Journal of Computational Mathematics
Vol.28, No.5, 2010, 711–724.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1003-m2809

VARIABLE MESH FINITE DIFFERENCE METHOD FOR SELF-ADJOINT SINGULARLY PERTURBED TWO-POINT BOUNDARY VALUE PROBLEMS *

Mohan K. Kadalbajoo and Devendra Kumar
Department of Mathematics & Statistics, I.I.T. Kanpur 208016, India
Email: kadal@iitk.ac.in and dpanwar@iitk.ac.in

Abstract

A numerical method based on finite difference method with variable mesh is given for self-adjoint singularly perturbed two-point boundary value problems. To obtain parameter-uniform convergence, a variable mesh is constructed, which is dense in the boundary layer region and coarse in the outer region. The uniform convergence analysis of the method is discussed. The original problem is reduced to its normal form and the reduced problem is solved by finite difference method taking variable mesh. To support the efficiency of the method, several numerical examples have been considered.

Mathematics subject classification: 45Exx, 65L10, 65L12, 65L50, 65L70.
Key words: Singularly perturbed boundary value problems, Finite difference method, Boundary layer, Parameter uniform-convergence, Variable mesh.

1. Introduction

The problem in which a small parameter multiplies to the highest derivative arise in various fields of science and engineering, for instance, fluid mechanics, fluid dynamics, elasticity, quantum mechanics, chemical reactor theory, hydrodynamics etc. A large number of papers and books have been published describing various methods for solving singular perturbation problems, see, e.g., Axelsson *et al.* [2], Bellman [3], Bender and Orszag [4], Cole and Kevorkian [5], Eckhaus [7], Hamker and Miller [10], O'Malley [13], Nayfeh [16], Van Dyke [22]

Niijima [17] gave uniformly second order accurate difference schemes for reaction-diffusion equations, whereas Miller [14] gave sufficient condition for the uniform first-order convergence of a general three-point difference scheme. Parameter-uniform numerical methods [9, 15] are methods whose numerical approximations U^N satisfy error bounds of the form

$$\|u_\epsilon - U^N\| \leq C\vartheta(N), \quad \vartheta(N) \to 0 \ \text{as} \ N \to \infty,$$

where u_ϵ is the solution of the continuous problem, $\|.\|$ is the maximum pointwise norm, N is the number of mesh points (independent of ϵ) used and C is a positive constant which is independent of both ϵ and N. In other words, the numerical approximations U^N converge to u_ϵ for all values of parameter ϵ in the range $0 < \epsilon \ll 1$.

It is well-known that standard discretization methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter ϵ is small. Therefore, it is important to develop suitable numerical methods for these problems, whose accuracy does not depend on the parameter value ϵ, i.e., the methods are convergent

* Received April 11, 2008 / Revised version received July 30, 2009 / Accepted October 20, 2009 / Published online May 1, 2010 /

ϵ-uniformly [6,8,20]. In this paper, the strategy and the proposed method based on a suitably designed fitted mesh has been shown to converge with $\vartheta(N) = N^{-1}\ln N$.

In this paper, we consider the following self-adjoint singularly perturbed two-point boundary value problem

$$Ly \equiv -\epsilon(p(x)y')' + q(x)y = f(x), \tag{1.1}$$

for $0 \le x \le 1$ with the natural boundary conditions

$$y(0) = \alpha, \quad y(1) = \beta, \tag{1.2}$$

where α, β are given constants and ϵ is a small positive parameter ($0 < \epsilon \ll 1$). Further assume that the coefficients $p(x)$, $q(x)$ and the function $f(x)$ are smooth and satisfy

$$p(x) \ge \eta_1 > 0, \quad p'(x) \ge 0, \quad q(x) \ge \eta_2 > 0.$$

Under these conditions, the operator L admits the maximum principle [18].

In general finding the numerical solution of a second order boundary value problem with y' term is more difficult as compare to a second order boundary value problem without y' term, therefore we first reduce (1.1) to its normal form and then the reduced problem is solved by finite difference scheme using arithmetic mesh.

Briefly, outline is as follows. In Section 2, we give description of the method. The derivation of the difference scheme has been given in Section 3. The idea how to choose the mesh has been given in Section 4, whereas the parameter uniform-convergence of the scheme is given in Section 5. To demonstrate the efficiency of the method some numerical experiments have been solved in Section 6 and finally the conclusion has been presented in Section 7.

2. Description of the Method

Eq. (1.1) can be rewritten as

$$y'' + P(x)y' + Q(x)y = F(x), \tag{2.1}$$

where

$$P(x) = \frac{p'(x)}{p(x)}, \quad Q(x) = -\frac{q(x)}{\epsilon p(x)} \quad \text{and} \quad F(x) = -\frac{f(x)}{\epsilon p(x)}.$$

By the transformation

$$y(x) = U(x)V(x), \tag{2.2}$$

Eq. (2.1) can be written as its normal form:

$$V''(x) + A(x)V(x) = G(x), \tag{2.3}$$

with

$$V(0) = \frac{y(0)}{U(0)} = \gamma, \quad V(1) = \frac{y(1)}{U(1)} = \delta, \quad \gamma, \delta \in \mathbb{R}, \tag{2.4}$$

where

$$A(x) = Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}(P(x))^2,$$
$$G(x) = F(x)\exp\left(\frac{1}{2}\int^x P(\zeta)\,d\zeta\right), \quad U(x) = \exp\left(-\frac{1}{2}\int^x P(\zeta)\,d\zeta\right).$$

Multiplying Eq. (2.3) throughout by $-\epsilon$ we get

$$-\epsilon V'' + W(x)V = Z(x), \quad \text{with} \quad V(0) = \gamma, \quad V(1) = \delta, \tag{2.5}$$

where

$$W(x) = -\epsilon A(x), \quad Z(x) = -\epsilon G(x).$$

Now we have

$$\begin{aligned} Z(x) &= -\epsilon\, G(x) \\ &= -\epsilon\, F(x) \exp\left(\frac{1}{2}\int^{x} P(\zeta)\, d\zeta\right) \\ &= \frac{f(x)}{p(x)} \exp\left(\frac{1}{2}\int^{x} P(\zeta)\, d\zeta\right). \end{aligned} \tag{2.6}$$

Eq. (2.6) shows that $Z(x)$ is independent of ϵ. However $W(x)$ may or may not depend on ϵ.

