Journal of Computational Mathematics
Vol.29, No.1, 2011, 1–15.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1006-m3213

# VARIATIONAL DISCRETIZATION OF PARABOLIC CONTROL PROBLEMS IN THE PRESENCE OF POINTWISE STATE CONSTRAINTS*

Klaus Deckelnick
*Institut für Analysis und Numerik, Otto–von–Guericke–Universität Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany*
*Email: klaus.deckelnick@mathematik.uni-magdeburg.de*
Michael Hinze
*Schwerpunkt Optimierung und Approximation, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany*
*Email: michael.hinze@uni-hamburg.de*

**Abstract**

We consider a parabolic optimal control problem with pointwise state constraints. The optimization problem is approximated by a discrete control problem based on a discretization of the state equation by linear finite elements in space and a discontinuous Galerkin scheme in time. Error bounds for control and state are obtained both in two and three space dimensions. These bounds follow from uniform estimates for the discretization error of the state under natural regularity requirements.

*Mathematics subject classification:* 49J20, 49K20, 35B37.
*Key words:* Parabolic optimal control problem, State constraints, Error estimates.

## 1. Introduction

Optimal control of time-dependent production processes plays an important role in many practical applications such as crystal growth [10,16,17] and cooling of glass melts [5,22]. These processes are frequently described by systems of partial differential equations involving the temperature as a system variable. A need to avoid overheating of the device or to prevent solidification/melting at the wrong places then naturally leads to pointwise bounds on the temperature variable. The introduction of pointwise state conditions however yields adjoint variables and multipliers which only admit low regularity complicating the analysis of the necessary first order conditions. These problems need to be taken into account in the numerical approximation and necessitate the development of tailored discrete concepts.

In the present work we consider an optimal control problem for the heat equation and with pointwise bounds on the state. The optimization problem is approximated using variational discretization [14] combined with linear finite elements in space and a discontinuous Galerkin scheme in time for the discretization of the state equation, compare [15, Chapter 3]. Our main result are $L^2$–error estimates for the optimal state and the optimal control. To derive these bounds, uniform estimates for the discretization error of the state under natural regularity requirements are proved. For the numerical analysis of the optimal control problem we use an approach which avoids error estimates for the adjoint state and which was developed in [7],

* Received September 23, 2009 / Revised version received March 11, 2010 / Accepted April 30, 2010 / Published online September 20, 2010 /

[15, Chapter 3] for the analysis of elliptic optimal control problems with state and gradient constraints.

Although a lot of contributions are known on elliptic optimal control problems with pointwise bounds on the state, see e.g. [4,6,7,15,18,19,23], to the best of the authors knowledge numerical analysis of parabolic optimal control problems with pointwise bounds in space-time for the state has not yet been considered in the literature. In this work we present the numerical analysis for our result of Theorem 4.1 which we already announced in [11]. However, there are some contributions on the analysis of related control problems. In [20] Lavrentiev regularization of state constrained parabolic control problems is investigated, optimal control problems with pointwise state constraints in time and averaged state constraints in space are considered in [1]. We note that numerical analysis for this particular setting is announced by Vexler in [11]. Optimality conditions for parabolic optimal control problems in the presence of state constraints are investigated in [8], where further references on analysis aspects of state constrained parabolic control problems can be found.

## 2. The Optimal Control Problem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded convex polygonal domain, $T > 0$, $\Omega_T := \Omega \times (0, T)$ and $\Gamma_T := \partial\Omega \times (0, T)$. Let us consider the initial boundary value problem

$$y_t - \Delta y = f, \quad \text{in } \Omega_T, \tag{2.1}$$

$$\frac{\partial y}{\partial \nu} = 0, \quad \text{on } \Gamma_T, \tag{2.2}$$

$$y(\cdot, 0) = y_0, \quad \text{in } \Omega. \tag{2.3}$$

It is well–known that for given $f \in L^2(0, T; L^2(\Omega)), y_0 \in H^1(\Omega)$ problem (2.1)–(2.3) has a unique solution $y \in C^0([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$. In what follows we shall keep the initial datum $y_0$ fixed and denote by $\hat{y}$ the solution of (2.1)–(2.3) corresponding to $f \equiv 0$. This allows us to write the solution of (2.1)–(2.3) in the form

$$y = \mathcal{G}(f) = \hat{y} + \mathcal{G}_0(f), \tag{2.4}$$

where $\mathcal{G}_0(f)$ is the linear operator that assigns to $f$ the solution of (2.1)–(2.3) for $y_0 \equiv 0$. Note that if $f \in L^2(0, T; H^1(\Omega))$ and

$$y_0 \in H^2(\Omega), \quad \text{with } \frac{\partial y_0}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \tag{2.5}$$

then we have

$$y \in W := \Big\{ w \in C^0([0, T]; H^2(\Omega)) \mid w_t \in L^2(0, T; H^1(\Omega)) \Big\},$$

and

$$\max_{0 \le t \le T} \|y(t)\|^2_{H^2} + \int_0^T \|y_t(t)\|^2_{H^1} dt \le C\Big( \|y_0\|^2_{H^2} + \int_0^T \|f(t)\|^2_{H^1} dt \Big). \tag{2.6}$$

We remark that $W \subset C^0(\overline{\Omega_T})$ since we have the continuous embedding $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$ for $d = 2, 3$.

Next, suppose that the functions $f_1, \cdots, f_m \in H^1(\Omega) \cap L^\infty(\Omega)$ are given and define $U := L^2(0, T; \mathbb{R}^m)$ as well as $B : U \to L^2(0, T; H^1(\Omega))$ by

$$(Bu)(x, t) := \sum_{i=1}^m u_i(t) f_i(x), \quad (x, t) \in \Omega_T. \tag{2.7}$$

This parametrization of the control is motivated by practical considerations. The functions $f_i$ represent given practical control actuations, whose impact is controlled through the time-dependent amplitudes $u_i$ which in our context play the role of control functions.

Note that (2.6) implies that $y = \mathcal{G}(Bu) \in W$ for $u \in U$ with

$$\max_{0\le t\le T} \|y(t)\|^2_{H^2} + \int_0^T \|y_t(t)\|^2_{H^1} dt \le C\Big(\|y_0\|^2_{H^2} + \int_0^T |u(t)|^2 dt\Big), \tag{2.8}$$

where the constant $C$ depends in addition on the $H^1$–norms of $f_1, \cdots, f_m$.

