$$

Likewise, we obtain that for A-mesh,

$$\begin{aligned}\epsilon^2\left|\left((E-\mathcal{R}_hE)',v_h'\right)\right| &\le c\epsilon^2\left((N^{-1}\sigma)^{r+1}|E|_{r+2,\Omega_0}+N^{-r-1}|E|_{r+2,\infty,\Omega_1\cup\Omega_2}\right)|v_h|_1\\ &\le c\epsilon^2\left((N^{-1}\epsilon|\ln\epsilon|)^{r+1}\epsilon^{-r-3/2}+N^{-r-1}\epsilon^{-r-2}\exp\left(-\alpha\sigma/\epsilon\right)\right)|v_h|_1\\ &\le c\epsilon N^{-r-1}\left(\epsilon^{1/2}|\ln\epsilon|^{r+1}+1\right)|v_h|_1\\ &\le c\epsilon N^{-r-1}|v_h|_1 \end{aligned}\tag{4.13}$$

and

$$\begin{aligned}\left|\left(a_0(E-\mathcal{R}_hE),v_h\right)\right| &\le c\left[(N^{-1}\sigma)^{r+1}|E|_{r+1,\Omega_0}+N^{-r-1}|E|_{r+1,\infty,\Omega_1\cup\Omega_2}\right]|v_h|_0\\ &\le c\left[(N^{-1}\epsilon|\ln\epsilon|)^{r+1}\epsilon^{-r-1/2}+N^{-r-1}\epsilon^{-r-1}\exp\left(-\alpha\sigma/\epsilon\right)\right]|v_h|_0\\ &\le cN^{-r-1}\left(\epsilon^{1/2}|\ln\epsilon|^{r+1}+1\right)|v_h|_0\\ &\le cN^{-r-1}|v_h|_0. \end{aligned}\tag{4.14}$$

From (4.11)-(4.14) we conclude that

$$|A_\epsilon(E-\mathcal{R}_hE,v_h)|\le c(N^{-1}\lambda)^{r+1}\|v_h\|_\epsilon.\tag{4.15}$$

Noting that the same argument applies to F and G, we thereby obtain

$$|A_\epsilon(u-\mathcal{R}_hu,v_h)|\le c(N^{-1}\lambda)^{r+1}\|v_h\|_\epsilon.\tag{4.16}$$

It follows from (2.2) and (4.8) that

$$A_\epsilon(u-\mathcal{R}_hu,u_h-\mathcal{R}_hu)=A_\epsilon(u_h-\mathcal{R}_hu,u_h-\mathcal{R}_hu).$$

Combining this with (2.5) and (4.16) gives the estimate (4.10). □

We now turn to an investigation of uniform superconvergence of the numerical solution by employing the interpolation post-processing technique, as shown in section 3. We can take the same procedure as that in section 3 but make a slight modification on the interpolation operator. To this end, we first define the finite-dimensional space $W_{\bar h}$ by

$$W_{\bar h}:=\left\{v\in H^1(I):\ v|_{J_i}\in P_{r+2}(J_i),\ i\in\mathbb{Z}_{2N}\right\}.\tag{4.17}$$

It is easily seen that $W_{\bar h}\subset C(I)$ by the Sobolev embedding theorem. We denote by $\mathbb{G}_i$ the set of $r+1$ Gauss-Lobatto points in $I_i, i\in\mathbb{Z}_{4N}$, i.e., $\mathbb{G}_i=\{x_{ik},k\in\mathbb{Z}_{r+1}\}$. We choose the r

distinct points $y_{ik}, k \in \mathbb{Z}_r$ in every subinterval $J_i, i \in \mathbb{Z}_{2N}$ such that $\{y_{ik}, k \in \mathbb{Z}_r\}$ is a subset of the set $(\mathbb{G}_{2i} \cup \mathbb{G}_{2i+1}) \setminus \{z_i, z_{i+1}\}$. The interpolation operator $\mathcal{T}_{\bar{h}}$ from $C(I)$ to $W_{\bar{h}}$ is defined by

$$(\mathcal{T}_{\bar{h}} v)(z_i) = v(z_i),\ i \in \mathbb{Z}_{2N+1} \ \text{ and } \ (\mathcal{T}_{\bar{h}} v)(y_{ik}) = v(y_{ik}),\ k \in \mathbb{Z}_r,\ i \in \mathbb{Z}_{2N}. \tag{4.18}$$

It is easy to verify that the interpolation operator $\mathcal{T}_{\bar{h}}$ is well defined. Following similar arguments as those in Lemmas 3.1 and 3.2, we have that for $v \in S_h$,

$$\|\mathcal{T}_{\bar{h}} v\|_\epsilon \le c\|v\|_\epsilon \tag{4.19}$$

and

$$\|u - \mathcal{T}_{\bar{h}} u\|_\epsilon \le c(N^{-1}\lambda)^{r+1}, \tag{4.20}$$

where u is the exact solution of problem (2.1) with $m = 1$.