Hypothesis: We assume that the coefficients $p(x)$ and $q(x)$ in Eq. (1.1) are such that

$$W(x) \geq \mu (= \sqrt{\epsilon}\ln N) > 0.$$

Now we shall solve the problem (2.5) by finite difference scheme taking a variable mesh.

3. Derivation of the Difference Scheme

In this difference scheme, we approximate y'' by a three point formula as follows: Let N be the number of mesh points in the interval $[0, 1]$ and let $x_0 = 0$, $x_i = \Sigma_{k=0}^{i-1} h_k$ $(1 \leq i \leq N), h_k = x_{k+1} - x_k,\ x_N = 1$. Let $d_i = h_i - h_{i-1}$, be the common mesh difference. For the sake of convenience, we can take $d_i = d\,(\text{a constant})\ \forall\ i$. Taking the Taylor's series expansion and neglecting the term of third and higher order, we get the following expansions for y_{i+1} and y_{i-1}

$$y_{i+1} \simeq y_i + h_i y_i' + \frac{h_i^2}{2} y_i'', \tag{3.1}$$

$$y_{i-1} \simeq y_i - h_{i-1} y_i' + \frac{h_{i-1}^2}{2} y_i''. \tag{3.2}$$

Multiplying Eq. (3.2) by d/h_{i-1}, we get

$$\frac{d}{h_{i-1}} y_{i-1} \simeq \frac{d}{h_{i-1}} y_i - d y_i' + \frac{d h_{i-1}}{2} y_i'', \tag{3.3}$$

adding Eqs. (3.2) and (3.3) and subtracting from Eq. (3.1), we get

$$y_i'' \simeq \frac{2}{h_i(h_i + h_{i-1})} \left[\left(1 + \frac{d}{h_{i-1}}\right) y_{i-1} - \left(2 + \frac{d}{h_{i-1}}\right) y_i + y_{i+1}\right], \tag{3.4}$$

approximating Eq. (2.5) with the help of Eq. (3.4), we get

$$\frac{-2\epsilon}{h_i(h_i + h_{i-1})} \left[\left(1 + \frac{d}{h_{i-1}}\right) V_{i-1} - \left(2 + \frac{d}{h_{i-1}}\right) V_i + V_{i+1}\right] + W_i V_i = Z_i,$$

this gives

$$\frac{-2\epsilon}{h_i(h_i+h_{i-1})}\left(1+\frac{d}{h_{i-1}}\right)V_{i-1}+\left\{\frac{2\epsilon}{h_i(h_i+h_{i-1})}\left(2+\frac{d}{h_{i-1}}\right)+W_i\right\}V_i \tag{3.5}$$

$$-\frac{2\epsilon}{h_i(h_i+h_{i-1})}V_{i+1}=Z_i, \tag{3.6}$$

with $V_0=\gamma$ and $V_N=\delta$. This can be written in matrix form as

$$MV=Z, \tag{3.7}$$

where $M=[m_{ij}]; 1\le i,j\le N-1$, is a tridiagonal matrix with

$$m_{i,i+1} = \text{coefficient of } V_{i+1} \text{ in Eq. (3.7)}, \quad i=1,\ldots,N-2,$$

$$m_{i,i} = \text{coefficient of } V_i \text{ in Eq. (3.7)}, \quad i=1,\ldots,N-1,$$

$$m_{i,i-1} = \text{coefficient of } V_{i-1} \text{ in Eq. (3.7)}, \quad i=1,\ldots,N-2,$$

$V=(V_1,V_2,V_3,\ldots V_{N-1})'$ and $Z=(Z_1,Z_2\ldots Z_{N-1})'$ are the column vectors. The tridiagonal system (3.7) can easily be solved by using an efficient algorithm called discrete invariant embedding [1].

4. Mesh Selection Strategy

It is well-known that on an equidistant mesh no scheme can attain convergence at all mesh points uniformly in ϵ, unless its coefficients have an exponential property. Therefore, unless we use a specially chosen mesh, we will not be able to get ϵ-uniform convergence at all the mesh points [20].

Let N be the number of mesh points in the interval [0,1] and let $d=h_i-h_{i-1}$, be the common mesh difference. Then we have

$$\begin{aligned} x_N-x_0 &= (x_N-x_{N-1})+(x_{N-1}-x_{N-2})+\ldots\ldots+(x_1-x_0)\\ &= \{h_0+(N-1)d\}+\{h_0+(N-2)d\}+\ldots\ldots+h_0\\ &= \{(N-1)+(N-2)+\ldots\ldots+1\}d+Nh_0\\ &= \frac{1}{2}N(N-1)d+Nh_0, \end{aligned}$$

this gives

$$d=\frac{2(1-Nh_0)}{N(N-1)} \tag{4.1}$$

Therefore for given values of N and h_0, we can choose d from Eq. (4.1) and subsequent $h_i'^s$ can be obtained by $h_i=h_{i-1}+d,\ i=1,\ldots,N$. For $d=0$, mesh will reduced to uniform mesh and the corresponding difference scheme will reduced to central finite difference scheme.

