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THE FINITE DIFFERENCE METHOD FOR DISSIPATIVE KLEIN–GORDON–SCHRÖDINGER EQUATIONS IN THREE SPACE DIMENSIONS*

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Abstract

A fully discrete finite difference scheme for dissipative Klein–Gordon–Schrödinger equations in three space dimensions is analyzed. On the basis of a series of the time-uniform priori estimates of the difference solutions and discrete version of Sobolev embedding theorems, the stability of the difference scheme and the error bounds of optimal order for the difference solutions are obtained in $H^2 \times H^2 \times H^1$ over a finite time interval. Moreover, the existence of a maximal attractor is proved for a discrete dynamical system associated with the fully discrete finite difference scheme.

Mathematics subject classification: 65M06, 35Q55, 65P99.
Key words: Dissipative Klein–Gordon–Schrödinger equations, Finite difference method, Error bounds, Maximal attractor.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$, we shall consider a finite difference approximation of dissipative Klein–Gordon–Schrödinger (KGS) equations [1]

$$i\psi_t + \Delta\psi + i\alpha\psi + \phi\psi = F, \qquad in \ \ \Omega, \ t > 0 \tag{1.1a}$$

$$\phi_{tt} + \beta\phi_t - \Delta\phi + \mu^2\phi = |\psi|^2 + G, \quad in \ \ \Omega, \ t > 0 \tag{1.1b}$$

with boundary condition

$$(\psi, \phi)|_{\partial\Omega} = 0, \quad t > 0, \tag{1.2}$$

and initial conditions

$$(\psi, \phi, \phi_t)(x, 0) = (\psi_0, \phi_0, \phi_1)(x), \quad in \ \ \Omega, \tag{1.3}$$

where ψ and ϕ represent a complex scalar neucleon field and a real meson field respectively, α, β and μ^2 are positive constants, F and G are given complex and real functions, respectively.

It is convenient to reduce (1.1) to an evolution equation of the first order in time. For this purpose, let $\varepsilon > 0$ be a fixed constant, satifying $\varepsilon \le \min\big(\beta/2, \mu^2/\beta\big)$. We introduce $\theta = \phi_t + \varepsilon\phi$.

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Then the problem (1.1)–(1.3) is equivalent to the following problem

$$i\psi_t + \Delta\psi + i\alpha\psi + \phi\psi = F, \quad in \;\; \Omega, \; t > 0, \tag{1.4a}$$

$$\phi_t + \varepsilon\phi - \theta = 0, \quad in \;\; \Omega, \; t > 0, \tag{1.4b}$$

$$\theta_t + (\beta - \varepsilon)\theta + \Big(\mu^2 - \varepsilon(\beta - \varepsilon) - \Delta\Big)\phi = |\psi|^2 + G, \quad in \;\; \Omega, \; t > 0, \tag{1.4c}$$

with boundary conditions

$$(\psi, \phi, \theta)|_{\partial\Omega} = 0, \quad t > 0, \tag{1.5}$$

and initial conditions

$$(\psi, \phi, \theta)(x, 0) = (\psi_0, \phi_0, \theta_0)(x), \quad in \;\; \Omega. \tag{1.6}$$

In the conservative case, i.e., $\alpha = \beta = 0$, $F = G = 0$, the system has been studied by many authors, see, e.g., [2, 3, 5, 13] and so on. In the dissipative case ($\alpha > 0, \beta > 0$), the long time behavior of infinite dimensional dynamical system $S(t)$ associated with the initial boundary value problem (1.1)–(1.3) has been studied in [1, 4, 9]. Biler [1] proved that the maximal attractor $\mathcal{A}$ exists in the weak topology of $H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$, which has finite Hausdorff and fractal dimension. Li [9] proved the existence of finite dimensional maximal attractor in the topology of $H^2(\Omega) \cap H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

At the same time, in numerical simulation of the continuous dynamical system, we are interested in study of the dynamical properties of the discrete dynamical system associated with the numerical scheme for problem (1.1). It is important that the discrete dynamical system can remain some properties of the continuous dynamical system such as dissipatedness. It is our purpose in this paper to consider a discrete dynamical system associated with the fully discrete finite difference scheme for problem (1.4)–(1.6). It will be proved that for each mesh size, the discrete dynamical system also possesses a maximal attractor. A similar problem was studied by many authors, see, e.g., [7, 11, 12, 14, 15].

The rest of this paper is organized as follows. After introducing some notations, in Sect. 2 we give several embedding theorems and interpolation inequalities for discrete functions, which are the analogues of embedding theorems and interpolation inequalities for the Sobolev space $W^{m,p}(\Omega)$. In Sect. 3, a fully discrete finite difference scheme is established for problem (1.4) with the homogeneous Dirichlet boundary condition (1.5). The existence of the solutions of the fully discrete finite difference scheme is proved by using the Leray-Schauder fixed point theorem. Then we establish some uniform bounds of the solutions in suitable norms. In Sect. 4, we obtain the stability and the convergence properties for the finite difference scheme over a finite time interval $(0, T]$. Finally, in Sect. 5, by regarding the fully discrete finite difference scheme as a discrete dynamical system $S_{h,\Delta t}(t_n)$ that is an approximation of the dynamical system $S(t)$, and by using the results in Sections 3 and 4, we prove the existence of an absorbing set and an attractor for the discrete dynamical system $S_{h,\Delta t}(t_n)$.

2. Some Notations and Lemmas

Assume that the domain Ω is the three-dimensional rectangular domain $(0, l_1) \times (0, l_2) \times (0, l_3)$, where l_i $(i = 1, 2, 3)$ are positive constants. Let us divide the domain $\overline{\Omega}$ into small grids by the parallel planes $x = ih_1$ $(0 \le i \le J_1), y = jh_2$ $(0 \le j \le J_2)$ and $z = kh_3$ $(0 \le k \le J_3)$, where h_1, h_2, h_3 are the spatial mesh lengths, J_1, J_2, J_3 are positive integers, and $J_i h_i = l_i$ $(i = 1, 2, 3)$. Let $\psi_h, \phi_h, u_h, v_h, \cdots$ denote complex-valued or real-valued discrete functions defined on the grid point set $\overline{\Omega}_h = \Big\{(x_i, y_j, z_k) = (ih_1, jh_2, kh_3); 0 \le i \le J_1, 0 \le j \le J_2, 0 \le k \le J_3\Big\}$. Set $\Omega_h = \overline{\Omega}_h \cap \Omega, \partial\Omega_h = \overline{\Omega}_h \cap \partial\Omega$, and let Δt denote the temporal mesh length, $t_n = n\Delta t, t_{n+\frac{1}{2}} = (n+\frac{1}{2})\Delta t, n = 0, 1, \cdots$. We employ δ_l and $\overline{\delta}_l$ to denote the forward difference quotient and the backward difference quotient operators of one order in x_l $(1\le l\le 3)$ directions respectively, and Δ_h and ∇_h to denote the discrete Laplace operator and the discrete gradient operator as follows

$$(\Delta_h u_h)_{i,j,k} = \Delta_h u_{i,j,k} = \sum_{l=1}^{3} \delta_l \overline{\delta}_l u_{i,j,k},$$
$$(\nabla_h u_h)_{i,j,k} = (\delta_1 u_{i,j,k}, \delta_2 u_{i,j,k}, \delta_3 u_{i,j,k}) .$$

Let $L^2(\Omega_h)$ be the space of complex-valued or real-valued discrete functions, equipped with the discrete L^2 inner product

$$(u_h, v_h) = \sum_{i=0}^{J_1}\sum_{j=0}^{J_2}\sum_{k=0}^{J_3} u_{i,j,k}\overline{v}_{i,j,k} h_1 h_2 h_3$$

and the associated norm

$$\|u_h\| = (u_h, u_h)^{\frac{1}{2}}.$$

Let $\delta^\alpha = \delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}$ denote a forward difference quotient operator of $|\alpha|$ order, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is an 3-index, α_i $(i = 1, 2, 3)$ are nonnegative integers, and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. We use the notations

$$|u_h|_k = \left(\sum_{|\alpha|=k}\sum_{i=0}^{J_1-\alpha_1}\sum_{j=0}^{J_2-\alpha_2}\sum_{k=0}^{J_3-\alpha_3} |\delta^\alpha u_{i,j,k}|^2 h_1 h_2 h_3\right)^{1/2} \quad \text{and} \quad \|u_h\|_k = \left(\sum_{l=0}^{k} |u_h|_l^2\right)^{1/2}$$

to denote respectively Sobolev's semi-norm and norm for the discrete function u_h. Let $H^k(\Omega_h)$ denote space of complex-valued or real-valued discrete functions, with the Sobolev's form norm $\|\cdot\|_k$. $H_0^1(\Omega_h)$ is the subspace of the space $H^1(\Omega_h)$ satisfying the homogeneous Dirichlet boundary condition. Two product spaces will be of special interest in the functional setting for the discrete dynamical system, namely

$$\mathbf{V}_h = H_0^1(\Omega_h) \times H_0^1(\Omega_h) \times L^2(\Omega_h),$$
$$\mathbf{W}_h = H^2(\Omega_h) \cap H_0^1(\Omega_h) \times H^2(\Omega_h) \cap H_0^1(\Omega_h) \times H_0^1(\Omega_h)$$

with the norms respectively

$$\|(\psi_h, \phi_h, \theta_h)\|_{\mathbf{V}_h} = \left(\|\psi_h\|_1^2 + \|\phi_h\|_1^2 + \|\theta_h\|^2\right)^{1/2},$$
$$\|(\psi_h, \phi_h, \theta_h)\|_{\mathbf{W}_h} = \left(\|\psi_h\|_2^2 + \|\phi_h\|_2^2 + \|\theta_h\|_1^2\right)^{1/2}.$$

In addition, we define the discrete L^p $(p \ge 1)$ norm and the discrete L^∞ norm for the discrete functions u_h as follows

$$\|u_h\|_{L^p} = \left(\sum_{i=0}^{J_1}\sum_{j=0}^{J_2}\sum_{k=0}^{J_3} |u_{i,j,k}|^p h_1 h_2 h_3\right)^{1/p}, \qquad \|u_h\|_\infty = \max_{i,j,k} |u_{i,j,k}|.$$

The summation by parts formula

$$\sum_{i=1}^{J_1-1}\sum_{j=1}^{J_2-1}\sum_{k=1}^{J_3-1}u_{i,j,k}\Delta_h v_{i,j,k}h_1h_2h_3=-\sum_{i=0}^{J_1-1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}(\nabla_h u_h)_{i,j,k}\cdot(\nabla_h v_h)_{i,j,k}h_1h_2h_3,\quad \forall u_h,v_h\in H_0^1(\Omega_h)$$

will be used frequently in the analysis below.

For brevity, we denote the space of complex-valued discrete functions and real-valued discrete functions by the same symbols. Finally, let C denote some positive constants independent of discretization parameters, and not necessarily the same at different places.

We shall use repeatedly discrete version of Sobolev embedding theorems and interpolation inequalities.