Let us denote by $\mathcal{M}(\overline{\Omega_T})$ the space of regular Borel measures on $\overline{\Omega_T}$. Given $\mu \in \mathcal{M}(\overline{\Omega_T})$ we consider the following backward parabolic problem

$$-\varphi_t - \Delta\varphi = \mu_{\Omega_T}, \qquad \text{in } \Omega_T, \tag{2.9}$$

$$\frac{\partial\varphi}{\partial\nu} = \mu_{\Gamma_T}, \qquad \text{on } \Gamma_T, \tag{2.10}$$

$$\varphi(\cdot, T) = \mu_T, \qquad \text{in } \Omega. \tag{2.11}$$

Here, $\mu_{\Omega_T} := \mu_{|\Omega_T}$, $\mu_{\Gamma_T} := \mu_{|\Gamma_T}$ and $\mu_T := \mu_{|\bar\Omega\times\{T\}}$.

**Theorem 2.1.** *There exists a unique function $\varphi$ that belongs to $L^s(0,T;W^{1,\sigma}(\Omega))$ for all $s,\sigma \in [1,2)$ with $\frac{2}{s} + \frac{d}{\sigma} > d+1$ and which solves (2.9)–(2.11) in the sense that*

$$\int_0^T (w_t - \Delta w, \varphi)dt + \int_0^T \int_{\partial\Omega} \frac{\partial w}{\partial\nu}\varphi do dt = \int_{\overline{\Omega_T}} w d\mu, \qquad \forall w \in W_0^\infty, \tag{2.12}$$

*where $W_0^\infty := \{w \in W \,|\, w(\cdot,0) = 0 \text{ in } \bar\Omega, w_t - \Delta w \in L^\infty(\Omega_T), \frac{\partial w}{\partial\nu} \in L^\infty(\Gamma_T)\}$. Here, $(\bullet,\bullet)$ denotes the inner product in $L^2(\Omega)$.*

*Proof.* See [8], Theorem 6.3. □

Note that $\varphi \in L^1(0,T;W^{1,1}(\Omega))$ so that all integrals in (2.12) exist. We consider the optimization problem

$$(TP) \quad \begin{cases} \min_{u\in U} J(u) := \dfrac{1}{2}\displaystyle\int_0^T \|y(\cdot,t) - \bar y(\cdot,t)\|^2 dt + \dfrac{\alpha}{2}\displaystyle\int_0^T |u(t)|^2 dt, \\ \text{s.t. } y = \mathcal{G}(Bu), \text{ and } y(x,t) \ge 0, (x,t) \in \overline{\Omega_T}, \end{cases} \tag{2.13}$$

where $\bar y \in H^1(0,T;L^2(\Omega))$ is given. From now on we shall assume (2.5) and that $\min_{x\in\bar\Omega} y_0(x) > 0$. It is not difficult to verify with the help of a comparison argument that the function $\hat y$ in (2.4) satisfies

$$\hat y(x,t) > 0, (x,t) \in \overline{\Omega_T}. \tag{2.14}$$

Since the state constraints form a convex set and the set of admissible controls is closed and convex one obtains the existence of a unique solution $u \in U$ to problem (2.13) by standard arguments.

**Theorem 2.2.** *The function $u \in U$ is the solution of (2.13) if and only if there exist $\mu \in \mathcal{M}(\overline{\Omega_T})$ and a function $p \in L^s(0,T;W^{1,\sigma}(\Omega))$ ($s,\sigma \in [1,2), \frac{2}{s} + \frac{d}{\sigma} > d+1$), such that with $y = \mathcal{G}(Bu)$ there holds*

$$\int_0^T (w_t - \Delta w, p)dt + \int_0^T \int_{\partial\Omega} \frac{\partial w}{\partial\nu} p do dt = \int_0^T (y - \bar y, w)dt + \int_{\overline{\Omega_T}} w d\mu, \qquad \forall w \in W_0^\infty, \tag{2.15}$$

$$\alpha u(t) + \big((p(\cdot,t), f_i)\big)_{i=1,\cdots,m} = 0, \qquad a.e.\ in\ (0,T), \tag{2.16}$$

$$\mu \le 0, \quad y(x,t) \ge 0, \quad (x,t) \in \overline{\Omega_T} \quad and \quad \int_{\overline{\Omega_T}} y d\mu = 0. \tag{2.17}$$

*Proof.* It is not difficult to see that the existence of $\mu$ and $p$ with the above properties implies that $u$ is a solution of (2.13). In order to show the reverse we apply Theorem 5.2 in [3] (compare also [2, Theorem 2]) with the choices $U = L^2(0,T;\mathbb{R}^m)$, $Z = C^0(\overline{\Omega_T})$, $K = U$ and

$$C = \Big\{ z \in Z \,|\, z(x,t) \ge 0, \quad \forall (x,t) \in \overline{\Omega_T} \Big\}.$$

Furthermore, let $G : U \to Z, G(v) := \mathcal{G}(Bv) = \hat{y} + \mathcal{G}_0(Bv)$. Clearly, $DG(u)v = \mathcal{G}_0(Bv)$ so that we obtain in particular with the choice $u_0 = 0$

$$G(u) + DG(u)(u_0 - u) = \hat{y} + \mathcal{G}_0(Bu) - \mathcal{G}_0(Bu) = \hat{y} \in \mathrm{int}(C),$$

by (2.14). According to Theorem 5.2 in [3] there exists $\mu \in \big(C^0(\overline{\Omega_T})\big)' = \mathcal{M}(\overline{\Omega_T})$ such that

$$\int_{\overline{\Omega_T}} (z - y) d\mu \le 0, \qquad \forall z \in C, \tag{2.18}$$

$$J'(u)v + \langle DG(u)^* \mu, v \rangle_U = 0, \quad \forall v \in U. \tag{2.19}$$

Standard measure theoretic arguments imply that $\mu \le 0$ and that supp$\mu \subset \{(x,t) \in \overline{\Omega_T} \,|\, y(x,t) = 0\}$ giving (2.17). Since $y_0 > 0$ in $\bar{\Omega}$ this yields in particular that supp$\mu \subset \bar{\Omega} \times (0,T]$. Next, we calculate

$$J'(u)v = \int_0^T (y - \bar{y}, y_v) dt + \alpha \int_0^T u \cdot v dt, \quad v \in U, \quad \text{where } y_v = \mathcal{G}_0(Bv). \tag{2.20}$$

Furthermore, since $DG(u)v = y_v$ we have

$$\langle DG(u)^* \mu, v \rangle_U = \int_{\overline{\Omega_T}} y_v d\mu.$$

Hence, combining (2.19) and (2.20) we derive

$$\int_0^T (y - \bar{y}, y_v) dt + \alpha \int_0^T u \cdot v dt + \int_{\overline{\Omega_T}} y_v d\mu = 0, \qquad \forall v \in U. \tag{2.21}$$