In the given mesh selection strategy, the boundary layer width plays an important role. According to Miller et al. [15], if the solution of the homogenous singular perturbation problem involves the functions of the type $\exp(-x/\epsilon)$, then width of boundary layer $\delta=\mathcal{O}(\epsilon\ln(1/\epsilon))$ whereas in case the solution involves the functions of the type $\exp(-x/\sqrt{\epsilon})$, then $\delta=\mathcal{O}(\sqrt{\epsilon}\ln(1/\epsilon))$. We used this fact while solving the examples. Our mesh selection procedure needs prior knowledge of δ, h_0 and d. We have chosen δ, as above whereas the other two parameters have been taken as $h_0 \ll \delta$ and d is given by Eq. (4.1). This selection of parameters implies that there must exist at least one point in the boundary layer region, i.e., boundary layer has been resolved with this choice of parameters.

Remark 4.1. If the boundary layer occurs at the left end then we take $d > 0$, this gives more mesh points near the boundary layer. Similarly, if the boundary layer occurs at the right end then we take $d < 0$, this gives more mesh points near the boundary layer.

Remark 4.2. If the boundary layer occurs at both ends then we take $d > 0$, for first half interval $[0, 1/2]$ and we take $d < 0$ for the second half interval $[1/2, 1]$. This gives more mesh points near the boundary layer.

Remark 4.3. If the boundary layer occurs at center then we take $d < 0$, for first half interval $[0, 1/2]$ and we take its mirror image for the second half interval $[1/2, 1]$. This gives more mesh points near the boundary layer.

5. Uniform-convergence of the Scheme

Let the problem (2.5) be denoted by P_ϵ and the corresponding discritized problem be denoted by P_ϵ^N, i.e.,

$$P_\epsilon \equiv \begin{cases} -\epsilon V'' + W(x)V = Z(x), \quad \text{with} \\ V(0) = \gamma, \quad V(1) = \delta, \end{cases}$$

and

$$P_\epsilon^N \equiv \begin{cases} -\epsilon \Delta^2 V(x_i) + W(x_i)V(x_i) = Z(x_i) \quad \text{with} \\ V(0) = \gamma, \quad V(1) = \delta. \end{cases}$$

Let L_ϵ be the operator corresponding to problem P_ϵ and let L_ϵ^N be the operator corresponding to problem P_ϵ^N, i.e.,

$$L_\epsilon \equiv -\epsilon \frac{d^2}{dx^2} + W(x)I,$$

and

$$L_\epsilon^N \equiv -\epsilon \Delta^2 + W_i I,$$

where

$$\Delta^2 V(x_i) = \frac{2}{h_i(h_i + h_{i-1})} \left[\left(1 + \frac{d}{h_{i-1}}\right) V_{i-1} - \left(2 + \frac{d}{h_{i-1}}\right) V_i + V_{i+1} \right].$$

In order to prove the uniform convergence of the method, we shall use the discrete maximum principle and the following lemmas:

Lemma 5.1 (Discrete Maximum Principle) *Assume that the mesh function ϕ_i satisfies $\phi_0 \geq 0$ and $\phi_N \geq 0$. Then $L_\epsilon^N \phi_i \geq 0$ for $1 \leq i \leq N-1$ implies that $\phi_i \geq 0$, for all $0 \leq i \leq N$.*

Proof. If possible suppose $\phi_j < 0$, for some j satisfying $1 \leq j \leq N-1$. Also suppose that $\phi_k = \min_{1 \leq i \leq N-1} \phi_i$. Then we have

$$\begin{aligned} L_\epsilon^N \phi_k &= -\epsilon \Delta^2 \phi_k + W_k \phi_k \\ &= -\frac{2\epsilon}{h_k(h_k + h_{k-1})} \left[\left(1 + \frac{d}{h_{k-1}}\right) \phi_{k-1} - \left(2 + \frac{d}{h_{k-1}}\right) \phi_k + \phi_{k+1} \right] + W_k \phi_k \\ &= -\frac{2\epsilon}{h_k(h_k + h_{k-1})} \left[\left\{ \left(1 + \frac{d}{h_{k-1}}\right) (\phi_{k-1} - \phi_k) \right\} + (\phi_{k+1} - \phi_k) \right] + W_k \phi_k. \end{aligned}$$

By the definition of ϕ_k, we have $\phi_{k-1} - \phi_k$ and $\phi_{k+1} - \phi_k$ are both positive. Consequently, it follows from the above equation that $L_\epsilon^N \phi_k < 0$, which contradicts the hypothesis and hence $\phi_i \geq 0$, for all $0 \leq i \leq N$. □

Lemma 5.2. *Let ϕ_i be any mesh function such that $\phi_0 = \phi_N = 0$. Then*

$$| \phi_i | \leq \frac{1}{\mu} \max_{1 \leq j \leq N-1} | L_\epsilon^N \phi_j |, \quad for \quad 0 \leq i \leq N. \tag{5.1}$$

Proof. Let the right hand side of (5.1) be M. Define two mesh functions ψ_i^+ and ψ_i^- such that

$$\psi_i^\pm = M \pm \phi_i.$$

Then $\psi_0^\pm = \psi_N^\pm = M > 0$ and for $1 \leq i \leq N-1$, we have

$$\begin{aligned} L_\epsilon^N \psi_i^\pm &= -\epsilon \Delta^2 \psi_i^\pm + W_i \psi_i^\pm \\ &= -\epsilon \Delta^2 (M \pm \phi_i) + W_i (M \pm \phi_i) \\ &= M W_i \pm L_\epsilon^N \phi_i \geq M\mu \pm L_\epsilon^N \phi_i \geq 0. \end{aligned}$$