Lemma 2.1. *For all* $u_h \in H_0^1(\Omega_h)$,

$$\|u_h\|_{L^6} \leq \frac{\sqrt[3]{36}}{2}\prod_{l=1}^{3}\|\delta_l u_h\|^{\frac{1}{3}} \leq \frac{\sqrt[6]{48}}{2}|u_h|_1. \tag{2.1}$$

Generally, for all $u_h \in H^1(\Omega_h)$,

$$\|u_h\|_{L^6} \leq C\Big(|u_h|_1 + \frac{1}{l}\|u_h\|\Big), \tag{2.2}$$

where $l = \min(l_1, l_2, l_3)$*, and the constant* C *is independent of mesh lengths* h_i $(i = 1, 2, 3)$*,* l *and the discrete function* u_h*.*

Proof. We only demonstrate (2.1), and (2.2) can be proved in a similar way. From $u_h|_{\partial\Omega_h} = 0$ we can easily see the relations

$$u_{i,j,k}^3 = \sum_{m=0}^{j-1}\Big(u_{i,m+1,k}^2 + u_{i,m+1,k}u_{i,m,k} + u_{i,m,k}^2\Big)\delta_2 u_{i,m,k}h_2,$$

$$u_{i,j,k}^3 = -\sum_{m=j}^{J_2-1}\Big(u_{i,m+1,k}^2 + u_{i,m+1,k}u_{i,m,k} + u_{i,m,k}^2\Big)\delta_2 u_{i,m,k}h_2.$$

By Cauchy's inequality, we have

$$\begin{aligned}2|u_{i,j,k}|^3 &\leq \frac{3}{2}\sum_{m=0}^{J_2-1}\Big(|u_{i,m+1,k}|^2+|u_{i,m,k}|^2\Big)|\delta_2 u_{i,m,k}|h_2\\ &\leq 3\left(\sum_{m=0}^{J_2-1}|u_{i,m,k}|^4h_2\right)^{\frac{1}{2}}\left(\sum_{m=0}^{J_2-1}|\delta_2 u_{i,m,k}|^2h_2\right)^{\frac{1}{2}},\end{aligned}$$

which yields

$$\max_{0\leq j\leq J_2}|u_{i,j,k}|^3 \leq \frac{3}{2}\left(\sum_{m=0}^{J_2-1}|u_{i,m,k}|^4h_2\right)^{\frac{1}{2}}\left(\sum_{m=0}^{J_2-1}|\delta_2 u_{i,m,k}|^2h_2\right)^{\frac{1}{2}}. \tag{2.3}$$

Multiplying both sides of (2.3) by h_3 and summing over k, then using Cauchy's inequality, we have

$$\sum_{k=0}^{J_3-1}\max_{0\leq j\leq J_2}|u_{i,j,k}|^3h_3 \leq \frac{3}{2}\left(\sum_{k=0}^{J_3-1}\sum_{m=0}^{J_2-1}|u_{i,m,k}|^4h_2h_3\right)^{\frac{1}{2}}\left(\sum_{k=0}^{J_3-1}\sum_{m=0}^{J_2-1}|\delta_2 u_{i,m,k}|^2h_2h_3\right)^{\frac{1}{2}}.$$

Similarly

$$\sum_{j=0}^{J_2-1}\max_{0\le k\le J_3}|u_{i,j,k}|^3h_2\le\frac{3}{2}\left(\sum_{j=0}^{J_2-1}\sum_{n=0}^{J_3-1}|u_{i,j,n}|^4h_2h_3\right)^{\frac{1}{2}}\left(\sum_{j=0}^{J_2-1}\sum_{n=0}^{J_3-1}|\delta_3u_{i,j,n}|^2h_2h_3\right)^{\frac{1}{2}}.$$

Hence

$$\begin{aligned}&\left(\sum_{k=0}^{J_3-1}\max_{0\le j\le J_2}|u_{i,j,k}|^3h_3\right)\left(\sum_{j=0}^{J_2-1}\max_{0\le k\le J_3}|u_{i,j,k}|^3h_2\right)\\ \le&\frac{9}{4}\left(\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|u_{i,j,k}|^4h_2h_3\right)\left(\sum_{k=0}^{J_3-1}\sum_{m=0}^{J_2-1}|\delta_2u_{i,m,k}|^2h_2h_3\right)^{\frac{1}{2}}\left(\sum_{j=0}^{J_2-1}\sum_{n=0}^{J_3-1}|\delta_3u_{i,j,n}|^2h_2h_3\right)^{\frac{1}{2}}.\end{aligned}$$

Next, we multiply both sides of the above inequality by h_1 and summing over i. Then using Cauchy's inequality, we have

$$\begin{aligned}&\sum_{i=0}^{J_1-1}\left(\sum_{k=0}^{J_3-1}\max_{0\le j\le J_2}|u_{i,j,k}|^3h_3\right)\left(\sum_{j=0}^{J_2-1}\max_{0\le k\le J_3}|u_{i,j,k}|^3h_2\right)h_1\\ \le&\frac{9}{4}\left(\max_{0\le i\le J_1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|u_{i,j,k}|^4h_2h_3\right)\left(\sum_{i,j,h}|\delta_2u_{i,j,k}|^2h_1h_2h_3\right)^{\frac{1}{2}}\left(\sum_{i,j,k}|\delta_3u_{i,j,k}|^2h_1h_2h_3\right)^{\frac{1}{2}},\end{aligned}\tag{2.4}$$

where the summation indices (i,j,k) above denote the summation for $0\le i\le J_1-1$, $0\le j\le J_2-1$ and $0\le k\le J_3-1$. On the other hand, note

$$\begin{aligned}|u_{i,j,k}|^4&\le\frac{1}{2}\sum_{l=0}^{J_1-1}\left|(u_{l+1,j,k}^2+u_{l,j,k}^2)(u_{l+1,j,k}+u_{l,j,k})\right||\delta_1u_{l,j,k}|h_1\\ &\le2\left(\sum_{l=0}^{J_1-1}|u_{l,j,k}|^6h_1\right)^{\frac{1}{2}}\left(\sum_{l=0}^{J_1-1}|\delta_1u_{l,j,k}|^2h_1\right)^{\frac{1}{2}},\end{aligned}$$

Consequently,

$$\max_{0\le i\le J_1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|u_{i,j,k}|^4h_2h_3\le2\left(\sum_{l,j,k}|u_{l,j,k}|^6h_1h_2h_3\right)^{\frac{1}{2}}\left(\sum_{l,j,k}|\delta_1u_{l,j,k}|^2h_1h_2h_3\right)^{\frac{1}{2}}.$$

Substituting the above inequality into (2.4), we have

$$\begin{aligned}&\sum_{i,j,k}|u_{i,j,k}|^6h_1h_2h_3=\sum_{i=0}^{J_1-1}\left(\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}|u_{i,j,k}|^3|u_{i,j,k}|^3h_2h_3\right)h_1\\ \le&\sum_{i=0}^{J_1-1}\left(\sum_{j=0}^{J_2-1}\max_{0\le k\le J_3}|u_{i,j,k}|^3h_2\right)\left(\sum_{k=0}^{J_3-1}\max_{0\le j\le J_2}|u_{i,j,k}|^3h_3\right)h_1\\ \le&\frac{9}{2}\left(\sum_{i,j,k}|u_{i,j,k}|^6h_1h_2h_3\right)^{\frac{1}{2}}\prod_{l=1}^{3}\left(\sum_{i,j,k}|\delta_lu_{i,j,k}|^2h_1h_2h_3\right)^{\frac{1}{2}},\end{aligned}$$

which yields the first inequality of (2.1). The second part of (2.1) can be obtained by using the theorem of the arithmetic and geometric mean. □

By Lemma 2.1, Hölder's inequality and the definition of the discrete L^p norm, we have

Lemma 2.2. *For all $u_h \in H_0^1(\Omega_h)$, and any $q \in [2,6]$,*

$$\|u_h\|_{L^q} \le \frac{48^{\frac{\mu}{6}}}{2^\mu}|u_h|_1^\mu \|u_h\|^{1-\mu} \quad \text{with} \quad \mu = \frac{3}{2} - \frac{3}{q}.$$

Generally, for all $u_h \in H^1(\Omega_h)$, and any $q \in [2,6]$,

$$\|u_h\|_{L^q} \le C\Big(|u_h|_1 + \frac{1}{l}\|u_h\|\Big)^\mu \|u_h\|^{1-\mu} \quad \text{with} \quad \mu = \frac{3}{2} - \frac{3}{q},$$

where the constant C is independent of mesh lengths h_i $(i = 1,2,3)$, l and the discrete function u_h.

Lemma 2.3. *For all $u_h \in H_0^1(\Omega_h) \cap H^2(\Omega_h)$, the following inequality holds*

$$\|u_h\|_\infty \le C_\Omega |u_h|_2,$$

where

$$C_\Omega = \frac{(l_1 l_2 l_3)^{\frac{1}{6}}}{2\sqrt{3}} \left(\sum_{k=1}^\infty k^{-\frac{4}{3}}\right)^{\frac{3}{2}}.$$

Proof. For any $u_h \in H_0^1(\Omega_h) \cap H^2(\Omega_h)$, we have the following expansion

$$u_h(x) = \sum_{k_1,k_2,k_3}{}' \hat{u}_{k_1,k_2,k_3} \zeta_{k_1}(x_1)\zeta_{k_2}(x_2)\zeta_{k_3}(x_3), \quad \forall\, x = (x_1, x_2, x_3) \in \overline{\Omega}_h, \tag{2.5}$$

where $\sum_{k_1,k_2,k_3}{}'$ denote summation over $1 \le k_1 \le J_1 - 1$, $1 \le k_2 \le J_2 - 1$ and $1 \le k_3 \le J_3 - 1$; $\zeta_{k_1}(x_1)$, $\zeta_{k_2}(x_2)$, $\zeta_{k_3}(x_3)$ are eigenfunctions of the operator $-\Delta_h$ with the homogeneous Dirichlet boundary conditions, and

$$\zeta_{k_i}(x_i) = \sqrt{\frac{2}{l_i}} \sin\Big(\frac{k_i \pi x_i}{l_i}\Big) \quad (i = 1,2,3)$$

are eigenfunctions of the difference quotient operators $-\delta_i \bar{\delta}_i$ with the homogeneous Dirichlet boundary conditions corresponding to the eigenvalues

$$\lambda_{k_i} = \Big(\frac{2J_i}{l_i}\Big)^2 \sin^2\Big(\frac{k_i \pi}{2J_i}\Big).$$

By the uniform boundedness of the eigenfunctions $\zeta_{k_1}(x_1)\zeta_{k_2}(x_2)\zeta_{k_3}(x_3)$ and Cauchy's inequality, (2.5) implies

$$\begin{aligned}
|u_h(x)|^2 &= \Big|\sum_{k_1,k_2,k_3}{}' \hat{u}_{k_1,k_2,k_3}\zeta_{k_1}(x_1)\zeta_{k_2}(x_2)\zeta_{k_3}(x_3)\Big|^2 \le \frac{8}{l_1 l_2 l_3}\Big(\sum_{k_1,k_2,k_3}{}' |\hat{u}_{k_1,k_2,k_3}|\Big)^2 \\
&\le \frac{8}{l_1 l_2 l_3} \sum_{k_1,k_2,k_3}{}' \Big(\lambda_{k_1}^2 + \lambda_{k_2}^2 + \lambda_{k_3}^2 + \lambda_{k_1}\lambda_{k_2} + \lambda_{k_1}\lambda_{k_3} + \lambda_{k_2}\lambda_{k_3}\Big)^{-1} \cdot \\
&\quad \times \sum_{k_1,k_2,k_3}{}' \Big(\lambda_{k_1}^2 + \lambda_{k_2}^2 + \lambda_{k_3}^2 + \lambda_{k_1}\lambda_{k_2} + \lambda_{k_1}\lambda_{k_3} + \lambda_{k_2}\lambda_{k_3}\Big)|\hat{u}_{k_1,k_2,k_3}|^2 \\
&= \frac{8}{l_1 l_2 l_3} \sum_{k_1,k_2,k_3}{}' \Big(\lambda_{k_1}^2 + \lambda_{k_2}^2 + \lambda_{k_3}^2 + \lambda_{k_1}\lambda_{k_2} + \lambda_{k_1}\lambda_{k_3} + \lambda_{k_2}\lambda_{k_3}\Big)^{-1} |u_h|_2^2 \quad \forall x \in \overline{\Omega}_h.
\end{aligned} \tag{2.6}$$

Since $0 \le \frac{k_i\pi}{2J_i} \le \frac{\pi}{2}$, by using the inequality $\sin x \ge \frac{2}{\pi}x,\ x \in [0, \frac{\pi}{2}]$ and the theorem of the arithmetic and geometric mean, we have

$$\left(\frac{8k_1k_2k_3}{l_1l_2l_3}\right)^{\frac{4}{3}} \le (\lambda_{k_1}^4\lambda_{k_2}^4\lambda_{k_3}^4)^{\frac{1}{6}} \le \frac{1}{6}\Big(\lambda_{k_1}^2 + \lambda_{k_2}^2 + \lambda_{k_3}^2 + \lambda_{k_1}\lambda_{k_2} + \lambda_{k_1}\lambda_{k_3} + \lambda_{k_2}\lambda_{k_3}\Big),$$

which gives

$$\Big(\lambda_{k_1}^2 + \lambda_{k_2}^2 + \lambda_{k_3}^2 + \lambda_{k_1}\lambda_{k_2} + \lambda_{k_1}\lambda_{k_3} + \lambda_{k_2}\lambda_{k_3}\Big)^{-1} \le \frac{1}{96}\left(\frac{l_1l_2l_3}{k_1k_2k_3}\right)^{\frac{4}{3}}.$$

Thus

$$\sum_{k_1,k_2,k_3}{}' \Big(\lambda_{k_1}^2 + \lambda_{k_2}^2 + \lambda_{k_3}^2 + \lambda_{k_1}\lambda_{k_2} + \lambda_{k_1}\lambda_{k_3} + \lambda_{k_2}\lambda_{k_3}\Big)^{-1} \le \frac{(l_1 l_2 l_3)^{\frac{4}{3}}}{96}\left(\sum_{k=1}^{\infty} k^{-\frac{4}{3}}\right)^3.$$

Substituting the above estimate into (2.6) completes the proof of the lemma. □

The following equivalent norms in $H_0^1(\Omega_h) \cap H^2(\Omega_h)$ is easily verified.