In view of Theorem 2.1 there exists a unique solution $p \in L^s(0,T;W^{1,\sigma}(\Omega))$ $(s, \sigma \in [1,2)$ with $\frac{2}{s} + \frac{d}{\sigma} > d+1)$ of the backward parabolic problem

$$-p_t - \Delta p = y - \bar{y} + \mu_{\Omega_T}, \qquad \text{in } \Omega_T, \tag{2.22}$$

$$\frac{\partial p}{\partial \nu} = \mu_{\Gamma_T}, \qquad \text{on } \Gamma_T, \tag{2.23}$$

$$p(\cdot, T) = \mu_T, \qquad \text{in } \Omega, \tag{2.24}$$

so that

$$\int_0^T (w_t - \Delta w, p) dt + \int_0^T \int_{\partial\Omega} \frac{\partial w}{\partial \nu} p do dt = \int_0^T (y - \bar{y}, w) dt + \int_{\overline{\Omega_T}} w d\mu, \qquad \forall w \in W_0^\infty. \tag{2.25}$$

It remains to verify (2.16). If $v \in C_0^\infty(0,T;\mathbb{R}^m)$ then $y_v = \mathcal{G}_0(Bv)$ belongs to $W_0^\infty$ because we have assumed that $f_i \in L^\infty(\Omega), i = 1, \cdots, m$. Hence we deduce from (2.21), (2.25) and the definition of $y_v$ that

$$\begin{aligned}
0 =& \int_0^T (y - \bar{y}, y_v)dt + \alpha \int_0^T u \cdot v dt + \int_{\overline{\Omega_T}} y_v d\mu \\
=& \int_0^T (y_{v,t} - \Delta y_v, p)dt + \alpha \int_0^T u \cdot v dt \\
=& \sum_{i=1}^m \int_0^T v_i \big\{(p(\cdot, t), f_i) + \alpha u_i\big\} dt.
\end{aligned}$$

Since $v \in C_0^\infty(0,T;\mathbb{R}^m)$ is arbitrary we obtain (2.16). □

## 3. Discretization

Let $\mathcal{T}_h$ be a quasi–uniform triangulation of $\Omega$ with maximum mesh size $h := \max_{S \in \mathcal{T}_h} \operatorname{diam}(S)$. Let us denote by $x_1, \cdots, x_J$ the set of nodes of $\mathcal{T}_h$. We consider the space of linear finite elements

$$X_h := \Big\{\phi_h \in C^0(\bar{\Omega}) \,|\, \phi_h \text{ is a linear polynomial on each } S \in \mathcal{T}_h\Big\}.$$

We denote by $I_h$ the usual Lagrange interpolation operator and by $P_h : L^2(\Omega) \to X_h$ the $L^2$–projection, i.e.

$$(z, \phi_h) = (P_h z, \phi_h), \qquad \forall \phi_h \in X_h.$$

Furthermore, let $R_h : H^1(\Omega) \to X_h$ be the Ritz–projection, defined by the relation

$$(\nabla R_h z, \nabla \phi_h) + (R_h z, \phi_h) = (\nabla z, \nabla \phi_h) + (z, \phi_h), \qquad \forall \phi_h \in X_h. \tag{3.1}$$

It is well–known that

$$\|z - R_h z\| + h\|\nabla(z - R_h z)\| \le Ch^m \|z\|_{H^m}, \qquad \forall z \in H^m(\Omega),\ m = 1, 2. \tag{3.2}$$

We shall also require a uniform bound on $z - R_h z$. Using interpolation and inverse estimates together with (3.2) we find for $z \in H^2(\Omega)$ that

$$\begin{aligned}
\|z - R_h z\|_{L^\infty} \le& \|z - I_h z\|_{L^\infty} + \|I_h z - R_h z\|_{L^\infty} \\
\le& Ch^{2-\frac{d}{2}} \|z\|_{H^2} + Ch^{-\frac{d}{2}} \|I_h z - R_h z\| \le Ch^{2-\frac{d}{2}} \|z\|_{H^2}.
\end{aligned} \tag{3.3}$$

Furthermore, we have the following estimate for functions $\phi_h \in X_h$,

$$\|\phi_h\|_{L^\infty} \le C\rho(d,h) \|\phi_h\|_{H^1}, \tag{3.4}$$

where

$$\rho(d,h) = \left\{ \begin{array}{ll} \sqrt{|\log h|}, & d = 2, \\ h^{-\frac{1}{2}}, & d = 3. \end{array} \right.$$

Next, let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ a time grid with $\tau_n := t_n - t_{n-1}, n = 1, \cdots, N$ and $\tau := \max_{1 \le n \le N} \tau_n$. We set

$$W_{h,\tau} := \Big\{\Phi : \bar{\Omega} \times [0,T] \to \mathbb{R} \,|\, \Phi(\cdot, t) \in X_h \text{ is constant in } t \in (t_{n-1}, t_n), 1 \le n \le N\Big\}.$$

For $Y,\Phi\in W_{h,\tau}$ we let

$$A(Y,\Phi):=\sum_{n=1}^N\tau_n(\nabla Y^n,\nabla\Phi^n)+\sum_{n=2}^N(Y^n-Y^{n-1},\Phi^n)+(Y^0_+,\Phi^0_+),$$

where $\Phi^n:=\Phi^n_-,\Phi^n_\pm=\lim_{s\to 0^\pm}\Phi(t_n+s)$. Given $u\in U$, our approximation $Y\in W_{h,\tau}$ of the solution $y$ of (2.1)–(2.3) is obtained by the following discontinuous Galerkin scheme:

$$A(Y,\Phi)=\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\big(Bu(t),\Phi^n\big)+(y_0,\Phi^0_+),\qquad\forall\Phi\in W_{h,\tau}.\tag{3.5}$$

The above solution will be denoted by $Y=\mathcal{G}_{h,\tau}(Bu)$. We have the following uniform error estimate.