Therefore by discrete maximum principle we have

$$\psi_i^\pm \geq 0, \text{ for } 0 \leq i \leq N,$$

and hence we have the desired estimate (5.1). □

Lemma 5.3. *Let $x_0 = 0$, $d = h_i - h_{i-1}$ be the common mesh difference,*

$$x_i = \sum_{k=0}^{i-1} h_k, \text{ for } 1 \leq i \leq N-1; \qquad x_N = 1.$$

Then

$$h_i \leq \frac{3}{N}, \quad for \quad 0 \leq i \leq N-1.$$

Proof. For $0 \leq i \leq N-1$, we have

$$\begin{aligned} h_i &= h_{i-1} + d = h_0 + id \\ &= \frac{1}{N}\left[1 - \frac{N(N-1)}{2} d\right] + id \leq \frac{1}{N} + id \\ &\leq \frac{1}{N} + i\frac{2}{N(N-1)} \leq \frac{3}{N}. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 5.4. *For every $\phi \in C^3(0,1)$, we have*

$$\left\| \left(\Delta^2 - \frac{d^2}{dx^2} \right) \phi \right\| \leq \frac{2}{N} \|\phi\|_3, \quad 0 \leq i \leq N, \tag{5.2}$$

where

$$\| \phi \|_j = \sup_{x \in (0,1)} \|\phi^{(j)}_{(x)}\|.$$

Proof. Note that $x_0 = 0$, $x_i = \Sigma_{k=0}^{i-1} h_k$, $i = 1, \ldots, N$, $h_k = x_{k+1} - x_k$, $x_N = 1$. Let d be the common mesh difference $h_i - h_{i-1}$. Taking Taylor's series expansion and neglecting the term of fourth and higher order, we get the following expansions for ϕ_{i+1} and ϕ_{i-1}, $0 \le i \le N-1$:

$$\phi_{i+1} \simeq \phi_i + h_i\phi_i' + \frac{h_i^2}{2}\phi_i'' + \frac{h_i^3}{6}\phi_i'''(\xi_1), \qquad x_i \le \xi_1 \le x_{i+1}, \tag{5.3a}$$

$$\phi_{i-1} \simeq \phi_i - h_{i-1}\phi_i' + \frac{h_{i-1}^2}{2}\phi_i'' - \frac{h_{i-1}^3}{6}\phi_i'''(\xi_2), \qquad x_{i-1} \le \xi_2 \le x_i. \tag{5.3b}$$

Multiplying Eq. (5.3b) by d/h_{i-1} gives

$$\frac{d}{h_{i-1}}\phi_{i-1} = \frac{d}{h_{i-1}}\phi_i - d\phi_i' + \frac{dh_{i-1}}{2}\phi_i'' - \frac{dh_{i-1}^2}{6}\phi_i'''(\xi_2). \tag{5.4}$$

Adding Eq. (5.3b) and Eq. (5.4) we get

$$\left(1 + \frac{d}{h_{i-1}}\right)\phi_{i-1} = \left(1 + \frac{d}{h_{i-1}}\right)\phi_i - h_i\phi_i' + \frac{h_i h_{i-1}}{2}\phi_i'' + \frac{h_i h_{i-1}^2}{6}\phi_i'''(\xi_2); \tag{5.5}$$

and adding Eq. (5.3a) and Eq. (5.5) yields

$$\begin{aligned}\left(\Delta^2 - \frac{d^2}{dx^2}\right)\phi(x_i) &= \frac{1}{3(h_i + h_{i-1})}[h_i^2\phi_i'''(\xi_1) - h_{i-1}^2\phi_i'''(\xi_2)],\\ &\le \frac{2h_i^2}{3(h_i + h_{i-1})}\|\phi\|_3 \le \frac{2}{3}h_i\|\phi\|_3.\\ &\le \frac{2}{N}\|\phi\|_3, \qquad \text{(using Lemma 5.3)}\end{aligned}$$

which lends to (5.2). □

Lemma 5.5. ([15]) *The solution u_ϵ of P_ϵ has the form*

$$u_\epsilon = v_\epsilon + w_\epsilon, \tag{5.6}$$

where the smooth component v_ϵ and singular component w_ϵ satisfy

$$| v_\epsilon^{(k)}(x) | \le C\,(1 + \epsilon^{-(k-2)/2} e(x,\mu)), \quad 0 \le k \le 3, \quad x \in [0,1], \tag{5.7a}$$

$$| w_\epsilon^{(k)}(x) | \le C\,\epsilon^{-k/2} e(x,\mu), \qquad 0 \le k \le 3, \quad x \in [0,1], \tag{5.7b}$$

where

$$e(x,\mu) = e^{-x\sqrt{\mu/\epsilon}} + e^{-(1-x)\sqrt{\mu/\epsilon}}.$$

Now we shall give the following theorem to show the uniform convergence of the proposed method.

Theorem 5.1. *The variable mesh finite difference scheme P_ϵ^N is ϵ-uniform for the problem P_ϵ. Moreover the solution u_ϵ of P_ϵ and the solution U_ϵ of P_ϵ^N satisfy the following ϵ-uniform error estimate*

$$\sup_{0<\epsilon\le 1} \| U_\epsilon - u_\epsilon \| \le CN^{-1}\ln N, \tag{5.8}$$

where C is a positive constant independent of ϵ.