Lemma 2.4. *For any discrete function $u_h \in H_0^1(\Omega_h) \cap H^2(\Omega_h)$, we have*

$$\frac{\sqrt{2}}{2}\|\Delta_h u_h\| \le |u_h|_2 \le \|\Delta_h u_h\|.$$

3. Finite Difference Scheme

We consider the following fully discrete finite difference scheme: find the complex function $\psi_h^n \in H_0^1(\Omega_h)$, and the real functions $\phi_h^n, \theta_h^n \in H_0^1(\Omega_h)$ for $n \ge 1$, such that

$$\begin{aligned}
&i\frac{e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^n - e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}}{\Delta t} + \frac{1}{2}\Delta_h\Big(e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^n + e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}\Big)\\
&\quad + \frac{1}{4}\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}\Big)\Big(e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^n + e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}\Big) = F_{i,j,k}, \qquad (3.1a)\\
&\frac{e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^n - e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}}{\Delta t} - \frac{1}{2}\Big(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_{i,j,k}^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_{i,j,k}^{n-1}\Big) = 0, \qquad (3.1b)\\
&\frac{e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_{i,j,k}^n - e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_{i,j,k}^{n-1}}{\Delta t} + \frac{\mu^2 - \varepsilon(\beta-\varepsilon)}{2}\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}\Big)\\
&\quad - \frac{1}{2}\Delta_h\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}\Big) = \frac{1}{2}\Big(|e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^n|^2 + |e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}|^2\Big) + G_{i,j,k}, \qquad (3.1c)
\end{aligned}$$

for $i=1,\cdots,J_1-1,\ j=1,\cdots,J_2-1,\ k=1,\cdots,J_3-1$.

The above scheme is supplemented with the initial values

$$(\psi_h^0, \phi_h^0, \theta_h^0) = (\psi_{0,h}, \phi_{0,h}, \phi_{1,h} + \varepsilon\phi_{0h}), \qquad (3.2)$$

where the complex-valued function $\psi_{0,h} \in H_0^1(\Omega_h)$, and the real-valued functions $\phi_{0,h}, \phi_{1,h} \in H_0^1(\Omega_h)$ are suitable approximations to ψ_0, ϕ_0, ϕ_1 which are given in (1.3), $\varsigma_h^n = \theta_h^n - \varepsilon\phi_h^n$ is the approximation to ϕ_t at time t_n. In this paper we make the following hypothesis on the functions F and G in the right hands of (3.1)

$$\max\Big(\|F_h\|_{L^4}, \|F_h\|, \|G_h\|\Big) \le C_{\{F,G\}}, \tag{H}$$

where $C_{\{F,G\}}$ is a constant independent of h_1, h_2 and h_3.

We are going to prove the existence of the solutions $(\psi_h^n, \phi_h^n, \theta_h^n)$ for the finite difference scheme (3.1)–(3.2). For any complex-valued discrete functions $\psi_h \in H_0^1(\Omega_h)$, any real-valued discrete functions $\phi_h \in H_0^1(\Omega_h)$, and any $\sigma \in [0,1]$, let us define the complex-valued discrete functions $\Psi_h \in H_0^1(\Omega_h)$, and the real-valued discrete function $\Phi_h \in H_0^1(\Omega_h)$ as follows

$$\begin{aligned}
&i\frac{e^{\frac{\alpha}{2}\Delta t}\Psi_{i,j,k} - e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}}{\Delta t} + \frac{\sigma}{2}\Delta_h\Big(e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k} + e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}\Big)\\
&\quad + \frac{\sigma}{4}\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k} + e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}\Big)\Big(e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k} + e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}\Big) = \sigma F_{i,j,k}, && (3.3\text{a})\\
&\frac{e^{\frac{\varepsilon}{2}\Delta t}\Phi_{i,j,k} - e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}}{\Delta t} - \frac{\sigma}{2}\Big(e^{\frac{\alpha-\varepsilon}{2}\Delta t}\theta_{i,j,k}^n + e^{-\frac{\alpha-\varepsilon}{2}\Delta t}\theta_{i,j,k}^{n-1}\Big) = 0, && (3.3\text{b})\\
&\frac{e^{\frac{\alpha-\varepsilon}{2}\Delta t}\theta_{i,j,k}^n - e^{-\frac{\alpha-\varepsilon}{2}\Delta t}\theta_{i,j,k}^{n-1}}{\Delta t} + \frac{\mu^2 - \varepsilon(\beta-\varepsilon)}{2}\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k} + e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}\Big)\\
&\quad - \frac{1}{2}\Delta_h\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k} + e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}\Big) = \frac{1}{2}\Big(|e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}|^2 + |e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}|^2\Big) + G_{i,j,k}, && (3.3\text{c})
\end{aligned}$$

for $i=1,\cdots,J_1-1,\ j=1,\cdots,J_2-1,\ k=1,\cdots,J_3-1$.

It defines a mapping $T : (\psi_h, \phi_h, \sigma) \in H_0^1(\Omega_h) \times H_0^1(\Omega_h) \times [0,1] \to (\Psi_h, \Phi_h) \in H_0^1(\Omega_h) \times H_0^1(\Omega_h)$. Obvious, the mapping T is continuous for every $(\psi_h, \phi_h, \sigma) \in H_0^1(\Omega_h) \times H_0^1(\Omega_h) \times [0,1]$. In order to obtain the existence of the solutions for the finite difference system (3.3), it is sufficient to prove the uniform boundedness for all the possible fixed point $\psi_h^{n-1+\sigma}, \phi_h^{n-1+\sigma}$ for the mapping σT with respect to the parameter $0 \le \sigma \le 1$ by using the Leray-Schauder fixed point theorem. Then the fixed point $\psi_h^{n-1+\sigma}, \phi_h^{n-1+\sigma}$ of the mapping σT satisfies that

$$\begin{aligned}
&i\frac{e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1+\sigma} - e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}}{\Delta t} + \frac{\sigma}{2}\Delta_h\Big(e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1+\sigma} + e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}\Big)\\
&\quad + \frac{\sigma}{4}\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1+\sigma} + e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}\Big)\Big(e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1+\sigma} + e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}\Big) = \sigma F_{i,j,k}, && (3.4\text{a})\\
&\frac{e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1+\sigma} - e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}}{\Delta t} - \frac{\sigma}{2}\Big(e^{\frac{\alpha-\varepsilon}{2}\Delta t}\theta_{i,j,k}^n + e^{-\frac{\alpha-\varepsilon}{2}\Delta t}\theta_{i,j,k}^{n-1}\Big) = 0, && (3.4\text{b})\\
&\frac{e^{\frac{\alpha-\varepsilon}{2}\Delta t}\theta_{i,j,k}^n - e^{-\frac{\alpha-\varepsilon}{2}\Delta t}\theta_{i,j,k}^{n-1}}{\Delta t} + \frac{\mu^2 - \varepsilon(\alpha-\varepsilon)}{2}\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1+\sigma} + e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}\Big)\\
&\quad - \frac{1}{2}\Delta_h\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1+\sigma} + e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}\Big) = \frac{1}{2}\Big(|e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1+\sigma}|^2 + |e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}|^2\Big) + G_{i,j,k}, && (3.4\text{c})
\end{aligned}$$

for $i=1,\cdots,J_1-1,\ j=1,\cdots,J_2-1,\ k=1,\cdots,J_3-1$.

We first multiply relation (3.4a) by $\Delta t(e^{\frac{\alpha}{2}\Delta t}\overline{\psi}_{i,j,k}^{n-1+\sigma} + e^{-\frac{\alpha}{2}\Delta t}\overline{\psi}_{i,j,k}^{n-1})h_1h_2h_3$, and sum over i,j,k. Then taking the imaginary part and using the summation by parts formula, we have

$$\|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}\|^2 = \|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\|^2 + \sigma\Delta t\,\mathsf{Im}(F_h, e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma} + e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}).$$

Using Cauchy's inequality gives

$$\begin{aligned} e^{\frac{\alpha}{2}\Delta t}\|\psi_h^{n-1+\sigma}\|^2 &\le e^{-\frac{3\alpha}{2}\Delta t}\|\psi_h^{n-1}\|^2 + \Delta t|(F_h,\psi_h^{n-1+\sigma})| + \Delta t e^{-\alpha\Delta t}|(F_h,\psi_h^{n-1})| \\ &\le e^{-\frac{3\alpha}{2}\Delta t}\|\psi_h^{n-1}\|^2 + \Delta t\|F_h\|\|\psi_h^{n-1+\sigma}\| + \Delta t e^{-\alpha\Delta t}\|F_h\|\|\psi_h^{n-1}\| \\ &\le e^{-\frac{3\alpha}{2}\Delta t}\|\psi_h^{n-1}\|^2 + \frac{\alpha}{2}\Delta t\|\psi_h^{n-1+\sigma}\|^2 + \frac{\Delta t}{2\alpha}\|F_h\|^2 \\ &\quad + \Delta t e^{-\alpha\Delta t}\frac{\alpha}{2}e^{-\frac{\alpha}{2}\Delta t}\|\psi_h^{n-1}\|^2 + \frac{\Delta t}{2\alpha}e^{-\frac{\alpha}{2}\Delta t}\|F_h\|^2, \end{aligned}$$

which implies that

$$\|\psi_h^{n-1+\sigma}\|^2 \le e^{-\alpha\Delta t}\|\psi_h^{n-1}\|^2 + \frac{\Delta t}{\alpha}\|F_h\|^2. \tag{3.5}$$

Next,we multiply both sides of (3.4a) by $4(e^{\frac{\alpha}{2}\Delta t}\overline{\psi}_{i,j,k}^{n-1+\sigma} - e^{-\frac{\alpha}{2}\Delta t}\overline{\psi}_{i,j,k}^{n-1})h_1h_2h_3$ and sum over i,j,k. Then taking the real part, we get

$$\begin{aligned} &2|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|_1^2 - (e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma} + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|^2 - |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2) \\ &\quad + 4\mathsf{Re}\left(F_h, e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}\right) = 2|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|_1^2 + 4\mathsf{Re}\left(F_h, e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\right). \end{aligned}$$