**Theorem 3.1.** *Let $u\in U, y=\mathcal{G}(Bu), Y=\mathcal{G}_{h,\tau}(Bu)$. Then*

$$\max_{1\le n\le N}\|y(\cdot,t_n)-Y^n\|_{L^\infty}\le C\rho(d,h)(h+\sqrt{\tau})\big(\|y_0\|_{H^2}+\|u\|_U\big).$$

*Proof.* We begin by deriving an error relation using standard arguments. Take $\Phi\in W_{h,\tau}$, multiply (2.1) by $\Phi^n\in X_h$ and integrate over $\Omega\times(t_{n-1},t_n)$: Abbreviating $y^n:=y(\cdot,t_n)$ we have

$$(y^n-y^{n-1},\Phi^n)+\int_{t_{n-1}}^{t_n}(\nabla y,\nabla\Phi^n)dt=\int_{t_{n-1}}^{t_n}\big(Bu(t),\Phi^n\big)dt,\qquad 1\le n\le N.\tag{3.6}$$

Next, let us introduce $\tilde Y\in W_{h,\tau}$ by

$$\tilde Y(\cdot,t):=R_hy^n,\qquad t\in(t_{n-1},t_n),1\le n\le N.\tag{3.7}$$

Using (3.6) along with (3.1) we derive by straightforward calculation

$$A(\tilde Y,\Phi)=\sum_{n=1}^N\int_{t_{n-1}}^{t_n}(Bu(t),\Phi^n)dt+(y_0,\Phi^0_+)+r(\Phi),\qquad\forall\Phi\in W_{h,\tau},$$

where

$$\begin{aligned}r(\Phi)&=\sum_{n=1}^N\tau_n\left(\nabla y^n-\frac{1}{\tau_n}\int_{t_{n-1}}^{t_n}\nabla y dt,\nabla\Phi^n\right)+\sum_{n=1}^N\tau_n(y^n-R_hy^n,\Phi^n)\\&\quad+\sum_{n=2}^N\big(R_h(y^n-y^{n-1})-(y^n-y^{n-1}),\Phi^n\big)+(R_hy^1-y^1,\Phi^1)\\&\equiv:\sum_{j=1}^4r_j(\Phi).\end{aligned}\tag{3.8}$$

As a consequence, the error $E:=\tilde Y-Y\in W_{h,\tau}$ satisfies

$$A(E,\Phi)=r(\Phi),\qquad\forall\Phi\in W_{h,\tau}.\tag{3.9}$$

Let us fix $l\in\{2,\cdots,N\}$ and define $\Phi\in W_{h,\tau}$ by

$$\Phi^n:=\begin{cases}0, & n=1\ \text{ or }\ n>l,\\ \dfrac{E^n-E^{n-1}}{\tau_n}, & 2\le n\le l.\end{cases}$$

Inserting $\Phi$ into (3.9) yields

$$\sum_{n=2}^{l}\tau_n\left\|\frac{E^n-E^{n-1}}{\tau_n}\right\|^2+\frac{1}{2}\|\nabla E^l\|^2-\frac{1}{2}\|\nabla E^1\|^2+\frac{1}{2}\sum_{n=2}^{l}\|\nabla(E^n-E^{n-1})\|^2=r(\Phi).$$

Let us estimate the integrals in the remainder term $r(\Phi)$. To begin,

$$\begin{aligned}|r_1(\Phi)|\leq&\sum_{n=2}^{l}\left\|\nabla y^n-\frac{1}{\tau_n}\int_{t_{n-1}}^{t_n}\nabla y(t)dt\right\|\,\|\nabla(E^n-E^{n-1})\|\\ \leq&\frac{1}{4}\sum_{n=2}^{l}\|\nabla(E^n-E^{n-1})\|^2+C\tau\int_0^T\|\nabla y_t\|^2dt.\end{aligned}$$

We infer from Young's inequality and (3.2) that

$$\begin{aligned}|r_2(\Phi)|\leq&\sum_{n=2}^{l}\|y^n-R_hy^n\|\,\|E^n-E^{n-1}\|\\ \leq&\frac{1}{4}\sum_{n=2}^{l}\tau_n\left\|\frac{E^n-E^{n-1}}{\tau_n}\right\|^2+Ch^4\max_{1\leq n\leq N}\|y^n\|_{H^2}^2.\end{aligned}$$

Finally, again by (3.2),

$$\begin{aligned}|r_3(\Phi)|\leq&Ch\sum_{n=2}^{l}\frac{1}{\tau_n}\|y^n-y^{n-1}\|_{H^1}\|E^n-E^{n-1}\|\\ \leq&Ch\sum_{n=2}^{l}\frac{1}{\sqrt{\tau_n}}\left(\int_{t_{n-1}}^{t_n}\|y_t\|_{H^1}^2dt\right)^{\frac{1}{2}}\|E^n-E^{n-1}\|\\ \leq&\frac{1}{4}\sum_{n=2}^{l}\tau_n\left\|\frac{E^n-E^{n-1}}{\tau_n}\right\|^2+Ch^2\int_0^T\|y_t\|_{H^1}^2dt.\end{aligned}$$

Since $r_4(\Phi)=0$ we obtain upon combining the above inequalities and recalling (2.8) that

$$\begin{aligned}&\frac{1}{2}\sum_{n=2}^{l}\tau_n\left\|\frac{E^n-E^{n-1}}{\tau_n}\right\|^2+\frac{1}{2}\|\nabla E^l\|^2\\ \leq&\frac{1}{2}\|\nabla E^1\|^2+C(h^2+\tau)\Big(\|y_0\|_{H^2}^2+\|u\|_U^2\Big).\end{aligned}\tag{3.10}$$

It remains to estimate $\|\nabla E^1\|^2$. We insert $\Phi\in W_{h,\tau}$ with $\Phi^1=\phi_h\in X_h, \Phi^n=0, n\geq 2$ into the error relation (3.9). After straightforward calculations we obtain

$$\tau_1(\nabla E^1,\nabla\phi_h)+(P_hy^1-Y^1,\phi_h)=\tau_1\left(\nabla y^1-\frac{1}{\tau_1}\int_{t_0}^{t_1}\nabla ydt,\nabla\phi_h\right)+\tau_1(y^1-R_hy^1,\phi_h)$$

for all $\phi_h\in X_h$. Choosing $\phi_h=P_hy^1-Y^1=E^1+(P_hy^1-R_hy^1)$ we have

$$\begin{aligned}&\tau_1\|\nabla E^1\|^2+\|P_hy^1-Y^1\|^2=\tau_1\left(\nabla y^1-\frac{1}{\tau_1}\int_{t_0}^{t_1}\nabla ydt,\nabla E^1+\nabla(P_hy^1-R_hy^1)\right)\\ &\quad+\tau_1\left(\nabla E^1,\nabla(R_hy^1-P_hy^1)\right)+\tau_1(y^1-R_hy^1,P_hy^1-Y^1)\\ \leq&\tau_1^{\frac{3}{2}}\Big(\int_{t_0}^{t_1}\|\nabla y_t\|^2dt\Big)^{\frac{1}{2}}\left(\|\nabla E^1\|+\|\nabla(R_hy^1-P_hy^1)\|\right)+\tau_1\|\nabla E^1\|\,\|\nabla(R_hy^1-P_hy^1)\|\\ &\quad+C\tau_1\|y^1-R_hy^1\|\,\|P_hy^1-Y^1\|\end{aligned}$$

$$\leq \frac{1}{2}\Big(\tau_1\big\|\nabla E^1\big\|^2 + \big\|P_h y^1 - Y^1\big\|^2\Big) + C\tau_1 h^2 \big\|y^1\big\|_{H^2}^2 + C\tau_1^2 \int_{t_0}^{t_1} \big\|\nabla y_t\big\|^2 dt.$$