Proof. Decompose the solution U_ϵ, of the discrete problem in smooth (R_ϵ) and singular (S_ϵ) components

$$U_\epsilon = R_\epsilon + S_\epsilon,$$

where R_ϵ is the solution of inhomogeneous problem

$$L_\epsilon^N R_\epsilon = Z, \quad R_\epsilon(a) = r_\epsilon(a), \quad R_\epsilon(b) = r_\epsilon(b), \tag{5.9}$$

and S_ϵ is the solution of the homogeneous problem

$$L_\epsilon^N S_\epsilon = 0, \quad S_\epsilon(a) = s_\epsilon(a), \quad S_\epsilon(b) = s_\epsilon(b). \tag{5.10}$$

Here r_ϵ and s_ϵ, are the regular component and singular component of the continuous problem P_ϵ and defined as:

$$u_\epsilon = r_\epsilon + s_\epsilon,$$

with $r_\epsilon = r_0 + \epsilon r_1$, where r_0 is the solution of the reduced problem, s_ϵ is the solution of the homogeneous problem

$$L_\epsilon s_\epsilon = 0, \qquad s_\epsilon(0) = u_0 - r_0(0), \quad s_\epsilon(1) = u_1 - r_0(1), \tag{5.11}$$

and r_1 satisfies

$$L_\epsilon r_1 = r_0'', \qquad r_1(0) = 0, \quad r_1(1) = 0. \tag{5.12}$$

Then we have $U_\epsilon - u_\epsilon = (R_\epsilon - r_\epsilon) + (S_\epsilon - s_\epsilon)$, this gives

$$\| U_\epsilon - u_\epsilon \| \leq \| (R_\epsilon - r_\epsilon) \| + \| (S_\epsilon - s_\epsilon) \| \tag{5.13}$$

Now we have

$$\begin{aligned} L_\epsilon^N (R_\epsilon - r_\epsilon) &= Z - L_\epsilon^N r_\epsilon = (L_\epsilon - L_\epsilon^N) r_\epsilon \\ &= -\epsilon \left(\frac{d^2}{dx^2} - \Delta^2 \right) r_\epsilon. \end{aligned}$$

Therefore Lemma 5.4 gives

$$\| L_\epsilon^N (R_\epsilon - r_\epsilon)(x_i) \| \leq \frac{2}{N} \epsilon \| r_\epsilon \|_3 .$$

Using Lemma 5.5 for the estimate of r_ϵ''' yields

$$\| L_\epsilon^N (R_\epsilon - r_\epsilon)(x_i) \| \leq C \sqrt{\epsilon} N^{-1} \leq C N^{-1}.$$

An application of Lemma 5.1 gives

$$\|(R_\epsilon - r_\epsilon)\| \leq CN^{-1}. \tag{5.14}$$

Now by the same argument as that for the smooth component, the error for the singular component of the solution is given by

$$\|L_\epsilon^N (S_\epsilon - s_\epsilon)(x_i)\| \leq \frac{2}{N} \epsilon \|s_\epsilon\|_3.$$

Using Lemma 5.5 for the estimate for s_ϵ''' we obtain

$$\| L_\epsilon^N (S_\epsilon - s_\epsilon)(x_i) \| \leq C \epsilon^{-\frac{1}{2}} N^{-1}.$$

Table 6.1: Maximum absolute error for Example 1, for different values of ϵ and N

ϵ	N=100		N=500		N=1000		N=1500	
	uniform mesh	variable mesh	uniform mesh	variable mesh	uniform mesh	variable mesh	uniform mesh	variable mesh
2^{-4}	1.20E-03	3.59E-04	4.66E-05	1.42E-05	1.17E-05	3.56E-06	5.18E-06	1.58E-06
2^{-6}	1.80E-03	4.64E-04	7.22E-05	1.84E-05	1.81E-05	4.60E-06	8.03E-06	2.04E-06
2^{-8}	5.70E-03	7.04E-04	2.30E-04	2.79E-05	5.74E-05	6.97E-06	2.55E-05	3.10E-06
2^{-10}	1.94E-02	1.30E-03	8.34E-04	5.21E-05	2.09E-04	1.30E-05	9.29E-05	5.80E-06
2^{-12}	6.74E-02	2.60E-03	3.20E-03	1.03E-04	7.93E-04	2.57E-05	3.53E-04	1.14E-05
2^{-14}	1.27E-01	5.10E-03	1.21E-02	2.05E-04	3.10E-03	5.13E-05	1.40E-03	2.28E-05
2^{-16}	9.12E-02	1.03E-02	4.47E-02	4.10E-04	1.19E-02	1.02E-04	5.40E-03	4.56E-05
2^{-18}	2.90E-02	2.07E-02	1.11E-01	8.20E-04	4.44E-02	2.05E-04	1.97E-02	9.10E-05
2^{-20}	7.40E-03	3.79E-02	1.11E-01	1.60E-03	1.11E-01	4.09E-04	7.11E-02	1.82E-04
2^{-22}	1.90E-03	7.55E-02	4.29E-02	3.20E-03	1.11E-01	8.20E-04	1.26E-01	3.64E-04
2^{-24}	4.65E-04	3.51E-02	1.12E-02	6.50E-03	4.27E-02	1.60E-03	8.18E-02	7.28E-04
E^N	1.27E-01	7.55E-02	1.11E-01	6.50E-03	1.11E-01	1.60E-03	1.26E-01	7.28E-04