Using (4.19) and (4.20), we establish in the next theorem the main result of this section, concerning uniform superconvergence of the Galerkin method on S-mesh or A-mesh for solving the second-order problem.

Theorem 4.2. *Let u be the exact solution of problem* (2.1) *with $m = 1$, and let u_h be the solution of Eq.* (4.8). *Assume that $\mathcal{T}_{\bar{h}}$ is the interpolation operator defined by* (4.18). *Then, for sufficiently large N, there exists a positive constant c independent of ϵ and N such that*

$$\|u - \mathcal{T}_{\bar{h}} u_h\|_\epsilon \le c(N^{-1}\lambda)^{r+1}. \tag{4.21}$$

Proof. The result follows from the same argument used in Theorem 3.3. □

5. Numerical Examples

In this section, we present two numerical examples to confirm the theoretical estimates obtained in the previous sections.

Example 1. Consider the following fourth-order reaction-diffusion problem

$$\begin{aligned} &\epsilon^2 u^{(4)}(x) - [(1 + x(1-x))u']' = f(x), \quad x \in (0,1), \\ &u(0) = u'(0) = u(1) = u'(1) = 0, \end{aligned} \tag{5.1}$$

where f is chosen such that

$$u(x) = \epsilon\left[\frac{\exp(-x/\epsilon) + \exp(-(1-x)/\epsilon)}{1 + \exp(-1/\epsilon)} - 1\right] + \frac{1 - \exp(-1/\epsilon)}{1 + \exp(-1/\epsilon)} x(1-x) + x^2(1-x)^2$$

is the exact solution of (5.1). In this case, $m = 2$. To obtain the superclose property of the numerical solution and the superconvergence property of the post-processed solution, we employ the cubic Hermit spline to solve the problem, as described in the sections 2 and 3. The theoretical orders of uniform convergence (or superconvergence) for the Galerkin method on S-mesh and that using the post-processing technique are 2 and 3, respectively, up to logarithmic factors. It can be seen from Table 5.1 that the numerical results confirm the theoretical estimates for three different values of ϵ. Plotted in Figure 5.1 are the convergent curves of errors $u - u_h$ and $u - \mathcal{P}_{\bar{h}} u_h$ in the weighted energy norm for $\epsilon = 4.605 \times 10^{-7}$, indicating the rates of $(N^{-1}\ln N)^2$ and $(N^{-1}\ln N)^3$, respectively. Numerical results demonstrate in Table 5.2 that the computed orders of convergence (or superconvergence) for the Galerkin method on A-mesh and that using the post-processing technique are consistent with the theoretical orders, which are 2 and 3, respectively.

Table 5.1 Numerical performance of Galkerin methods on S-mesh for the 4th-order problem

ϵ	N	$\|u-u_h\|_\epsilon$	Order of conv.	$\|u-\mathcal{P}_{\bar{h}}u_h\|_\epsilon$	Order of conv.
1.623e-3	16	4.0053e-4	-	1.3340e-4	-
	32	1.5767e-4	1.3450	3.3882e-5	1.9772
	64	5.6952e-5	1.4691	7.4558e-6	2.1841
	128	1.9404e-5	1.5534	1.4920e-6	2.3211
	256	6.3386e-6	1.6141	2.7973e-7	2.4151
	512	2.0059e-6	1.6599	5.0099e-8	2.4812
2.451e-5	16	4.9407e-5	-	1.6420e-5	-
	32	1.9384e-5	1.3498	4.1639e-6	1.9794
	64	6.9991e-6	1.4696	9.1506e-7	2.1860
	128	2.3845e-6	1.5535	1.8284e-7	2.3233
	256	7.7895e-7	1.6141	3.4194e-8	2.4188
	512	2.4650e-7	1.6599	6.2037e-9	2.4625
4.605e-7	16	7.9204e-6	-	2.3904e-6	-
	32	2.7069e-6	1.5489	5.7510e-7	2.0554
	64	9.6156e-7	1.4932	1.2258e-7	2.1952
	128	3.2695e-7	1.5563	2.5067e-8	2.3247
	256	1.0678e-7	1.6144	4.6872e-9	2.4190
	512	3.3788e-8	1.6601	8.5294e-10	2.4582

Table 5.2 Numerical performance of Galerkin methods on A-mesh for the 4th-order problem