Using the identity

$$\begin{aligned} &\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma} + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|^2 - |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right) \\ =&2\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|^2\right) - 2\left(e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right) \\ &- \left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma} - e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|^2 + |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right), \end{aligned}$$

we obtain

$$\begin{aligned} &2|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|_1^2 - 2\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|^2\right) + 4\mathsf{Re}\left(F_h, e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}\right) \\ =& 2|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|_1^2 - 2\left(e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right) + 4\mathsf{Re}\left(F_h, e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\right) \\ &- \left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma} - e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|^2 + |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right). \end{aligned} \tag{3.6}$$

Multiplying (3.4c) by $\sigma(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_{i,j,k}^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_{i,j,k}^{n-1})h_1h_2h_3\Delta t$, summing the resulting equation over i,j,k and using (3.4b), we have

$$\begin{aligned} &\sigma\|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n\|^2 + (\mu^2 - \varepsilon(\beta-\varepsilon))\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}\|^2 + |e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}|_1^2 - \left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}\right) \\ =& \sigma\|e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}\|^2 + (\mu^2 - \varepsilon(\beta-\varepsilon))\|e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|^2 + |e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}|_1^2 - \left(G_h, e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\right) \\ &+ \left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma} - e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|^2 + |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right). \end{aligned} \tag{3.7}$$

Adding both sides of the identities (3.6) and (3.7), gives

$$A^n \le B^{n-1}, \tag{3.8}$$

where

$$\begin{aligned} A^n =& 2|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|_1^2 - 2\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|^2\right) + 4\mathsf{Re}(F_h, e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}) \\ &+ (\mu^2 - \varepsilon(\beta-\varepsilon))\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}\|^2 + |e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}|_1^2 - \left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}\right), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} B^{n-1} =& 2|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|_1^2 - 2\left(e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right) + 4\mathsf{Re}\left(F_h, e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\right) \\ &+ \|e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}\|^2 + (\mu^2 - \varepsilon(\beta-\varepsilon))\|e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|^2 \\ &+ |e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}|_1^2 - \left(G_h, e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\right). \end{aligned} \tag{3.10}$$

By Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned}\left|2\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n+1-\sigma}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|^2\right)\right| &\le 2\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}\|_{L^6}\|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}\|^2_{L^{12/5}}\\ &\le C|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}|_1|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|_1^{\frac{1}{2}}\|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}\|^{\frac{3}{2}}\\ &\le \frac{1}{2}|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}|_1^2+|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|_1^2+C\|\psi_h^{n-1+\sigma}\|^6,\end{aligned}$$

$$\left|4Re\left(F_h, e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}\right)\right| \le 4\|F_h\|\|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}\| \le 2\Big(\|F_h\|^2+\|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}\|^2\Big),$$

$$\left|\left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}\right)\right| \le \|G_h\|\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}\| \le \frac{\mu^2-\varepsilon(\beta-\varepsilon)}{2}\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}\|^2+C\|G_h\|^2.$$

We derive a lower bounded on A^n from the above estimates

$$\begin{aligned}A^n \ge\; &|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1+\sigma}|_1^2+\frac{\mu^2-\varepsilon(\beta-\varepsilon)}{2}\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}\|^2+\frac{1}{2}|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1+\sigma}|_1^2\\ &-C\Big(\|\psi_h^{n-1+\sigma}\|^6+\|F_h\|^2+\|\psi_h^{n-1+\sigma}\|^2+\|G_h\|^2\Big).\end{aligned}$$

Combining (3.5) and (3.11), we see that $\|\psi_h^{n-1+\sigma}\|_1^2+\|\phi_h^{n-1+\sigma}\|_1^2$ is uniformly bounded with respect to the parameter $0\le\sigma\le 1$. Therefore, the solution $(\psi_h^n, \phi_h^n, \theta_h^n)$ of the fully discrete finite difference scheme (3.1)–(3.2) exists. The uniqueness of the solution of the fully discrete finite difference scheme is will be proved in Theorem 4.1 in Section 4.

Below we provide some t-independent priori estimates for the solutions $(\psi_h^n, \phi_h^n, \theta_h^n)$ of the finite difference scheme (3.1)–(3.2).

Lemma 3.1. *For the solution ψ_h^n of the fully discrete finite difference scheme* (3.1)–(3.2)*, the following priori estimates hold:*

$$\|\psi_h^n\|^2 \le e^{-\alpha n\Delta t}\|\psi_h^0\|^2+\alpha^{-2}e^{\alpha\Delta t}\|F_h\|^2(1-e^{-\alpha n\Delta t}), \qquad n\ge 0.$$

Consequently, there exists a constant $C_0=\max\left(\|\psi_h^0\|, \alpha^{-1}e^{\frac{\alpha}{2}\Delta t}\|F_h\|\right)$ such that

$$\sup_{n\ge 0}\|\psi_h^n\|\le C_0.$$

Proof. Multiplying relation (3.1a) by $\Delta t(e^{\frac{\alpha}{2}\Delta t}\overline{\psi}_{i,j,k}^n+e^{-\frac{\alpha}{2}\Delta t}\overline{\psi}_{i,j,k}^{n-1})h_1h_2h_3$, and summing over i,j,k, then taking the imaginary part, we have

$$\begin{aligned}&\mathsf{Re}(e^{\frac{\alpha}{2}\Delta t}\psi_h^n-e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}, e^{\frac{\alpha}{2}\Delta t}\psi_h^n+e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1})\\ &\quad+\frac{1}{2}\Delta t\mathsf{Im}(\Delta_h(e^{\frac{\alpha}{2}\Delta t}\psi_h^n+e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}), e^{\frac{\alpha}{2}\Delta t}\psi_h^n+e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1})\\ &=\Delta t\mathsf{Im}(F_h, e^{\frac{\alpha}{2}\Delta t}\psi_h^n+e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}).\end{aligned} \tag{3.11}$$

It can be verified that

$$\mathsf{Re}(e^{\frac{\alpha}{2}\Delta t}\psi_h^n-e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}, e^{\frac{\alpha}{2}\Delta t}\psi_h^n+e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1})=\|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|^2-\|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\|^2.$$

Using the summation by parts gives

$$\mathsf{Im}(\Delta_h(e^{\frac{\alpha}{2}\Delta t}\psi_h^n+e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}), e^{\frac{\alpha}{2}\Delta t}\psi_h^n+e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1})=0.$$

Therefore, (3.12) can be rewritten as follows

$$e^{\frac{\alpha}{2}\Delta t}\|\psi_h^n\|^2 = e^{-\frac{3\alpha}{2}\Delta t}\|\psi_h^{n-1}\|^2 + \Delta t \mathsf{Im}(F_h, \psi_h^n + e^{-\alpha\Delta t}\psi_h^{n-1}).$$

It follows that

$$\begin{aligned} e^{\frac{\alpha}{2}\Delta t}\|\psi_h^n\|^2 \le & e^{-\frac{3\alpha}{2}\Delta t}\|\psi_h^{n-1}\|^2 + \Delta t|(F_h, \psi_h^n)| + \Delta t e^{-\alpha\Delta t}|(F_h, \psi_h^{n-1})| \\ \le & e^{-\frac{3\alpha}{2}\Delta t}\|\psi_h^{n-1}\|^2 + \Delta t\|F_h\|\|\psi_h^n\| + \Delta t e^{-\alpha\Delta t}\|F_h\|\|\psi_h^{n-1}\| \\ \le & e^{-\frac{3\alpha}{2}\Delta t}\|\psi_h^{n-1}\|^2 + \frac{\alpha}{2}\Delta t\|\psi_h^n\|^2 + \frac{\Delta t}{2\alpha}\|F_h\|^2 \\ & + \Delta t e^{-\alpha\Delta t}\frac{\alpha}{2}e^{-\frac{\alpha}{2}\Delta t}\|\psi_h^{n-1}\|^2 + \frac{\Delta t}{2\alpha}e^{-\frac{\alpha}{2}\Delta t}\|F_h\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} \|\psi_h^n\|^2 \le & e^{-\alpha\Delta t}\|\psi_h^{n-1}\|^2 + \frac{\Delta t}{\alpha}\|F_h\|^2 \le \cdots \\ \le & e^{-\alpha n\Delta t}\|\psi_h^0\|^2 + \frac{1}{\alpha^2}e^{\alpha\Delta t}\|F_h\|^2(1 - e^{-\alpha n\Delta t}), \quad n \ge 0. \end{aligned}$$

□

Lemma 3.2. Assume that the hypothesis (H) is true, and $(\psi_h^0, \phi_h^0, \theta_h^0) \in \mathbf{V}_h$. Then the solution $(\psi_h^n, \phi_h^n, \theta_h^n)$ of the fully discrete finite difference scheme (3.1)–(3.2) is uniformly bounded in $\mathbf{V}_h$. Moreover,

$$\limsup_{n\to+\infty} \|(\psi_h^n, \phi_h^n, \theta_h^n)\|_{\mathbf{V}_h} \le \rho_1,$$

where ρ_1 is a constant independent of the initial value $(\psi_h^0, \phi_h^0, \theta_h^0)$ and the discretization parameters $h_1, h_2, h_3, \Delta t$.

Proof. We first multiply both sides of (3.1a) by $4(e^{\frac{\alpha}{2}\Delta t}\overline{\psi}_{i,j,k}^n - e^{-\frac{\alpha}{2}\Delta t}\overline{\psi}_{i,j,k}^{n-1})h_1h_2h_3$ and summing over i, j, k. Then taking the real part, gives

$$\begin{aligned} & 2|e^{\frac{\alpha}{2}\Delta t}\psi_h^n|_1^2 - (e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^n|^2 - |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2) + 4Re\left(F_h, e^{\frac{\alpha}{2}\Delta t}\psi_h^n\right) \\ =& 2|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|_1^2 + 4Re\left(F_h, e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\right). \end{aligned} \tag{3.12}$$

Using the identity

$$\begin{aligned} & \left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^n|^2 - |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right) \\ =& 2\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n, |e^{\frac{\alpha}{2}\Delta t}\psi_h^n|^2\right) - 2\left(e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right) \\ & - \left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n - e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^n|^2 + |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right), \end{aligned}$$

we obtain

$$\begin{aligned} & 2|e^{\frac{\alpha}{2}\Delta t}\psi_h^n|_1^2 - 2\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n, |e^{\frac{\alpha}{2}\Delta t}\psi_h^n|^2\right) + 4\mathsf{Re}\left(F_h, e^{\frac{\alpha}{2}\Delta t}\psi_h^n\right) \\ =& 2|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|_1^2 - 2\left(e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right) + 4\mathsf{Re}\left(F_h, e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\right) \\ & - \left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n - e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^n|^2 + |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right). \end{aligned} \tag{3.13}$$

Next, multiplying (3.1c) by $(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_{i,j,k}^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_{i,j,k}^{n-1})h_1h_2h_3\Delta t$, and summing over i, j, k, and using (3.1b), we have

$$\begin{aligned} & \|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n\|^2 + (\mu^2 - \varepsilon(\beta-\varepsilon))\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^2 + |e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2 - \left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\right) \\ =& \|e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}\|^2 + (\mu^2 - \varepsilon(\beta-\varepsilon))\|e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|^2 + |e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}|_1^2 - \left(G_h, e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\right) \\ & + \left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n - e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^n|^2 + |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right). \end{aligned} \tag{3.14}$$

Adding both sides of the identities (3.14) and (3.15), gives

$$E_1^n = G_1^{n-1}, \tag{3.15}$$

where E_1^n and G_1^{n-1} are defined by

$$\begin{aligned} E_1^n =& 2|e^{\frac{\alpha}{2}\Delta t}\psi_h^n|_1^2 - 2\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n, |e^{\frac{\alpha}{2}\Delta t}\psi_h^n|^2\right) + 4\mathsf{Re}(F_h, e^{\frac{\alpha}{2}\Delta t}\psi_h^n) + \|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n\|^2 \\ &+ (\mu^2 - \varepsilon(\beta-\varepsilon))\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^2 + |e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2 - \left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\right), \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} G_1^{n-1} =& 2|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|_1^2 - 2\left(e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right) + 4\mathsf{Re}\left(F_h, e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\right) \\ &+ \|e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}\|^2 + (\mu^2 - \varepsilon(\beta-\varepsilon))\|e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|^2 \\ &+ |e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}|_1^2 - \left(G_h, e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\right). \end{aligned} \tag{3.17}$$