Now since $\mu = \sqrt{\epsilon}\ln N$, we have

$$\| L_\epsilon^N(S_\epsilon - s_\epsilon)(x_i) \| \le CN^{-1}\ln N,$$

and an application of Lemma 5.1 to the function $S_\epsilon - s_\epsilon$ gives

$$\| (S_\epsilon - s_\epsilon) \| \le CN^{-1}\ln N, \tag{5.15}$$

Combining the relations (5.13) - (5.15) yields the desired estimate (5.8). □

6. Test Examples and Numerical Results

Example 1. First, we consider the following singular perturbation problem [19]

$$-\epsilon y'' + \frac{4}{(x+1)^4}\Big(1+\sqrt{\epsilon}(x+1)\Big)y = f(x), \quad y(0)=2, \quad y(1)=-1, \tag{6.1}$$

where $f(x)$ is chosen such that the exact solution is given by

$$y(x) = -\cos\left(\frac{4\pi x}{x+1}\right) + \frac{3[\exp(-2x/\sqrt{\epsilon}(x+1)) - \exp(-1/\sqrt{\epsilon})]}{1-\exp(-1/\sqrt{\epsilon})}. \tag{6.2}$$

Table 6.2: Results of Stynes [19] for Example 1(maximum errors)

N	$\epsilon=(1/N)^{0.25}$	$\epsilon=(1/N)^{0.5}$	$\epsilon=(1/N)^{0.75}$	$\epsilon=(1/N)^{1.0}$
16	9.5E-02	7.8E-02	6.6E-02	6.4E-02
32	2.3E-02	1.8E-02	1.6E-02	1.7E-02
64	5.6E-03	4.2E-03	4.0E-03	4.2E-03
128	1.3E-03	1.0E-03	1.0E-03	1.3E-03
256	3.1E-04	2.5E-04	2.6E-04	3.7E-04

Example 2. Next, we consider the problem [21]

$$-\epsilon y'' + y = f(x), \quad y(0) = 0, \quad y(1) = 0,$$

where $f(x)$ is chosen such that the exact solution of the problem is given by

$$y(x) = \exp(x) + \exp(-x/\sqrt{\epsilon}) - x\Big(\exp(1) + \exp(-1/\sqrt{\epsilon})\Big) - 2(1-x).$$

Example 3. Next, we consider the problem having two boundary layers at each end point [23]:

$$-\epsilon y'' + (1 + x(1-x))y = f(x), \quad y(0) = 0, \quad y(1) = 0,$$

where $f(x)$ is chosen such that the exact solution is given by

$$y(x) = 1 + (x-1)\exp(-x/\sqrt{\epsilon}) - x\exp(-(1-x)/\sqrt{\epsilon}).$$

Table 6.3: Results of Patidar [11] for Example 1(maximum errors)

N	$\epsilon = (1/N)^{0.25}$	$\epsilon = (1/N)^{0.5}$	$\epsilon = (1/N)^{0.75}$	$\epsilon = (1/N)^{1.0}$
16	5.8E-02	4.8E-02	4.8E-02	4.80E-02
32	1.3E-02	1.2E-02	1.2E-02	1.2E-02
64	3.2E-03	3.1E-03	3.0E-03	3.4E-03
128	7.8E-04	1.0E-03	7.6E-04	1.0E-03
256	1.9E-04	1.9E-04	2.1E-04	3.1E-04

Table 6.4: Results of Lubuma [12] for Example 1(maximum errors)

N	$\epsilon = (1/N)^{0.25}$	$\epsilon = (1/N)^{0.5}$	$\epsilon = (1/N)^{0.75}$	$\epsilon = (1/N)^{1.0}$
16	3.8E-02	2.5E-02	1.6E-02	1.4E-02
32	9.6E-03	6.3E-03	4.3E-03	7.9E-03
64	2.4E-03	1.6E-03	1.1E-03	2.4E-03
128	6.0E-04	3.9E-04	2.7E-04	6.2E-04
256	1.5E-04	9.8E-05	6.9E-05	1.6E-04

Table 6.5: Our results for Example 1(maximum errors)

N	$\epsilon = (1/N)^{0.25}$	$\epsilon = (1/N)^{0.5}$	$\epsilon = (1/N)^{0.75}$	$\epsilon = (1/N)^{1.0}$
16	2.0E-02	1.7E-02	1.5E-02	1.4E-02
32	4.7E-03	4.0E-03	3.4E-03	4.1E-03
64	1.1E-03	9.1E-04	9.3E-04	1.1E-03
128	2.6E-04	2.0E-04	2.4E-04	3.2E-04
256	6.1E-05	5.0E-05	6.4E-05	9.6E-05

Table 6.6: Maximum absolute error for Example 2, for different values of ϵ and N

ϵ	N=100		N=500		N=1000		N=1500	
	uniform mesh	variable mesh	uniform mesh	variable mesh	uniform mesh	variable mesh	uniform mesh	variable mesh
2^{-4}	2.48E-05	2.38E-05	9.90E-07	9.44E-07	2.48E-07	2.36E-07	1.10E-07	1.05E-07
2^{-6}	9.81E-05	5.31E-05	3.93E-06	2.10E-06	9.82E-07	5.26E-07	4.36E-07	2.34E-07
2^{-8}	3.91E-04	1.07E-04	1.57E-05	4.26E-06	3.92E-06	1.06E-06	1.74E-06	4.73E-07
2^{-10}	1.60E-03	2.15E-04	6.27E-05	8.53E-06	1.57E-05	2.13E-06	6.98E-06	9.48E-07
2^{-12}	5.90E-03	4.31E-04	2.51E-04	1.71E-05	6.27E-05	4.26E-06	2.79E-05	1.90E-06
2^{-14}	2.15E-02	8.61E-04	9.98E-04	3.41E-05	2.51E-04	8.53E-06	1.12E-04	3.79E-06
2^{-16}	4.12E-02	1.70E-03	3.90E-03	6.82E-05	9.98E-04	1.71E-05	4.45E-04	7.58E-06
2^{-18}	2.95E-02	3.40E-03	1.47E-02	1.36E-04	3.90E-03	3.41E-05	1.80E-03	1.52E-05
2^{-20}	9.30E-03	6.90E-03	3.69E-02	2.73E-04	1.47E-02	6.82E-05	6.50E-03	3.03E-05
E^N	2.95E-02	6.90E-03	3.69E-02	2.73E-04	1.47E-02	6.82E-05	6.50E-03	3.03E-05

Table 6.7: Comparisons of maximum errors for Example 2 with those in [21], for $\epsilon = 5^{-6}$.