Hence, recalling (2.8),

$$\big\|\nabla E^1\big\|^2 + \frac{1}{\tau_1}\big\|P_h y^1 - Y^1\big\|^2 \leq C(h^2+\tau)\Big(\big\|y_0\big\|_{H^2}^2 + \big\|u\big\|_U^2\Big). \tag{3.11}$$

Inserting (3.11) into (3.10) we deduce that

$$\sum_{n=2}^{N} \tau_n \left\|\frac{E^n - E^{n-1}}{\tau_n}\right\|^2 + \max_{1\leq n\leq N} \|\nabla E^n\|^2 \leq C(h^2+\tau)\Big(\|y_0\|_{H^2}^2 + \|u\|_U^2\Big). \tag{3.12}$$

Furthermore, we infer from (3.11) and (3.12) for $1 \leq n \leq N$,

$$\begin{aligned}
\big\|E^n\big\| \leq& \big\|E^1\big\| + \sum_{i=2}^{n} \big\|E^i - E^{i-1}\big\| \\
\leq& \big\|P_h y^1 - Y^1\big\| + \big\|R_h y^1 - P_h y^1\big\| + \left(\sum_{i=2}^{n} \tau_i \left\|\frac{E^i - E^{i-1}}{\tau_i}\right\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \tau_i\right)^{\frac{1}{2}} \\
\leq& C(h+\sqrt{\tau})\Big(\big\|y_0\big\|_{H^2} + \big\|u\big\|_U\Big).
\end{aligned}$$

Combining this estimate with (3.12) we obtain

$$\max_{n=1,\cdots,N} \big\|E^n\big\|_{H^1} \leq C\big(h+\sqrt{\tau}\big)\Big(\big\|y_0\big\|_{H^2} + \big\|u\big\|_U\Big). \tag{3.13}$$

Finally, we infer with the help of (3.3), (3.4), (2.8) and (3.13) that

$$\begin{aligned}
\big\|y^n - Y^n\big\|_{L^\infty} \leq& \big\|y^n - R_h y^n\big\|_{L^\infty} + \big\|E^n\big\|_{L^\infty} \\
\leq& C h^{2-\frac{d}{2}} \big\|y^n\big\|_{H^2} + C\rho(d,h)\big\|E^n\big\|_{H^1} \\
\leq& C\rho(d,h)(h+\sqrt{\tau})\Big(\big\|y_0\big\|_{H^2} + \big\|u\big\|_U\Big),
\end{aligned}$$

which completes the proof. □

**Remark 3.2.** Note that the error bound in Theorem 3.1 is derived under the condition that the right hand side and hence the time derivative of the solution is only square integrable in time. Classical results known from the literature require higher regularity requirements, see e.g. [9, Theorem 1.2] and thus are not applicable in our case. A situation that is comparable to ours in that the time derivative only belongs to some $L^q$–space is considered in [21]. For a function

$$y \in W_q^{2,1}(\Omega_T) := \Big\{z \in L^q(0,T;W^{2,q}(\Omega)), z_t \in L^q(0,T;L^q(\Omega))\Big\}, \qquad (q>2),$$

with $y = 0$ on $\Gamma_T$ the following parabolic projection is analyzed: $Y^0 = I_h y_0$ and

$$\frac{1}{\tau}(Y^n - Y^{n-1}, \phi_h) + (\nabla Y^n, \nabla\phi_h) = \frac{1}{\tau}(y^n - y^{n-1}, \phi_h) + \left(\frac{1}{\tau}\int_{t_{n-1}}^{t_n} \nabla y(\cdot,t)dt, \nabla\phi_h\right),$$

for all $\phi_h \in X_{h0} := X_h \cap H_0^1(\Omega), 1 \leq n \leq N$. Here, $y^n = y(\cdot, t_n)$. It is shown in [21, Theorem 4.1] that

$$\max_{1\leq n\leq N} \big\|y^n - Y^n\big\|_{L^\infty} \leq Cq^2|\log h|^2\Big(h^{2-4/q} + \tau^{1-2/q}\Big)\|y\|_{W_q^{2,1}}, \tag{3.14}$$

provided that $d = 2$ and $\tau \geq C^*|\log h|^3 h^2$. We expect that the techniques in [21] can be applied to the scheme (3.5) and Neumann boundary conditions provided that the solution has the necessary regularity.

**Remark 3.3.** In what follows we shall assume that the time step is coupled to the spatial grid size $h$ in such a way that $\tau = o(\rho(d,h)^{-2})$ as $h \to 0$. With this choice we infer from Theorem 3.1 that

$$\max_{1\leq n\leq N} \left\|y(\cdot,t_n) - Y^n\right\|_{L^\infty} \to 0, \quad h \to 0.$$

We use the variational approach of [14] in order to discretize our optimal control problem as follows:

$$(TP)_h \quad \begin{cases} \min_{u\in U} J_{h,\tau}(u) := \dfrac{1}{2}\displaystyle\sum_{n=1}^{N} \tau_n \left\|Y^n - \bar{y}^n\right\|^2 + \frac{\alpha}{2}\int_0^T |u(t)|^2 dt, \\ \text{s.t. } Y = \mathcal{G}_{h,\tau}(Bu) \text{ and } Y^n(x_j) \geq 0, 1 \leq j \leq J, 1 \leq n \leq N. \end{cases} \tag{3.15}$$