Using the Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned} \left|2\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n, |e^{\frac{\alpha}{2}\Delta t}\psi_h^n|^2\right)\right| &\le 2\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|_{L^6}\|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|_{L^{12/5}}^2 \\ &\le C|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1|e^{\frac{\alpha}{2}\Delta t}\psi_h^n|_1^{\frac{1}{2}}\|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|^{\frac{3}{2}} \\ &\le \frac{1}{2}|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2 + |e^{\frac{\alpha}{2}\Delta t}\psi_h^n|_1^2 + C\|\psi_h^n\|^6, \end{aligned}$$

$$\left|4\mathsf{Re}\left(F_h, e^{\frac{\alpha}{2}\Delta t}\psi_h^n\right)\right| \le 4\|F_h\|\|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|,$$

$$\left|\left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\right)\right| \le \|G_h\|\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\| \le \frac{\mu^2 - \varepsilon(\beta-\varepsilon)}{2}\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^2 + C\|G_h\|^2.$$

We derive lower and upper bounded on E_1^n from the above estimates

$$\begin{aligned} E_1^n \ge& |e^{\frac{\alpha}{2}\Delta t}\psi_h^n|_1^2 + \|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n\|^2 + \frac{\mu^2 - \varepsilon(\beta-\varepsilon)}{2}\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^2 + \frac{1}{2}|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2 \\ &- C\left(\|\psi_h^n\|^6 + \|F_h\|\|\psi_h^n\| + \|G_h\|^2\right), \end{aligned} \tag{3.18}$$

$$\begin{aligned} E_1^n \le& 3|e^{\frac{\alpha}{2}\Delta t}\psi_h^n|_1^2 + \|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n\|^2 + \frac{3(\mu^2 - \varepsilon(\beta-\varepsilon))}{2}\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^2 + \frac{3}{2}|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2 \\ &+ C\left(\|\psi_h^n\|^6 + \|F_h\|\|\psi_h^n\| + \|G_h\|^2\right). \end{aligned} \tag{3.19}$$

Let $\kappa = \min(\alpha, \varepsilon)$, then we can rewrite G_1^{n-1} as follows

$$\begin{aligned} G_1^{n-1} =& 2e^{-2\alpha\Delta t}|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|_1^2 - 2e^{-\kappa\Delta t}\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right) \\ &+ 4e^{-\kappa\Delta t}\mathsf{Re}\left(F_h, e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\right) + e^{-2(\beta-\varepsilon)\Delta t}\|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}\|^2 \\ &+ e^{-2\varepsilon\Delta t}(\mu^2 - \varepsilon(\beta-\varepsilon))\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|^2 + e^{-2\varepsilon\Delta t}|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}|_1^2 \\ &- e^{\kappa\Delta t}\left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\right) + 2\left(e^{\kappa\Delta t} - e^{-(2\alpha-\varepsilon)\Delta t}\right)\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, |e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right) \\ &+ 4\left(e^{-\alpha\Delta t} - e^{-\kappa\Delta t}\right)\mathsf{Re}\left(F_h, e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\right) + \left(e^{-\kappa\Delta t} - e^{-\varepsilon\Delta t}\right)\left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\right). \end{aligned} \tag{3.20}$$

We estimate the last three terms of (3.21). By the inequality $1+x \le e^x\ \forall x \in \mathbb{R}$, Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned}
&\left|2(e^{\kappa\Delta t}-e^{-(2\alpha-\varepsilon)\Delta t})\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1},|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2\right)\right|\\
&\quad\le C\Delta t\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|_{L^6}\|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\|^2_{L^{12/5}}\\
&\quad\le C\Delta t|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}|_1|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|_1^{\frac{1}{2}}\|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\|^{\frac{3}{2}}\\
&\quad\le \varepsilon\Delta t e^{-2\varepsilon\Delta t}|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}|_1^2+2\alpha\Delta t e^{-2\alpha\Delta t}|e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|_1^2+C\Delta t\|\psi_h^{n-1}\|^6,
\end{aligned}$$

$$\left|4(e^{-\alpha\Delta t}-e^{-\kappa\Delta t})\mathrm{Re}\left(F_h,e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\right)\right|\le C\Delta t\|F_h\|\|\psi_h^{n-1}\|,$$

$$\begin{aligned}
&\left|(e^{-\kappa\Delta t}-e^{-\varepsilon\Delta t})\left(G_h,e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\right)\right|\\
&\le C\Delta t\|G_h\|\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|\le \varepsilon\Delta t e^{-2\varepsilon\Delta t}(\mu^2-\varepsilon(\beta-\varepsilon))\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|^2+C\Delta t\|G_h\|^2.
\end{aligned}$$

It follows from the above these estimates and (3.21) and (3.16) that

$$E_1^n\le e^{-\kappa\Delta t}E_1^{n-1}+C\Delta t\Big(\|\psi_h^{n-1}\|^6+\|F_h\|\|\psi_h^{n-1}\|+\|G_h\|^2\Big),\quad n\ge 1. \tag{3.21}$$

Using the above inequality repeatedly gives

$$\begin{aligned}
E_1^n&\le e^{-\kappa n\Delta t}E_1^0+C\Delta t\sum_{l=0}^{n-1}e^{-\kappa l\Delta t}\Big(\|\psi_h^{n-1-l}\|^6+\|F_h\|\|\psi_h^{n-1-l}\|+\|G_h\|^2\Big)\\
&\le e^{-\kappa n\Delta t}E_1^0+C\Big(C_0^6+C_0\|F_h\|+\|G_h\|^2\Big),\quad n\ge 0.
\end{aligned} \tag{3.22}$$

Hence, from (3.19) and (3.23), for $(\psi_h^0,\phi_h^0,\theta_h^0)\in\mathbf{V}_h$, and $\|(\psi_h^0,\phi_h^0,\theta_h^0)\|_{\mathbf{V}_h}\le R$, there exists a positive constant $C(R)$, such that

$$\|(\psi_h^n,\phi_h^n,\theta_h^n)\|_{\mathbf{V}_h}\le C(R),\quad n\ge 0.$$

If initial value ψ_h^0 satisfied that $\|\psi_h^0\|\le R$. Then by Lemma 3.1, there exists a positive integer $N=N(R)$, such that

$$\|\psi^n\|\le \alpha^{-1}e^{\alpha\Delta t}C_{\{F,G\}}+1,\quad n\ge N.$$

For $n\ge N$, using (3.22) repeatedly gives

$$\begin{aligned}
E_1^n&\le e^{-\kappa(n-N)\Delta t}E_1^N+C\Delta t\sum_{l=0}^{n-1-N}e^{-\kappa l\Delta t}\Big(\|\psi_h^{n-1-l}\|^6+\|F_h\|\|\psi_h^{n-1-l}\|+\|G_h\|^2\Big)\\
&\le e^{-\kappa(n-N)\Delta t}E_1^N+C\Big(C_{\{F,G\}}^6+C_{\{F,G\}}^2+1\Big),\quad n\ge N.
\end{aligned} \tag{3.23}$$

From (3.19) and (3.24), we obtain that

$$\begin{aligned}
&|e^{\frac{\alpha}{2}\Delta t}\psi_h^n|_1^2+\|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n\|^2+\frac{\mu^2-\varepsilon(\beta-\varepsilon)}{2}\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^2+\frac{1}{2}|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2\\
&\le e^{-\kappa(n-N)\Delta t}E_1^N+C\Big(C_{\{F,G\}}^6+C_{\{F,G\}}^2+1\Big),\quad n\ge N.
\end{aligned}$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *Assume that the hypothesis* (H) *is true, and* $(\psi_h^0, \phi_h^0, \theta_h^0) \in \mathbf{W}_h$. Then the solution $(\psi_h^n, \phi_h^n, \theta_h^n)$ of the fully discrete finite difference scheme (3.1)–(3.2) is uniformly bounded in $\mathbf{W}_h$. Moreover,

$$\limsup_{n\to+\infty} \|(\psi_h^n, \phi_h^n, \theta_h^n)\|_{\mathbf{W}_h} \le \rho_2,$$

where ρ_2 is a constant independent of the initial value $(\psi_h^0, \phi_h^0, \theta_h^0)$ and the discretization parameters $h_1, h_2, h_3, \Delta t$.

Proof. Multiplying both sides of (3.1a) by $4(e^{\frac{\alpha}{2}\Delta t}\Delta_h\overline{\psi}_{i,j,k}^n - e^{-\frac{\alpha}{2}\Delta t}\Delta_h\overline{\psi}_{i,j,k}^{n-1})h_1h_2h_3$ and summing over i,j,k, then taking the real part, we get

$$\begin{aligned}&2\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2 + Re((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})(e^{\frac{\alpha}{2}\Delta t}\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}), e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n - e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1})\\ &- 4Re\left(F_h, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right) = 2\|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2 - 4Re\left(F_h, e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right). \qquad (3.24)\end{aligned}$$

Note that

$$\begin{aligned}&\mathsf{Re}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})(e^{\frac{\alpha}{2}\Delta t}\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}), e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n - e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right)\\ =&4\mathsf{Re}\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n e^{\frac{\alpha}{2}\Delta t}\psi_h^n, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right) - 4\mathsf{Re}\left(e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}, e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right)\\ &- 2\mathsf{Re}(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n - e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, e^{\alpha\Delta t}\overline{\psi}_h^n\Delta_h\psi_h^n + e^{-\alpha\Delta t}\overline{\psi}_h^{n-1}\Delta_h\psi_h^{n-1})\\ &- \mathsf{Re}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})(e^{\frac{\alpha}{2}\Delta t}\psi_h^n - e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}), e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right),\end{aligned}$$

while by (3.1b)

$$\begin{aligned}&- 2\mathsf{Re}(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n - e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}, e^{\alpha\Delta t}\overline{\psi}_h^n\Delta_h\psi_h^n + e^{-\alpha\Delta t}\overline{\psi}_h^{n-1}\Delta_h\psi_h^{n-1})\\ =&- \Delta t\mathsf{Re}(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}, e^{\alpha\Delta t}\overline{\psi}_h^n\Delta_h\psi_h^n + e^{-\alpha\Delta t}\overline{\psi}_h^{n-1}\Delta_h\psi_h^{n-1}),\end{aligned}$$

and by (3.1a)

$$\begin{aligned}&- \mathsf{Re}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})(e^{\frac{\alpha}{2}\Delta t}\psi_h^n - e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}), e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right)\\ =&\frac{1}{4}\Delta t\mathsf{Im}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})^2(e^{\frac{\alpha}{2}\Delta t}\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}), e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right)\\ &- \Delta t\mathsf{Im}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})f_h, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right).\end{aligned}$$

Using the above results and (3.25) gives

$$\begin{aligned}&2\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2 + 4\mathsf{Re}\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n e^{\frac{\alpha}{2}\Delta t}\psi_h^n, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right) - 4\mathsf{Re}\left(F_h, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right)\\ =&2\|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2 + 4\mathsf{Re}\left(e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}, e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right) - 4\mathsf{Re}\left(F_h, e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right)\\ &+ \Delta t\mathsf{Re}(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}, e^{\alpha\Delta t}\overline{\psi}_h^n\Delta_h\psi_h^n + e^{-\alpha\Delta t}\overline{\psi}_h^{n-1}\Delta_h\psi_h^{n-1}),\\ &- \frac{1}{4}\Delta t\mathsf{Im}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})^2(e^{\frac{\alpha}{2}\Delta t}\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}), e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right)\\ &+ \Delta t\mathsf{Im}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})f_h, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right). \qquad (3.25)\end{aligned}$$