	N=20	N=40	N=80	N=160
Scheme in [21]	3.3E-02	9.4E-03	1.6E-03	1.9E-04
Our scheme	2.1E-02	5.1E-03	1.0E-03	9.1E-05

Table 6.8: Comparisons of maximum absolute error for Example 3 with those in [11]

	Results in [11]		Our Results	
ϵ	uniform mesh	variable mesh	uniform mesh	variable mesh
1/8	2.30E-05	2.60E-05	2.27E-05	4.59E-05
1/16	3.10E-05	3.90E-05	3.06E-05	2.44E-05
1/32	4.50E-05	5.30E-05	4.54E-05	2.92E-05
1/64	7.80E-05	7.00E-05	7.84E-05	3.74E-05
1/128	1.50E-04	9.00E-05	1.45E-04	4.58E-05
1/256	2.80E-04	1.10E-04	2.74E-04	5.29E-05
1/512	5.30E-04	1.40E-04	5.26E-04	5.47E-05
1/1024	1.00E-03	2.00E-04	1.02E-03	5.82E-05
1/2048	2.00E-03	3.00E-04	1.99E-03	1.28E-04

Table 6.9: Maximum absolute error for Example 4, for different values of ϵ and N

ϵ	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048
2^{-4}	1.23E-02	1.35E-02	1.40E-02	1.42E-02	1.43E-02	1.43E-02	1.44E-02
2^{-8}	3.03E-03	2.89E-03	2.88E-03	2.88E-03	2.88E-03	2.88E-03	2.88E-03
2^{-12}	1.11E-02	2.60E-03	5.71E-04	2.98E-04	2.87E-04	2.84E-04	2.83E-04
2^{-16}	3.49E-02	1.06E-02	2.56E-03	6.39E-04	1.54E-04	3.37E-05	2.17E-05
2^{-20}	3.89E-02	3.48E-02	1.04E-02	2.54E-03	6.39E-04	1.59E-04	3.91E-05
2^{-24}	8.76E-02	3.80E-02	3.47E-02	1.03E-02	2.55E-03	6.36E-04	1.56E-04

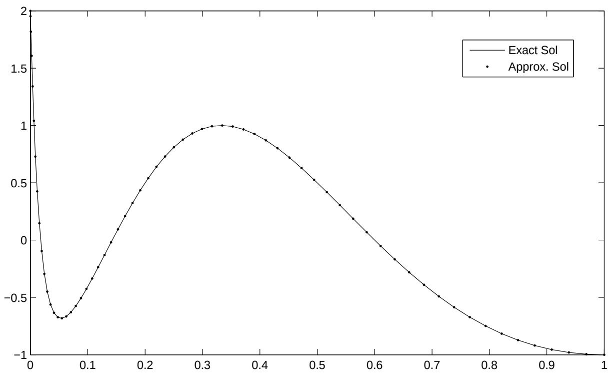

Figure 1. Exact solution and approximate solution of Example 1, for $\epsilon = 10^{-3}$ and $N = 64$.

Example 4. Finally, we consider the problem having two boundary layers at each end point:

$$-\epsilon\Big((1+x^2)y'\Big)' + \Big(1 + x(1-x)\Big)y = f(x), \quad y(0) = 0, \quad y(1) = 0,$$

where $f(x)$ is chosen such that the exact solution is of the form:

$$y(x) = 1 + (x-1)\exp(-x/\sqrt{\epsilon}) - x\exp(-(1-x)/\sqrt{\epsilon}).$$

It is noted that the normal form is given by

$$-\epsilon V'' + W(x)V = Z(x),$$

where

$$W(x) = \frac{[1 + x(1-x) + \epsilon(1+x^2)]}{(1+x^2)}, \quad Z(x) = f(x)/\sqrt{1+x^2}.$$

In this section, the maximum absolute error at nodal points is given by

$$E_\epsilon^N = \max_i \mid y(x_i) - y_i \mid,$$

and the ϵ-uniform maximum nodal error is defined by

$$E^N = \max_\epsilon E_\epsilon^N, \qquad 0 < \epsilon \ll 1.$$

7. Discussion and Conclusion

We have described a practical method for solving self-adjoint singularly perturbed two-point boundary value problems. The method is shown to be uniformly convergent, i.e., independent of perturbation parameter ϵ. Several examples have been solved to demonstrate the efficiency of the presented method.

Numerical results for Examples 1 and 2 are presented in Tables 6.1 and 6.6 with uniform mesh and with variable mesh for different values of parameter ϵ and the number of mesh points N. The numerical results presented in Tables 6.1 and 6.6 clearly indicate that the proposed scheme with uniform mesh is not uniformly convergent for sufficiently small value of ϵ and

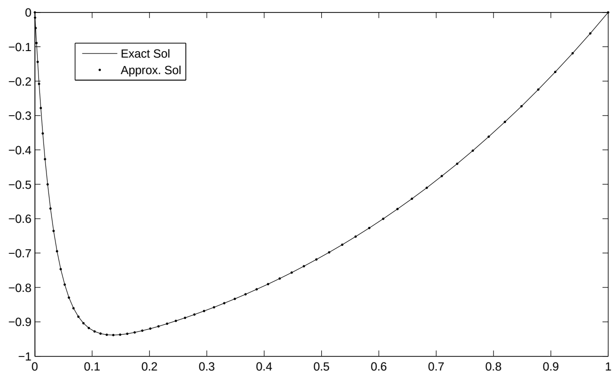

Figure 2. Exact solution and approximate solution of Example 1, for $\epsilon = 10^{-3}$ and $N = 64$.