ϵ	N	$\|u-u_h\|_\epsilon$	Order of conv.	$\|u-\mathcal{P}_{\bar{h}}u_h\|_\epsilon$	Order of conv.
1.623e-3	16	2.0342e-3	-	1.2751e-3	-
	32	5.3484e-4	1.9273	2.0179e-4	2.6597
	64	1.3551e-4	1.9807	2.7065e-5	2.8984
	128	3.9991e-5	1.9952	3.4500e-6	2.9718
	256	8.5050e-6	1.9988	4.3432e-7	2.9898
	512	2.1267e-6	1.9997	5.4734e-8	2.9882
2.451e-5	16	6.1682e-4	-	4.7794e-4	-
	32	1.7430e-4	1.8233	9.6966e-5	2.3013
	64	4.5136e-5	1.9492	1.4397e-5	2.7517
	128	1.1388e-5	1.9868	1.8896e-6	2.9296
	256	2.8536e-6	1.9967	2.3920e-7	2.9818
	512	7.1381e-7	1.9992	2.9997e-8	2.9953
4.605e-7	16	1.4193e-4	-	1.1675e-4	-
	32	4.3401e-4	1.7094	2.9115e-5	2.0036
	64	1.1563e-5	1.9082	4.8154e-6	2.5853
	128	2.9405e-6	1.9754	6.6256e-7	2.8723
	256	7.3832e-7	1.9937	8.4799e-8	2.9659
	512	1.8478e-7	1.9984	1.0664e-8	2.9913

Example 2. In this example, we consider the second-order reaction-diffusion problem

$$\begin{aligned} &-\epsilon^2 u''(x) + [2+\cos(x)]\,u(x) = f(x), \quad x \in (0,1), \\ &u(0)) = u(1) = 0. \end{aligned} \tag{5.2}$$

We choose f so that problem (5.2) has the exact solution

$$u(x) = \exp(-x/\epsilon) + \exp(-(1-x)/\epsilon) + x(1-x) - (1+\exp(-1/\epsilon)).$$

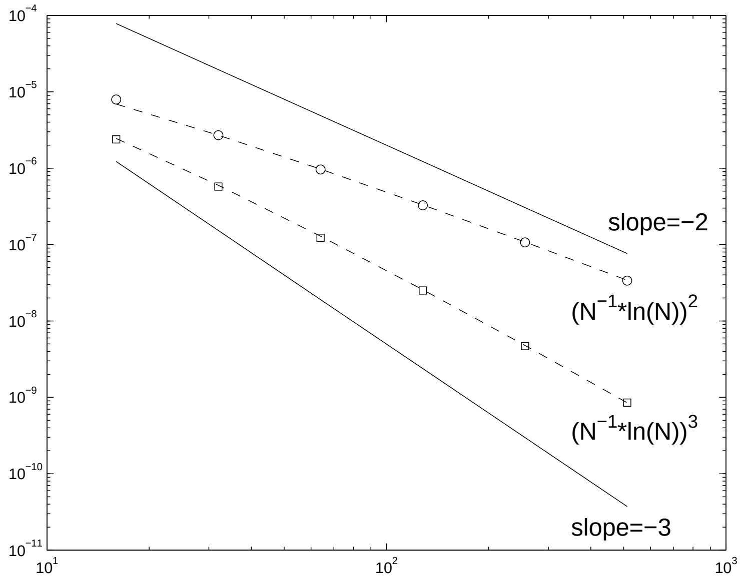

Fig. 5.1. The original numerical solution and the post-processed solution based on the Shishkin mesh; $\epsilon = 4.605 \times 10^{-7}$

In this case, $m = 1$. For both of the meshes we use the piecewise polynomials of degree 2, that is, $r = 2$. Numerical results presented in Tables 5.1 and 5.2 are about errors and orders of convergence in the weighted energy norm for the original numerical solution and the post-processed solution, corresponding to S-mesh and A-mesh respectively. It is clear that the numerical results confirm the theoretical estimates established in section 4 in terms of both of the meshes.

Table 5.3 Numerical performance of Galerkin methods on S-mesh for the 2nd-order problem