Multiplying relation (3.1c) by $-(e^{\frac{\beta-\varepsilon}{2}\Delta t}\Delta_h\theta_{i,j,k}^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\Delta_h\theta_{i,j,k}^{n-1})h_1h_2h_3\Delta t$, summing over i,j,k, and using the relation (3.1b), we have

$$\begin{aligned}&|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n|_1^2 + (\mu^2 - \varepsilon(\beta-\varepsilon))|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2 + \|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\|^2 + 2\left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\right)\\ =&|e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}|_1^2 + (\mu^2 - \varepsilon(\beta-\varepsilon))|e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}|_1^2 + \|e^{-\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n-1}\|^2 + 2\left(G_h, e^{-\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n-1}\right)\\ &- \frac{1}{2}\Delta t(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}, \Delta_h|e^{\frac{\alpha}{2}\Delta t}\psi_h^n|^2 + \Delta_h|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2). \qquad (3.26)\end{aligned}$$

Adding both sides of the identities (3.26) and (3.27), we obtain

$$E_2^n = G_2^{n-1}, \tag{3.27}$$

where E_2^n and G_2^{n-1} are defined by

$$\begin{aligned} E_2^n =\, & 2\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2 + 4\mathsf{Re}\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n e^{\frac{\alpha}{2}\Delta t}\psi_h^n, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right) - 4\mathsf{Re}\left(F_h, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right) + |e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n|_1^2 \\ & + (\mu^2 - \varepsilon(\beta-\varepsilon))|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2 + \|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\|^2 + 2\left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\right), \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} G_2^{n-1} =\, & 2\|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2 + 4\mathsf{Re}\left(e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1} e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}, e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right) \\ & - 4\mathsf{Re}\left(F_h, e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right) + |e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}|_1^2 + (\mu^2 - \varepsilon(\beta-\varepsilon))|e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}|_1^2 \\ & + \|e^{-\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n-1}\|^2 + 2\left(G_h, e^{-\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n-1}\right) \\ & + \Delta t\mathsf{Re}(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}, e^{\alpha\Delta t}\overline{\psi}_h^n\Delta_h\psi_h^n + e^{-\alpha\Delta t}\overline{\psi}_h^{n-1}\Delta_h\psi_h^{n-1}) \\ & - \frac{1}{4}\Delta t\mathsf{Im}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})^2(e^{\frac{\alpha}{2}\Delta t}\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}), e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right) \\ & + \Delta t\mathsf{Im}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})f_h, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right) \\ & - \frac{1}{2}\Delta t(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}, \Delta_h|e^{\frac{\alpha}{2}\Delta t}\psi_h^n|^2 + \Delta_h|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2). \end{aligned} \tag{3.29}$$

Let $\kappa_1 = \min\left(\frac{\alpha}{4}, \varepsilon\right)$. Then we can rewrite G_2^{n-1} as follows

$$\begin{aligned} G_2^{n-1} =\, & 2e^{-\frac{\alpha}{2}\Delta t}\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^{n-1}\|^2 + 4e^{-\kappa_1\Delta t}\mathsf{Re}\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1} e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right) \\ & - 4e^{-\kappa_1\Delta t}\mathsf{Re}\left(F_h, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right) + e^{-\frac{3(\beta-\varepsilon)}{2}\Delta t}|e^{\frac{\beta-\varepsilon}{4}\Delta t}\theta_h^{n-1}|_1^2 \\ & + (\mu^2 - \varepsilon(\beta-\varepsilon))e^{-2\varepsilon\Delta t}|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}|_1^2 \\ & + e^{-2\varepsilon\Delta t}\|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n-1}\|^2 + 2e^{-\kappa_1\Delta t}\left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n-1}\right) + \sum_{i=1}^{7} I_i, \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} I_1 &= 4(e^{-(\varepsilon+2\alpha)\Delta t} - e^{-\kappa_1\Delta t})\mathsf{Re}\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1} e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right), \\ I_2 &= 4(e^{-\kappa_1\Delta t} - e^{-\alpha\Delta t})\mathsf{Re}\left(F_h, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right), \\ I_3 &= 2(e^{-\varepsilon t} - e^{-\kappa_1\Delta t})\left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n-1}\right), \\ I_4 &= \Delta t\mathsf{Re}(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}, e^{\alpha\Delta t}\overline{\psi}_h^n\Delta_h\psi_h^n + e^{-\alpha\Delta t}\overline{\psi}_h^{n-1}\Delta_h\psi_h^{n-1}), \\ I_5 &= \frac{1}{4}\Delta t\mathsf{Im}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})^2(e^{\frac{\alpha}{2}\Delta t}\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}), e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right), \\ I_6 &= \Delta t\mathsf{Im}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})F_h, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right), \\ I_7 &= -\frac{1}{2}\Delta t(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}, \Delta_h|e^{\frac{\alpha}{2}\Delta t}\psi_h^n|^2 + \Delta_h|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}|^2). \end{aligned}$$

We estimate the last seven terms of (3.31). By the inequality $1 + x \le e^x\ \forall x \in \mathbb{R}$, Hölder's inequality and Lemma 2.2, we have

$$
\begin{aligned}
|I_1| \le& C\Delta t\|\phi_h^{n-1}\|_{L^4}\|\psi_h^{n-1}\|_{L^4}\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\| \\
&\le \varepsilon_1\Delta t\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^{n-1}\|^2 + C\Delta t\|\phi_h^{n-1}\|_1^2\|\psi_h^{n-1}\|_1^2, \\
|I_2| \le& \varepsilon_1\Delta t\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^{n-1}\|^2 + C\Delta t\|F_h\|^2, \\
|I_3| \le& \varepsilon_3\Delta t\|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n-1}\|^2 + C\Delta t\|G_h\|^2, \\
|I_4| \le& C\Delta t\|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}\|\Big(\|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\| + \|e^{-\frac{3\alpha}{2}\Delta t}\psi_h^{n-1}\|\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|\Big) \\
&\le \varepsilon_2\Delta t\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^n\|^2 + \varepsilon_1\Delta t\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^{n-1}\|^2 + C\Delta t\Big(\|\theta_h^n\|^2 + \|\theta_h^{n-1}\|^2)(\|\psi_h^n\|^2 + \|\psi_h^{n-1}\|^2\Big), \\
|I_5| \le& \frac{1}{4}\Delta t\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|_{L^6}^2\|e^{\frac{\alpha}{2}\Delta t}\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\|_{L^6}\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\| \\
&\le \varepsilon_2\Delta t\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2 + \varepsilon_1\Delta t\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2 + C\Delta t\Big(|\phi_h^n|_1^4 + |\phi_h^{n-1}|_1^4)(|\psi_h^n|_1^2 + |\psi_h^{n-1}|_1^2\Big), \\
|I_6| \le& C\Delta t\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n + e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|_{L^4}\|F_h\|_{L^4}\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n + e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\| \\
&\le \varepsilon_2\Delta t\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^n\|^2 + \varepsilon_1\Delta t\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^{n-1}\|^2 + C\Delta t\Big(\|\phi_h^n\|_1^2 + \|\phi_h^{n-1}\|_1^2\Big)\|F_h\|_{L^4}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|I_7| \le& C\Delta t\|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}\|\Big(\|e^{\frac{\alpha}{2}\Delta t}\nabla_h\psi_h^n\|_{L^4}^2 + \|e^{-\frac{\alpha}{2}\Delta t}\nabla_h\psi_h^{n-1}\|_{L^4}^2\Big) \\
&+ C\Delta t\|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n + e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}\|_{L^4}\Big(\|\psi_h^n\|_{L^4} + \|\psi_h^{n-1}\|_{L^4})(\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\| + \|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|\Big) \\
\le& \varepsilon_2\Delta t\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^n\|^2 + \varepsilon_1\Delta t\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^{n-1}\|^2 + C\Delta t\Big(\|\theta_h^n\|^2 + \|\theta_h^{n-1}\|^2\Big) \\
&+ C\Delta t\Big(|e^{\frac{\beta-\varepsilon}{4}\Delta t}\theta_h^n|_1^{\frac{3}{2}}\|e^{\frac{\beta-\varepsilon}{4}\Delta t}\theta_h^n\|^{\frac{1}{2}} + |e^{\frac{\beta-\varepsilon}{4}\Delta t}\theta_h^{n-1}|_1^{\frac{3}{2}}\|e^{\frac{\beta-\varepsilon}{4}\Delta t}\theta_h^{n-1}\|^{\frac{1}{2}}\Big)\Big(\|\psi_h^n\|_1^2 + \|\psi_h^{n-1}\|_1^2\Big) \\
\le& \varepsilon_2\Delta t\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^n\|^2 + \varepsilon_1\Delta t\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^{n-1}\|^2 + \varepsilon_4\Delta t|e^{\frac{\beta-\varepsilon}{4}\Delta t}\theta_h^n|_1^2 + \varepsilon_5\Delta t|e^{\frac{\beta-\varepsilon}{4}\Delta t}\theta_h^{n-1}|_1^2 \\
&+ C\Delta t\Big(\|\theta_h^n\|^2 + \|\theta_h^{n-1}\|^2)(1 + \|\psi_h^n\|_1^8 + \|\psi_h^{n-1}\|_1^8\Big).
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_1 &= \frac{e^{-\frac{\alpha}{4}\Delta t} - e^{-\frac{\alpha}{2}\Delta t}}{3\Delta t}, \quad \varepsilon_2 = \frac{e^{\frac{\alpha}{2}\Delta t} - 1}{2\Delta t}, \quad \varepsilon_3 = \frac{e^{-\varepsilon\Delta t} - e^{-2\varepsilon\Delta t}}{\Delta t}, \\
\varepsilon_4 &= \frac{e^{\frac{\beta-\varepsilon}{2}\Delta t} - 1}{\Delta t}, \qquad \varepsilon_5 = \frac{e^{-(\beta-\varepsilon)\Delta t} - e^{-\frac{3(\beta-\varepsilon)}{2}\Delta t}}{\Delta t}.
\end{aligned}
$$

Substituting the above estimates into (3.28), we derive that

$$
\widetilde{E}_2^n \le e^{-\kappa_1\Delta t}\widetilde{E}_2^{n-1} + \Delta t\widetilde{C}^n, \qquad n = 1, 2, \cdots \tag{3.31}
$$

where

$$
\begin{aligned}
\widetilde{E}_2^n =& 2\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^n\|^2 + 4\mathsf{Re}\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n e^{\frac{\alpha}{2}\Delta t}\psi_h^n, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right) - 4\mathsf{Re}\left(F_h, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right) \\
&+ |e^{\frac{\beta-\varepsilon}{4}\Delta t}\theta_h^n|_1^2 + (\mu^2 - \varepsilon(\beta-\varepsilon))|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2 + \|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\|^2 + 2\left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\right),
\end{aligned}
$$

and

$$
\widetilde{C}^n \triangleq \widetilde{C}\Big(\|\phi_h^{n-1}\|_1, \|\phi_h^n\|_1, \|\psi_h^{n-1}\|_1, \|\psi_h^n\|_1, \|\theta_h^{n-1}\|, \|\theta_h^n\|, \|F_h\|_{L^4}, \|G_h\|\Big).
$$

Now we estimate lower and upper bounded for $\widetilde{E}_2^n$. By Cauchy's inequality and Lemma 2.2, we have

$$\begin{aligned}
&\left|4Re\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n e^{\frac{\alpha}{2}\Delta t}\psi_h^n, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right)\right| \\
&\quad\le\ 4\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|_{L^4}\|e^{\frac{3\alpha}{4}\Delta t}\psi_h^n\|_{L^4}\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^n\| \\
&\quad\le\ 4|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^{\frac{3}{4}}\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^{\frac{1}{4}}|e^{\frac{3\alpha}{4}\Delta t}\psi_h^n|_1^{\frac{3}{4}}\|e^{\frac{3\alpha}{4}\Delta t}\psi_h^n\|^{\frac{1}{4}}\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^n\| \\
&\quad\le\ \frac{1}{2}\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^n\|^2+8|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^{\frac{3}{2}}\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^{\frac{1}{2}}|e^{\frac{3\alpha}{4}\Delta t}\psi_h^n|_1^{\frac{3}{2}}\|e^{\frac{3\alpha}{4}\Delta t}\psi_h^n\|^{\frac{1}{2}}, \\
&\left|4Re\left(F_h, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right)\right| \le 4\|e^{\frac{\alpha}{4}\Delta t}F_h\|\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^n\| \le \frac{1}{2}\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^n\|^2+8\|e^{\frac{\alpha}{4}\Delta t}F_h\|^2, \\
&\left|2\left(G_h, e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\right)\right| \le 2\|G_h\|\|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\| \le \frac{1}{2}\|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\|^2+2\|G_h\|^2.
\end{aligned}$$