As a minimization problem for a quadratic functional over a closed and convex domain, $(TP)_h$ has a unique solution $u_h \in U$. Furthermore, using [3, Theorem 5.2] again, we conclude that there exist $\mu_j^n \in \mathbb{R}, 1 \leq j \leq J, 1 \leq n \leq N$ as well as $P \in W_{h,\tau}$ such that

$$A(\Phi, P) = \sum_{n=1}^{N} \tau_n (Y^n - \bar{y}^n, \Phi^n) + \sum_{n=1}^{N}\sum_{j=1}^{J} \Phi^n(x_j)\mu_j^n, \qquad \forall\, \Phi \in W_{h,\tau}, \tag{3.16}$$

$$\alpha u_h(t) + \big((P^n, f_i)\big)_{i=1,\cdots,m} = 0, \qquad a.e. \text{ in } (t_{n-1}, t_n), \tag{3.17}$$

$$\mu_j^n \leq 0,\ Y^n(x_j) \geq 0, \quad \text{and} \quad \sum_{n=1}^{N}\sum_{j=1}^{J} Y^n(x_j)\mu_j^n = 0. \tag{3.18}$$

Here we note that in view of (3.17) variational discretization automatically yields an optimal control $u_h$ which is piecewise constant in time. An inspection of the arguments above then shows that replacing the control space $U$ in problem (3.15) by the space of piecewise constant functions over the time grid in the present situation would give the same optimal control $u_h$.

Let us define the measure $\mu_{h,\tau} \in \mathcal{M}(\overline{\Omega_T})$ by

$$\int_{\overline{\Omega_T}} f d\mu_{h,\tau} := \sum_{n=1}^{N}\sum_{j=1}^{J} f(x_j, t_n)\mu_j^n, \qquad f \in C^0(\overline{\Omega_T}).$$

As a first result for (3.15) we prove that the sequence of optimal controls, states and measures $\mu_{h,\tau}$ are uniformly bounded.

**Lemma 3.4.** *Let $u_h \in U$ be the optimal solution of* (3.15) *with corresponding state $Y = \mathcal{G}_{h,\tau}(Bu_h)$ and adjoint variables $P \in W_{h,\tau}$ and $\mu_{h,\tau} \in \mathcal{M}(\overline{\Omega_T})$. Then there exists $h_0 > 0$ such that*

$$\sum_{n=1}^{N} \tau_n \left\|Y^n\right\|^2 + \int_0^T |u_h(t)|^2 dt + \sum_{n=1}^{N}\sum_{j=1}^{J} |\mu_j^n| \leq C, \qquad \textit{for all} \quad 0 < h \leq h_0.$$

*Proof.* We infer from (2.14) that there exists $\delta > 0$ such that

$$\hat{y} = \mathcal{G}(0) \geq \delta, \qquad \text{in } \overline{\Omega_T}.$$

Theorem 3.1 and Remark 3.3 then imply that $\hat{Y} := \mathcal{G}_{h,\tau}(0) \in W_{h,\tau}$ satisfies

$$\hat{Y}^n(x_j) \geq \frac{\delta}{2}, \qquad 1 \leq j \leq J,\ 1 \leq n \leq N,\ 0 < h \leq h_0. \tag{3.19}$$

Using (3.18) and (3.16) we obtain

$$\begin{aligned}
&\sum_{n=1}^{N}\sum_{j=1}^{J} \hat{Y}^n(x_j)\,|\mu_j^n| = \sum_{n=1}^{N}\sum_{j=1}^{J}(Y^n(x_j) - \hat{Y}^n(x_j))\mu_j^n \\
=& -\sum_{n=1}^{N}\tau_n(Y^n - \bar{y}^n, Y^n - \hat{Y}^n) + A(Y - \hat{Y}, P) \\
=& \sum_{n=1}^{N}\tau_n \int_\Omega \big(-(Y^n)^2 + Y^n\hat{Y}^n + \bar{y}^n Y^n - \bar{y}^n\hat{Y}^n\big) + \sum_{n=1}^{N}\sum_{i=1}^{m}\tau_n\, u_{h,i|(t_{n-1},t_n)}(P^n, f_i) \\
\leq& -\frac{1}{2}\sum_{n=1}^{N}\tau_n \big\|Y^n\big\|^2 - \alpha\int_0^T |u_h(t)|^2 dt + C
\end{aligned}$$

recalling that $Y = \mathcal{G}_{h,\tau}(Bu_h), \hat{Y} = \mathcal{G}_{h,\tau}(0)$ as well as (3.17). Combining this estimate with (3.19) implies the result. □

## 4. Error Estimate

**Theorem 4.1.** *Let $u$ be the solution of $(TP)$, $u_h$ the solution of $(TP)_h$ with corresponding states $y = \mathcal{G}(Bu)$ and $Y = \mathcal{G}_{h,\tau}(Bu_h)$. Then*

$$\sum_{n=1}^{N}\tau_n\big\|y(\cdot,t_n) - Y^n\big\|^2 + \int_0^T |u(t) - u_h(t)|^2 dt \leq C\rho(d,h)\big(h + \sqrt{\tau}\big) + C\tau.$$

*Proof.* Let us write

$$\begin{aligned}
&\alpha\int_0^T |u(t) - u_h(t)|^2 dt \\
=&\alpha\int_0^T u(t)\cdot(u(t) - u_h(t))dt - \alpha\int_0^T u_h(t)\cdot(u(t) - u_h(t))dt \\
\equiv& I + II.
\end{aligned} \tag{4.1}$$

In order to deal with $I$ we choose a sequence $(v_k)_{k\in\mathbb{N}}, v_k \in C_0^\infty(0,T;\mathbb{R}^m)$ such that $v_k \to u - u_h$ in $L^2(0,T;\mathbb{R}^m)$ as $k \to \infty$. Furthermore, let $y^h := \mathcal{G}(Bu_h)$ and $z_k := \mathcal{G}_0(Bv_k)$. Note that $z_k \in W_0^\infty$ in view of the smoothness of $v_k$ and the fact that $f_i \in L^\infty(\Omega), i = 1,\cdots,m$. In addition, (2.8) yields

$$\begin{aligned}
\big\|(y - y^h) - z_k\big\|_{C^0(\overline{\Omega_T})} \leq& C\max_{0\leq t\leq T}\big\|(y - y^h)(\cdot,t) - z_k(\cdot,t)\big\|_{H^2} \\
\leq& C\Big(\int_0^T |(u - u_h)(t) - v_k(t)|^2 dt\Big)^{\frac{1}{2}} \to 0, k \to \infty.
\end{aligned} \tag{4.2}$$