ϵ	N	$\|u-u_h\|_\epsilon$	Order of conv.	$\|u-\mathcal{T}_{\bar{h}}u_h\|_\epsilon$	Order of conv.
1.612e-4	16	1.2626e-4	-	5.6610e-5	-
	32	4.9693e-5	1.3453	1.4270e-5	1.9881
	64	1.7949e-5	1.4691	3.1262e-6	2.1905
	128	6.1151e-6	1.5535	6.2388e-7	2.3251
	256	1.9976e-6	1.6141	1.1663e-7	2.4193
	512	6.3216e-7	1.6599	2.0769e-8	2.4894
2.217e-6	16	1.4808e-5	-	6.6383e-6	-
	32	5.8277e-6	1.3454	1.6734e-6	1.9880
	64	2.1049e-6	1.4692	3.6661e-7	2.1905
	128	7.1714e-7	1.5534	7.3163e-8	2.3251
	256	2.3427e-7	1.6141	1.3677e-8	2.4194
	512	7.4135e-8	1.6599	2.4357e-9	2.4893
6.423e-8	16	2.5204e-6	-	1.1299e-6	-
	32	9.9194e-7	1.3453	2.8483e-7	1.9880
	64	3.5828e-7	1.4692	6.2401e-8	2.1905
	128	1.2207e-7	1.5534	1.2453e-8	2.3251
	256	3.9875e-8	1.6142	2.3281e-9	2.4193
	512	1.2619e-8	1.6599	4.1480e-10	2.4887

Table 5.4 Numerical performance of Galkerin methods on A-mesh for the 2nd-order problem

ϵ	N	$\|u-u_h\|_\epsilon$	Order of conv.	$\|u-\mathcal{T}_{\bar{h}}u_h\|_\epsilon$	Order of conv.
1.612e-4	16	1.1254e-3	-	1.2175e-3	-
	32	3.0701e-4	1.8741	2.0397e-4	2.5775
	64	7.8632e-5	1.9651	2.8138e-5	2.8578
	128	1.9781e-5	1.9910	3.6138e-6	2.9609
	256	4.9530e-6	1.9977	4.5487e-7	2.9900
	512	1.2387e-7	1.9995	5.6959e-8	2.9975
2.217e-6	16	2.6054e-4	-	3.4630e-4	-
	32	7.7098e-5	1.7567	6.9262e-5	2.3219
	64	2.0292e-5	1.9258	1.0485e-5	2.7237
	128	5.1429e-6	1.9803	1.3887e-6	2.9165
	256	1.2902e-6	1.9950	1.7626e-7	2.9780
	512	3.2283e-7	1.9987	2.2118e-8	2.9944
6.423e-8	16	6.4261e-5	-	9.4864e-5	-
	32	2.0420e-5	1.6540	2.1409e-5	2.1476
	64	5.5273e-6	1.8853	3.5143e-6	2.6069
	128	1.4123e-6	1.9685	4.8052e-7	2.8706
	256	3.5507e-7	1.9919	6.1551e-8	2.9647
	512	8.8892e-8	1.9980	7.7424e-9	2.9909

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References

[1] S.C. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods, Springer-Verlag, New York, 1994.

[2] Y. Chen, J. Lin and Q. Lin, Superconvergence of discontinuous Galerkin methods for nonstationary hyperbolic equations, *J. Comput. Math.*, **20** (2002), 429-436.

[3] J. Li, Convergence analysis of finite element methods for singularly perturbed problems, *Comput. Math. Appl.*, **40** (2000), 735-745.

[4] Q. Lin and N. Yan, The Construction and Analysis of High Efficiency Finite Element Methods, Hebei University Press, China, 1996.

[5] Q. Lin, S. Zhang and N. Yan, An acceleration method for integral equations by using interpolation post-processing, *Adv. Comput. Math.*, **9** (1998), 117-129.

[6] T. Linss, Analysis of a Galerkin finite element method on a Bakhvalov-Shishkin mesh for a linear convection-diffusion problem, *IMA. J. Numer. Anal.*, **20** (2000), 621-632.

[7] T. Linss, Uniform superconvergence of a Galerkin finite element method on Shishkin-type meshes, *Numer. Meth. Part. D. E.*, **16** (2000), 426-440.

[8] S.T. Liu and Y. Xu, Galerkin methods based on Hermite splines for singular perturbation problems, *SIAM J. Numer. Anal.*, **43** (2006), 2607-2623.

[9] R.E. O'Malley, Introduction to Singular Perturbations, Academic Press, New York, 1974.

[10] H.G. Roos, M. Stynes and L. Tobiska, Robust Numerical Methods for Singularly Perturbed Differential Equations, 2nd edition, Springer-Verlag, Berlin, 2008.

[11] L. L. Schumaker, Spline Functions: Basic Theory, John Wiley & Sons, New York, 1981.

[12] G. Sun and M. Stynes, Finite element methods for singularly perturbed two-point boundary value problems, I: reaction-diffusion-type problems, *IMA J. Numer. Anal.*, **15** (1995), 117-139.

[13] G. Sun and M. Stynes, Finite element methods for singularly perturbed two-point boundary value problems, II: convection-diffusion-type problems, *IMA J. Numer. Anal.*, **15** (1995), 197-219.

[14] Z. Zhang, Finite element superconvergent approximation for one-dimensional singularly perturbed problems, *Numer. Meth. Part. D. E.*, **18** (2002), 374-395.