From the above estimates, we derive that

$$\begin{aligned}
\widetilde{E}_2^n \ge\ &\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^n\|^2+|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n|_1^2+(\mu^2-\varepsilon(\beta-\varepsilon))|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2+\frac{1}{2}\|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\|^2 \\
&-8|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^{\frac{3}{2}}\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^{\frac{1}{2}}|e^{\frac{3\alpha}{4}\Delta t}\psi_h^n|_1^{\frac{3}{2}}\|e^{\frac{3\alpha}{4}\Delta t}\psi_h^n\|^{\frac{1}{2}}-8\|e^{\frac{\alpha}{4}\Delta t}F_h\|^2-2\|G_h\|^2, \qquad (3.32)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{E}_2^n \le\ &3\|e^{\frac{\alpha}{4}\Delta t}\Delta_h\psi_h^n\|^2+|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n|_1^2+(\mu^2-\varepsilon(\beta-\varepsilon))|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2+\frac{3}{2}\|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\|^2 \\
&+8|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^{\frac{3}{2}}\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^{\frac{1}{2}}|e^{\frac{3\alpha}{4}\Delta t}\psi_h^n|_1^{\frac{3}{2}}\|e^{\frac{3\alpha}{4}\Delta t}\psi_h^n\|^{\frac{1}{2}}+8\|e^{\frac{\alpha}{4}\Delta t}F_h\|^2+2\|G_h\|^2. \qquad (3.33)
\end{aligned}$$

Using repeatedly the inequality (3.32), we derive that

$$\widetilde{E}_2^n \le e^{-\kappa_1(n-k)\Delta t}\widetilde{E}_2^k+\Delta t\sum_{l=k+1}^{n}e^{-\kappa_1(n-l)\Delta t}\widetilde{C}^l, \quad n>k, \quad k=0,1,\cdots.$$

Using the result of Lemma 3.2, the proof of Lemma 3.3 is complete. □

By Lemmas 2.3 and 3.3, we have

Corollary 3.1. *Assume that conditions of Lemma* 3.3 *are satisfied. Then* ψ_h^n, ϕ_h^n *in* $L^\infty(\Omega_h)$, $(e^{\frac{\alpha}{2}\Delta t}\psi_h^n-e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1})/\Delta t$, $(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n-e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1})/\Delta t$ *in* $L^2(\Omega_h)$, $(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n-e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})/\Delta t$ *in* $H_0^1(\Omega_h)$ *are uniformly bounded.*

4. Stability and Convergence of the Difference Scheme

Let $(\psi_{k,h}^n, \phi_{k,h}^n, \theta_{k,h}^n)$ $(k=1,2)$ be two solutions of the fully discrete finite difference scheme (3.1)–(3.2) associated to the initial data $(\psi_{k,h}^0, \phi_{k,h}^0, \theta_{k,h}^0)$ and $(\psi_h^n, \phi_h^n, \theta_h^n) = (\psi_{1,h}^n-\psi_{2,h}^n, \phi_{1,h}^n-\phi_{2,h}^n, \theta_{1,h}^n-\theta_{2,h}^n)$ with $(\psi_h^0, \phi_h^0, \theta_h^0) = (\psi_{1,h}^0-\psi_{2,h}^0, \phi_{1,h}^0-\phi_{2,h}^0, \theta_{1,h}^0-\theta_{2,h}^0)$. Then $(\psi_h^n, \phi_h^n, \theta_h^n)$

satisfies that

$$
\begin{aligned}
& i\frac{e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n}-e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}}{\Delta t}+\frac{1}{2}\Delta_h\Big(e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n}+e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}\Big)\\
&\quad+\frac{1}{4}\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{1,i,j,k}^{n}+e^{-\frac{\varepsilon}{2}\Delta t}\phi_{1,i,j,k}^{n-1}\Big)\Big(e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n}+e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}\Big)\\
&\quad+\frac{1}{4}\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n}+e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}\Big)\Big(e^{\frac{\alpha}{2}\Delta t}\psi_{2,i,j,k}^{n}+e^{-\frac{\alpha}{2}\Delta t}\psi_{2,i,j,k}^{n-1}\Big)=0, \qquad (4.1a)\\
& \frac{e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n}-e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}}{\Delta t}-\frac{1}{2}\Big(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_{i,j,k}^{n}+e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_{i,j,k}^{n-1}\Big)=0, \qquad (4.1b)\\
& \frac{e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_{i,j,k}^{n}-e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_{i,j,k}^{n-1}}{\Delta t}+\frac{\mu^2-\varepsilon(\beta-\varepsilon)}{2}\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n}+e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}\Big)\\
&\quad-\frac{1}{2}\Delta_h\Big(e^{\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n}+e^{-\frac{\varepsilon}{2}\Delta t}\phi_{i,j,k}^{n-1}\Big)=\frac{1}{2}\Big(e^{\frac{\alpha}{2}\Delta t}\psi_{1,i,j,k}^{n}e^{\frac{\alpha}{2}\Delta t}\overline{\psi_{i,j,k}^{n}}+e^{\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n}e^{\frac{\alpha}{2}\Delta t}\overline{\psi_{2,i,j,k}^{n}}\Big)\\
&\quad+\frac{1}{2}\Big(e^{-\frac{\alpha}{2}\Delta t}\psi_{1,i,j,k}^{n-1}e^{-\frac{\alpha}{2}\Delta t}\overline{\psi_{i,j,k}^{n-1}}+e^{-\frac{\alpha}{2}\Delta t}\psi_{i,j,k}^{n-1}e^{-\frac{\alpha}{2}\Delta t}\overline{\psi_{2,i,j,k}^{n-1}}\Big), \qquad (4.1c)
\end{aligned}
$$

for $1\le i\le J_1-1$, $1\le j\le J_2-1$ and $1\le k\le J_3-1$.

Multiply (4.1a) by $\Delta t(e^{\frac{\alpha}{2}\Delta t}\overline{\psi_{i,j,k}^{n}}+e^{-\frac{\alpha}{2}\Delta t}\overline{\psi_{i,j,k}^{n-1}})h_1h_2h_3$, sum over i,j,k, and take the imaginary part. Then by Corollary 3.1 and Cauchy's inequality, we have

$$
\begin{aligned}
\|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|^2 \le&\|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\|^2+C\Delta t\Big(\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^2+\|e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|^2\\
&+\|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|^2+\|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\|^2\Big). \qquad (4.2)
\end{aligned}
$$

Multiplying both sides of (4.1a) by $4(e^{\frac{\alpha}{2}\Delta t}\Delta_h\overline{\psi_{i,j,k}^{n}}-e^{-\frac{\alpha}{2}\Delta t}\Delta_h\overline{\psi_{i,j,k}^{n-1}})h_1h_2h_3$, summing over i,j,k, and taking the real part, we get

$$
2\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2+J_1+J_2=2\|e_{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2, \qquad (4.3)
$$

where

$$
\begin{aligned}
J_1&=\mathsf{Re}((e^{\frac{\varepsilon}{2}\Delta t}\phi_{1h}^n+e^{-\frac{\varepsilon}{2}\Delta t}\phi_{1h}^{n-1})(e^{\frac{\alpha}{2}\Delta t}\psi_h^n+e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}),e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n-e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}),\\
J_2&=\mathsf{Re}((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n+e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})(e^{\frac{\alpha}{2}\Delta t}\psi_{2h}^n+e^{-\frac{\alpha}{2}\Delta t}\psi_{2h}^{n-1}),e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n-e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}).
\end{aligned}
$$

Note that

$$
\begin{aligned}
J_1=&4\mathsf{Re}\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_{1h}^ne^{\frac{\alpha}{2}\Delta t}\psi_h^n,e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right)-4Re\left(e^{-\frac{\varepsilon}{2}\Delta t}\phi_{1h}^{n-1}e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1},e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right)\\
&-2\mathsf{Re}(e^{\frac{\varepsilon}{2}\Delta t}\phi_{1h}^n-e^{-\frac{\varepsilon}{2}\Delta t}\phi_{1h}^{n-1},e^{\alpha\Delta t}\overline{\psi_h^n}\Delta_h\psi_h^n+e^{-\alpha\Delta t}\overline{\psi_h^{n-1}}\Delta_h\psi_h^{n-1})\\
&-\mathsf{Re}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_{1h}^n+e^{-\frac{\varepsilon}{2}\Delta t}\phi_{1h}^{n-1})(e^{\frac{\alpha}{2}\Delta t}\psi_h^n-e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}),e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n+e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right),\\
J_2=&4\mathsf{Re}\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^ne^{\frac{\alpha}{2}\Delta t}\psi_{2h}^n,e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right)-4Re\left(e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}e^{-\frac{\alpha}{2}\Delta t}\psi_{2h}^{n-1},e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right)\\
&-2\mathsf{Re}(e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n-e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1},e^{\alpha\Delta t}\overline{\psi_{2h}^n}\Delta_h\psi_h^n+e^{-\alpha\Delta t}\overline{\psi_{2h}^{n-1}}\Delta_h\psi_h^{n-1})\\
&-\mathsf{Re}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n+e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1})(e^{\frac{\alpha}{2}\Delta t}\psi_{2h}^n-e^{-\frac{\alpha}{2}\Delta t}\psi_{2h}^{n-1}),e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n+e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right).
\end{aligned}
$$

It follows from (4.3) and the above results that

$$
\begin{aligned}
&2\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2+4Re\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_{1h}^ne^{\frac{\alpha}{2}\Delta t}\psi_h^n+e^{\frac{\varepsilon}{2}\Delta t}\phi_h^ne^{\frac{\alpha}{2}\Delta t}\psi_{2h}^n,e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right)\\
=&2\|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2+4\mathsf{Re}\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_{1h}^{n-1}e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}+e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}e^{\frac{\alpha}{2}\Delta t}\psi_{2h}^{n-1},e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right)+\sum_{j=1}^{5}T_j, \qquad (4.4)
\end{aligned}
$$

where

$$T_1 = 4(e^{-(\varepsilon+2\alpha)\Delta t}-1)Re\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_{1h}^{n-1}e^{\frac{\alpha}{2}\Delta t}\psi_h^{n-1}+e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}e^{\frac{\alpha}{2}\Delta t}\psi_{2h}^{n-1}, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right),$$

$$T_2 = 2\mathrm{Re}(e^{\frac{\varepsilon}{2}\Delta t}\phi_{1h}^{n}-e^{-\frac{\varepsilon}{2}\Delta t}\phi_{1h}^{n-1}, e^{\alpha\Delta t}\overline{\psi_h^n}\Delta_h\psi_h^n+e^{-\alpha\Delta t}\overline{\psi_h^{n-1}}\Delta_h\psi_h^{n-1}),$$

$$T_3 = \mathrm{Re}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_{1h}^{n}+e^{-\frac{\varepsilon}{2}\Delta t}\phi_{1h}^{n-1})(e^{\frac{\alpha}{2}\Delta t}\psi_h^n-e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}), e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n+e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right),$$

$$T_4 = 2\mathrm{Re}(e^{\frac{\varepsilon}{2}\Delta t}\phi_{h}^{n}-e^{-\frac{\varepsilon}{2}\Delta t}\phi_{h}^{n-1}, e^{\alpha\Delta t}\overline{\psi_{2h}^n}\Delta_h\psi_h^n+e^{-\alpha\Delta t}\overline{\psi_{2h}^{n-1}}\Delta_h\psi_h^{n-1}),$$

$$T_5 = \mathrm{Re}\left((e^{\frac{\varepsilon}{2}\Delta t}\phi_{h}^{n}+e^{-\frac{\varepsilon}{2}\Delta t}\phi_{h}^{n-1})(e^{\frac{\alpha}{2}\Delta t}\psi_{2h}^n-e^{-\frac{\alpha}{2}\Delta t}\psi_{2h}^{n-1}), e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n+e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\right).$$