Hence, we infer from (2.16), the definition of $z_k$, (2.15) and (4.2) that

$$I = \alpha\lim_{k\to\infty}\int_0^T u(t)\cdot v_k(t)dt = -\lim_{k\to\infty}\int_0^T\sum_{i=1}^{m} v_{k,i}(t)\,(p(\cdot,t), f_i)dt$$

$$
\begin{aligned}
&= - \lim_{k\to\infty} \int_0^T (Bv_k, p)dt = - \lim_{k\to\infty} \int_0^T (z_{k,t} - \Delta z_k, p)dt \\
&= - \lim_{k\to\infty} \Big\{ \int_0^T (y - \bar{y}, z_k)dt + \int_{\overline{\Omega_T}} z_k d\mu \Big\} \\
&= \int_0^T (y - \bar{y}, y^h - y)dt + \int_{\overline{\Omega_T}} (y^h - y) d\mu.
\end{aligned}
$$

Recalling (2.17) we may continue

$$
\begin{aligned}
I =& \sum_{n=1}^N \tau_n (y^n - \bar{y}^n, y^{h,n} - y^n) + \int_{\overline{\Omega_T}} (y^h)^- d\mu \\
&+ \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \Big\{ (y - \bar{y}, y^h - y) - (y^n - \bar{y}^n, y^{h,n} - y^n) \Big\} dt \\
\equiv& I_1 + I_2 + I_3.
\end{aligned}
$$

Here, we have abbreviated $y^- = \min(y, 0)$. Let us start with the second term. For $(x,t) \in \bar{\Omega} \times (t_{n-1}, t_n)$ we deduce upon recalling (3.3), (3.4), Theorem 3.1 and the fact that $Y^n(x) \geq 0, x \in \bar{\Omega}$

$$
\begin{aligned}
&\big|(y^h)^-(x,t)\big| \leq \big|(y^h)^-(x,t) - (y^h)^-(x,t_n)\big| + \big|(y^h)^-(x,t_n) - (Y^n)^-(x)\big| \\
\leq& \big|y^h(x,t) - y^h(x,t_n)\big| + \big|y^h(x,t_n) - Y^n(x)\big| \\
\leq& 2 \max_{0\leq s\leq T} \big\|y^h(\cdot,s) - R_h y^h(\cdot,s)\big\|_{L^\infty} + \big\|R_h y^h(\cdot,t) - R_h y^h(\cdot,t_n)\big\|_{L^\infty} + \big\|y^{h,n} - Y^n\big\|_{L^\infty} \\
\leq& Ch^{2-\frac{d}{2}} \max_{0\leq s\leq T} \big\|y^h(\cdot,s)\big\|_{H^2} + C\rho(d,h)\big\|R_h y^h(\cdot,t) - R_h y^h(\cdot,t_n)\big\|_{H^1} \\
&\quad + C\rho(d,h)(h + \sqrt{\tau})\Big(\big\|y_0\big\|_{H^2} + \big\|u_h\big\|_U\Big) \\
\leq& C\rho(d,h)(h + \sqrt{\tau})\big(\big\|y_0\big\|_{H^2} + \big\|u_h\big\|_U\big) + C\rho(d,h)\sqrt{\tau_n}\Big(\int_{t_{n-1}}^{t_n} \big\|R_h y_t^h\big\|_{H^1}^2 dt\Big)^{\frac{1}{2}} \\
\leq& C\rho(d,h)(h + \sqrt{\tau})\big(\big\|y_0\big\|_{H^2} + \big\|u_h\big\|_U\big) \\
\leq& C\rho(d,h)(h + \sqrt{\tau}),
\end{aligned} \tag{4.3}
$$

in view of (2.8) and Lemma 3.4. By continuity this estimate also holds at the points $t = t_n, n = 0, \cdots, N$. An elementary calculation shows that

$$
|I_3| \leq C\tau \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \Big(\|y_t\| + \|y_t^h\| + \|\bar{y}_t\|\Big)\Big(\|y\| + \|y^h\| + \|\bar{y}\|\Big) dt \leq C\tau. \tag{4.4}
$$

Inserting the estimates (4.3) and (4.4) into our formula for $I$ we have

$$
I \leq \sum_{n=1}^N \tau_n (y^n - \bar{y}^n, y^{h,n} - y^n) + C\rho(d,h)(h + \sqrt{\tau}) + C\tau. \tag{4.5}
$$

Next, let us introduce $\tilde{Y} = \mathcal{G}_{h,\tau}(Bu)$. Then,(3.16)$-$(3.18) imply that

$$
\begin{aligned}
II =& \sum_{n=1}^N \sum_{i=1}^m (P^n, f_i) \int_{t_{n-1}}^{t_n} (u_i - u_{h,i})(t) dt \\
=& \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (B(u - u_h), P^n) dt = A(\tilde{Y} - Y, P)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} \tau_n (Y^n - \bar{y}^n, \tilde{Y}^n - Y^n) + \sum_{n=1}^{N}\sum_{j=1}^{J} \Big(\tilde{Y}^n(x_j) - Y^n(x_j)\Big)\mu_j^n \\
&\le \sum_{n=1}^{N} \tau_n (Y^n - \bar{y}^n, \tilde{Y}^n - Y^n) + \max_{1\le n\le N, 1\le j\le J} |(\tilde{Y}^n)^-(x_j)| \sum_{n=1}^{N}\sum_{j=1}^{J} |\mu_j^n|.
\end{aligned}
$$

Recalling that $y \ge 0$ in $\overline{\Omega_T}$ we have for $1 \le j \le J, 1 \le n \le N$

$$
\begin{aligned}
|(\tilde{Y}^n)^-(x_j)| =& |(\tilde{Y}^n)^-(x_j) - y^-(x_j, t_n)| \le |\tilde{Y}^n(x_j) - y(x_j, t_n)| \\
\le& \big\|\tilde{Y}^n - y(\cdot, t_n)\big\|_{L^\infty} \le C\rho(d,h)(h + \sqrt{\tau})\Big(\big\|y_0\big\|_{H^2} + \big\|u\big\|_U\Big) \\
\le& C\rho(d,h)(h + \sqrt{\tau}),
\end{aligned}
$$

again by Theorem 3.1. As a result,

$$
II \le \sum_{n=1}^{N} \tau_n (Y^n - \bar{y}^n, \tilde{Y}^n - Y^n) + C\rho(d,h)(h + \sqrt{\tau}). \tag{4.6}
$$