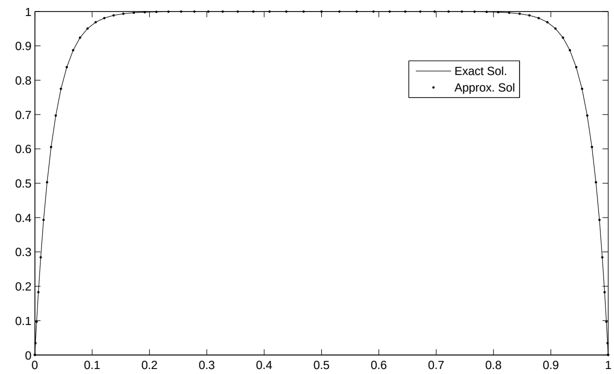

Figure 3. Exact solution and approximate solution of Example 1, for $\epsilon = 10^{-3}$ and $N = 64$.

the maximal nodal error increases as ϵ decreases. To overcome this drawback, we have used a variable mesh as described above. The numerical results displayed in Tables 6.1 and 6.6 clearly indicate that the proposed method based on a finite difference scheme with given mesh is ϵ-uniformly convergent. The proposed numerical method is accurate of order $\mathcal{O}(N^{-1}\ln N)$. Tables 6.2–6.5 and 6.7–6.8 shows, how the present method is more efficient than the methods given in [11,12,19,21]. To further corroborate the applicability of the proposed method, graphs between exact solution and approximate solutions have been plotted for the first three examples for a fixed $\epsilon = 10^{-3}$ and $N = 64$. It is observsed that the numerical solutions are in very good agreement of the exact solution.

Acknowledgments. The authors are grateful to the referees for their comments and suggestions to improve the quality of the paper.

References

[1] E. Angel and R. Bellman, Dynamic Programming and Partial Differential Equations, Academic Press, New York, 1972.

[2] O. Axelsson, L. S. Frank, and A. Vander Sluis, Analytical and Numerical Approaches to Asymptotic Problems in Analysis, North Holland, Amsterdam, 1981.

[3] R. Bellman, Perturbation Techniques in Mathematics, Physics and Engineering, Holt Rinehart & Winston 1964.

[4] C.M. Bender and S.A. Orszag, Advanced Mathematical Methods for Scientist and Engineers, McGraw-Hill, New York, 1978.

[5] J.D. Cole and J. Kevorkian, Perturbation Methods in Applied Mathematics, Springer, New York, 1979.

[6] E.P. Doolan, J.J. H. Miller and W.H.A. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, 1980.

[7] W. Eckhaus, Asymptotic Analysis of Singular Perturbations, North Holland, New York, 1979.

[8] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, R.E. O'Riordan and G.I. Shishkin, Robust Computational Techniques for Boundary Layers, Chapman-Hall/CRC, New York, 2000.

[9] P.A. Farrell, E. O'Riordan and G.I. Shishkin, A class of singularly perturbed semilinear differential equations with interior layers, *Math. Comput.*, **74** (2005), 1759–1776.

[10] P.W. Hemker and J.J.H. Miller, Numerical Analysis of Singular Perturbation Problems, Academic Press, New York, 1979.

[11] M.K. Kadalbajoo and K.C. Patidar, Spline approximation method for solving self-adjoint singular perturbation problems on non-uniform grids, *J. Comput. Anal. Appl.*, **5** (2003), 425-451.

[12] J.M.-S. Lubuma and K.C. Patidar, Uniformly convergent non-standard finite difference methods for self-adjoint singular perturbation problems, *J. Comput. Appl. Math.*, **191** (2006), 228-238.

[13] R.E. O'Malley, Introduction to Singular Perturbations, Academic Press, New York, 1974.

[14] J.J.H. Miller, On a convergence, uniformly in ϵ, of finite difference schemes for a two point boundary singular perturbation problems, *Proc. Conf. Math. Ins. Catholic Univ.Nijmegen, 1978*, Academic Press, New York, 1979, 467-474.

[15] J.J.H. Miller, E. O'Riordan and G.I. Shishkin, Fitted Numerical Methods for Singular Perturbation Problems, World Scientific, Singapore, 1996.

[16] A.H. Nayfeh, Perturbation Methods, Wiley, New York, 1973.

[17] K. Niijima, On a three point difference scheme for a singular perturbation problem without a first derivative term, I, II., *Mem. Numer. Math*, **7** (1980), 1-27.

[18] M.H. Protter and H.F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, New Jersey, 1967.

[19] E. O'Riordan and M. Stynes, A uniformly accurate finite element method for a singularly perturbed one-dimensional reaction-diffusion problem, *Math. Comput.*, **47** (1986), 555-570.

[20] H.G. Roos, M. Stynes and L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations, Convection-Diffusion and Flow Problems, Springer Verlag, Berlin, 1996.

[21] A.H. Schatz and L.B. Wahlbin, On the finite element method for singularly perturbed reaction diffusion problems in two and one dimension, *Math. Comput.*, **40** (1983), 47-89.

[22] M. Vandyke, Perturbation Methods in Fluid Mechanics, Parabolic, Stanford, Calif, 1975.

[23] V. Vukoslavčević and K. Surla, Finite element method for solving self adjoint singularly perturbed boundary value problems, *Math. Montisnigri*, **8** (1996), 89-96.