Using (3.1a),(3.1b),(4.1a),(4.1b), Lemmas 2.3 and 2.4 gives

$$|T_1| \le C\Delta t\Big(\|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\|^2+\|e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|^2+\|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2\Big),$$

$$|T_2| \le C\Delta t\Big(\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n}\|^2+\|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2\Big),$$

$$|T_3| \le C\Delta t\Big(\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n}\|^2+\|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2+\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n}\|^2+\|e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|^2\Big),$$

$$|T_4| \le C\Delta t\Big(\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n}\|^2+\|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2+\|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n}\|^2+\|e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}\|^2\Big),$$

$$|T_5| \le C\Delta t\Big(\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n}\|^2+\|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2+\|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n}\|^2+\|e^{-\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n-1}\|^2\Big).$$

Let

$$\xi^n = 2\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2+4\mathrm{Re}\left(e^{\frac{\varepsilon}{2}\Delta t}\phi_{1h}^{n}e^{\frac{\alpha}{2}\Delta t}\psi_h^n+e^{\frac{\varepsilon}{2}\Delta t}\phi_h^{n}e^{\frac{\alpha}{2}\Delta t}\psi_{2h}^n, e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\right).$$

We can derive lower and upper bounded on ξ^n

$$\begin{aligned}\xi^n &\ge 2\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2-4\Big(\|e^{\frac{\varepsilon}{2}\Delta t}\phi_{1,h}^n\|_\infty\|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|+\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|\|e^{\frac{\alpha}{2}\Delta t}\psi_{2,h}^n\|_\infty\Big)\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|\\ &\ge \|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2-M\Big(\|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|^2+\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^2\Big), \end{aligned} \tag{4.5a}$$

$$\xi^n \le 3\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2+M\Big(\|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|^2+\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^2\Big), \tag{4.5b}$$

where

$$M = \max_{n\Delta t\le T}(8\|e^{\frac{\varepsilon}{2}\Delta t}\phi_{1,h}^n\|_\infty^2, 8\|e^{\frac{\alpha}{2}\Delta t}\psi_{2,h}^n\|_\infty^2).$$

From (4.4) and above estimates, we obtain that

$$\begin{aligned}\xi^n \le&\xi^{n-1}+C\Delta t\Big(\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2+\|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2+\|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\|^2\\ &+\|e^{-\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n-1}\|^2+\|e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}\|^2+\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^2+\|e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}\|^2\\ &+\|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n\|^2+\|e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}\|^2\Big).\end{aligned} \tag{4.6}$$

Multiplying (4.1c) by $-(e^{\frac{\beta-\varepsilon}{2}\Delta t}\Delta_h\theta_{i,j,k}^n+e^{-\frac{\beta-\varepsilon}{2}\Delta t}\Delta_h\theta_{i,j,k}^{n-1})h_1h_2h_3\Delta t$, summing over i,j,k, and using (4.1b), we have

$$\begin{aligned}&|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n|_1^2+(\mu^2-\varepsilon(\beta-\varepsilon))|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2+\|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\|^2\\ =&|e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}|_1^2+(\mu^2-\varepsilon(\beta-\varepsilon))|e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}|_1^2+T_6,\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}T_6 = &\|e^{-\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n-1}\|^2+\frac{1}{2}\Delta t(\nabla_h(e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n+e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}),\\ &\nabla_h(e^{\frac{\alpha}{2}\Delta t}\psi_{1,h}^n e^{\frac{\alpha}{2}\Delta t}\overline{\psi_h^n}+e^{\frac{\alpha}{2}\Delta t}\psi_h^n e^{\frac{\alpha}{2}\Delta t}\overline{\psi_{2,h}^n}+e^{-\frac{\alpha}{2}\Delta t}\psi_{1,h}^{n-1}e^{-\frac{\alpha}{2}\Delta t}\overline{\psi_h^{n-1}}\\ &+e^{-\frac{\alpha}{2}\Delta t}\psi_h^{n-1}e^{-\frac{\alpha}{2}\Delta t}\overline{\psi_{2,h}^{n-1}})).\end{aligned}$$

Using Lemmas 2.3 and 2.4, we have

$$|T_6| \leq C\Delta t\Big(\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2 + \|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2 + |e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n|_1^2 + |e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}|_1^2\Big).$$

Substituting the above estimate to (4.7) gives

$$\begin{aligned}
&|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n|_1^2 + (\mu^2-\varepsilon(\beta-\varepsilon))|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2 + \|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\|^2\\
\leq&|e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}|_1^2 + (\mu^2-\varepsilon(\beta-\varepsilon))|e^{-\frac{\varepsilon}{2}\Delta t}\phi_h^{n-1}|_1^2 + \|e^{-\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^{n-1}\|^2\\
&+ C\Delta t(\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2 + \|e^{-\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^{n-1}\|^2 + |e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n|_1^2 + |e^{-\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^{n-1}|_1^2).
\end{aligned} \tag{4.8}$$

Multiply both sides of (4.2) and (4.8) by $2M$ and M_1 respectively, then add them up with the inequality (4.6). Using the resulting result repeatedly for n gives

$$E^n \leq E^0 + C\Delta t\sum_{k=0}^{n}\Big(\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^k\|^2+\|e^{\frac{\alpha}{2}\Delta t}\psi_h^k\|^2+|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^k|_1^2+\|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^k\|^2+\|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^k\|^2\Big), \tag{4.9}$$

where

$$\begin{aligned}
&E^n = 2M\|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|^2 + \xi^n + M_1|e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n|_1^2 + M_1(\mu^2-\varepsilon(\beta-\varepsilon))|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n|_1^2 + M_1\|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\|^2,\\
&M_1 = \frac{2M}{\lambda_{1,h}(\mu^2-\varepsilon(\beta-\varepsilon))}, \qquad \lambda_{1,h} = \inf_{v_h\in H_0^1(\Omega_h)}\frac{|v_h|_1^2}{\|v_h\|^2} \geq \frac{512}{(l_1l_2l_3)^2}.
\end{aligned}$$

By using (4.5) and choosing constant M_1, there exist positive constants M_2 and M_3, such that

$$E^n \geq M_2\Big(\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^2 + \|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|^2 + |e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n|_1^2 + \|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\|^2 + \|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2\Big), \tag{4.10}$$

$$E^n \leq M_3\Big(\|e^{\frac{\varepsilon}{2}\Delta t}\phi_h^n\|^2 + \|e^{\frac{\alpha}{2}\Delta t}\psi_h^n\|^2 + |e^{\frac{\beta-\varepsilon}{2}\Delta t}\theta_h^n|_1^2 + \|e^{\frac{\varepsilon}{2}\Delta t}\Delta_h\phi_h^n\|^2 + \|e^{\frac{\alpha}{2}\Delta t}\Delta_h\psi_h^n\|^2\Big). \tag{4.11}$$

Combining the inequality (4.9)–(4.11), and using Grönwall's Lemma, yields the following result.

Theorem 4.1. *Under the conditions of Lemma* 3.3*, the fully discrete finite difference scheme* (3.1)–(3.2) *is stable on* $\mathbf{W}_h$ *over finite time interval* $(0,T]$*. Moreover, the solution* $(\psi_h^n, \phi_h^n, \theta_h^n)$ *of the fully discrete finite difference scheme* (3.1)–(3.2) *is unique.*

Theorem 4.2. *Assume* $\psi(x,t), \phi(x,t), \phi_t(x,t) \in C^{4,3}$ *for the solution of problem* (1.1)–(1.3)*. Then the finite difference scheme* (3.1)–(3.2) *possesses truncation errors of order* $\mathcal{O}(h_1^2 + h_2^2 + h_3^2 + \Delta t^2)$.

Proof. The result can be obtained by using the standard Taylor expansion techniques. □

By using the same procedures as the proof of Theorem 4.1, we can obtain the following result.

Theorem 4.3. *Assume that the conditions of Lemma* 3.3 *are satisfied, the initial values* $(\psi_h^0, \phi_h^0, \theta_h^0)$ *of the fully discrete finite difference scheme* (3.1)–(3.2) *satisfies that*

$$\|\psi_0 - \psi_h^0\|_2 + \|\phi_0 - \phi_h^0\|_2 + \|\theta_0 - \theta_h^0\|_1 = \mathcal{O}(h_1^2 + h_2^2 + h_3^2),$$

and the solution $(\psi(x,t), \phi(x,t))$ *of the problem* (1.1)–(1.3) *satisfies that* $\psi(x,t), \phi(x,t) \in C^{4,3}$*. Then the solution* $(\psi_h^n, \phi_h^n, \theta_h^n)$ *of the fully discrete finite difference scheme* (3.1)–(3.2) *converges to the solution* $(\psi(x,t), \phi(x,t), \theta(x,t))$ *of the problem* (1.4)–(1.6) *with order* $\mathcal{O}(h_1^2+h_2^2+h_3^2+\Delta t^2)$ *in the* $\mathbf{W}_h$ *norm over finite time interval* $(0,T]$.

5. The Existence of Attractor of the Discrete Dynamical System

In this section, let us put problem (3.1)–(3.2) in the framework of dissipative dynamical system. For fixed h_1, h_2, h_3 and Δt let us define the mapping

$$S_{h,\Delta t}(t_n) : (\psi_h^0, \phi_h^0, \theta_h^0) \longrightarrow (\psi_h^n, \phi_h^n, \theta_h^n) \qquad \text{for all } n \in \mathcal{Z}^+,$$

where $(\psi_h^n, \phi_h^n, \theta_h^n)$ is the solution of the fully discrete finite difference scheme (3.1)–(3.2). It maps $\mathbf{W}_h$ into itself, and has the usual semigroup properties as follows

$$\begin{cases} S_{h,\Delta t}(t_n + t_m) = S_{h,\Delta t}(t_n) S_{h,\Delta t}(t_m) & \forall\ m, n \in \mathcal{Z}^+, \\ S_{h,\Delta t}(0) = I, \end{cases}$$

where I is the identity. The system $S(t)$ is approximated by the discrete dynamical system $S_{h,\Delta t}(t_n)$. Now we prove the existence of the attractor $\mathcal{A}_{h,\Delta t}$ for the discrete systems $S_{h,\Delta t}(t_n)$ on $\mathbf{W}_h$. Theorem 4.1 shows that, for every $n \geq 0$, $S_{h,\Delta t}(t_n)$ is a continuous operator from the finite dimensional space $\mathbf{W}_h$ into itself. By Lemma 3.3, there exists a bounded set B^h which is absorbing in $\mathbf{W}_h$ under $S_{h,\Delta t}(t_n)$. Using Theorem 1.1 in [10], we obtain the main result as follows

Theorem 5.1. *Assume that the conditions of Lemma* 3.3 *are satisfied. Then the discrete dynamical system $S_{h,\Delta t}(t_n)$ associated with the fully discrete finite difference scheme* (3.1)–(3.2) *possess a maximal attractor $\mathcal{A}_{h,\Delta t}$ in $\mathbf{W}_h$, and*

$$\mathcal{A}_{h,\Delta t} = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} S_{h,\Delta t}(t_m) B^h}.$$

Remark 5.1. As the parameters $\alpha, \beta > 0$, the discrete system can well remain the dissipative properties of the original system. As the parameters $\alpha = \beta = 0$, the discrete system can also remain conservation properties of the original system.

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