Inserting (4.5) and (4.6) into (4.1) we have

$$
\begin{aligned}
&\alpha \int_0^T |u(t) - u_h(t)|^2 dt \\
\le& \sum_{n=1}^{N} \tau_n (y^n - \bar{y}^n, y^{h,n} - y^n) + \sum_{n=1}^{N} \tau_n (Y^n - \bar{y}^n, \tilde{Y}^n - Y^n) + C\rho(d,h)(h + \sqrt{\tau}) + C\tau \\
=& \sum_{n=1}^{N} \tau_n \int_\Omega \Big\{ -(y^n - Y^n)^2 + (Y^n - \bar{y}^n)(\tilde{Y}^n - y^n) + (y^n - \bar{y}^n)(y^{h,n} - Y^n)\Big\} \\
&\quad + C\rho(d,h)(h + \sqrt{\tau}) + C\tau \\
\le& -\sum_{n=1}^{N} \tau_n \big\|y^n - Y^n\big\|^2 + C\rho(d,h)(h + \sqrt{\tau}) + C\tau,
\end{aligned}
$$

where we once more used Theorem 3.1. The proof is complete. □

**Remark 4.2.** The order of convergence obtained in Theorem 4.1 is essentially determined by the error bound in Theorem 3.1, which in turn is limited by our regularity assumptions on the control $u$. A situation in which one can expect better error bounds occurs when the solution of the state equation enjoys better regularity properties, for example in the case where additional control constraints of the form

$$
a \le u_i(t) \le b, \qquad \text{a.e. in } (0,T), i = 1, \cdots, m,
$$

are prescribed. Here, $a < b$ are given constants. Then, $Bu \in L^\infty(\Omega_T)$ so that parabolic regularity theory implies that $y \in W_q^{2,1}(\Omega_T)$ for all $q < \infty$. Recalling Remark 3.2 it seems possible to us that an error bound of the form

$$
\sum_{n=1}^{N} \tau_n \big\|y(\cdot, t_n) - Y^n\big\|^2 + \int_0^T |u(t) - u_h(t)|^2 dt \le C_\epsilon \Big(h^{2-\epsilon} + \tau^{1-\epsilon}\Big), \qquad (\epsilon > 0),
$$

can be proved in this situation.

## 5. Numerical Experiments

In order to construct a test example we consider the numerical approximation of problem (2.13) for $\Omega := B_1(0) \subset \mathbb{R}^2$, $T = 1$ and $\alpha = 0.1$. We set $y_0 \equiv 0$ and allow an additional right hand side in the state equation, i.e. $y = \mathcal{G}(Bu + f)$, where we choose $(Bu)(x,t) = u(t)$, so that $m = 1$ and $f_1 \equiv 1$ in (2.7). Furthermore we replace the state constraint $y \geq 0$ in $\overline{\Omega_T}$ by the constraint $y \leq y_b$ in $\overline{\Omega_T}$. Note that (2.17) then has to be replaced by the condition

$$\mu \geq 0, \quad y(x,t) \leq y_b(x,t),\ (x,t) \in \overline{\Omega_T}, \quad \int_{\overline{\Omega_T}} (y - y_b) d\mu = 0. \tag{5.1}$$

Let us introduce

$$y(x,t) = \cos\left(\pi|x|\right)\sin(\pi t) \quad \text{and} \quad u(t) = \frac{2}{\alpha} t(t-1)$$

as well as $y_b(x,t) = \max(0.5, y(x,t))$. Clearly, $y \leq y_b$ in $\overline{\Omega_T}$, $0 = y(x,0) < y_b(x,0), x \in \bar{\Omega}$ and $y = \mathcal{G}(Bu + f)$ provided that

$$f(x,t) = \pi \cos\left(\pi|x|\right)\cos(\pi t) + \left(\pi^2 \cos\left(\pi|x|\right) + \pi \sin\left(\pi|x|\right)\frac{1}{|x|}\right)\sin(\pi t) - u(t).$$

Next, setting

$$\begin{aligned} &p(x,t) = \sin\left(\frac{\pi}{2}|x|^2\right) t(1-t), \\ &d\mu = \mu(x,t) dx dt, \quad \mu(x,t) = \max(y(x,t) - 0.5, 0), \quad \mu_{\Gamma_T} = \mu_T = 0, \end{aligned}$$

we see that (2.16) and (5.1) hold. Finally, (2.15) is satisfied if we choose

$$\bar{y}(x,t) = y(x,t) + \mu(x,t) + (1-2t)\sin\left(\frac{\pi}{2}|x|^2\right) + t(1-t)\left(2\pi\cos\left(\frac{\pi}{2}|x|^2\right) - \pi^2|x|^2\sin\left(\frac{\pi}{2}|x|^2\right)\right).$$

In view of Theorem 2.2, $u$ is the exact solution of problem $(TP)$. For the numerical solution of problem (3.15) we use the penalization method, see e.g. [12]. In this approach the state constraints $Y^n(x_j) \leq y_b(x_j, t_n)$, $(1 \leq j \leq J, 1 \leq n \leq N)$ are relaxed by adding a penalization term to the cost functional. The relaxed optimization problem then reads

$$(TP)_h^\gamma \quad \begin{cases} \min_{u \in U} J_{h,\tau}^\gamma(u) := \dfrac{1}{2}\displaystyle\sum_{n=1}^N \tau_n \left\|Y^n - \bar{y}^n\right\|^2 + \dfrac{\alpha}{2}\int_0^T |u(t)|^2 dt \\ \qquad\qquad\qquad + \dfrac{\gamma}{2}\displaystyle\sum_{n=1}^N \tau_n \left\|\max(0, Y^n - I_h(y_b^n)\right\|^2 \\ \text{s.t. } Y = \mathcal{G}_{h,\tau}(Bu + f). \end{cases}$$

Here, $I_h : C^0(\bar{\Omega}) \to X_h$ denotes the spatial Lagrange interpolation operator.

Table 5.1 summarizes our numerical findings for the errors

$$\eta_u = \left(\int_0^T |u(t) - u_h(t)|^2 dt\right)^{\frac{1}{2}} \quad \text{and} \quad \eta_y = \left(\sum_{n=1}^N \tau_n \left\|y(\cdot, t_n) - Y^n\right\|^2\right)^{\frac{1}{2}}.$$

Here we used the coupling $\tau = 0.5h^2$. In both cases we observe an experimental order of convergence of approximately 1. This clearly exceeds the values predicted by Theorem 4.1 and might be explained with the help of Remark 4.2.

Table 5.1: Experimental order of convergence.

| $h$ | $\eta_u$ | eoc | $\eta_y$ | eoc |
|---|---|---|---|---|
| 0,5710 | 0,7299 | 0 | 0,3785 | 0 |
| 0,3956 | 0,3738 | 0,965 | 0,1986 | 1,756 |
| 0,3022 | 0,2192 | 0,953 | 0,1295 | 1,587 |
| 0,2050 | 0,0986 | 0,991 | 0,0773 | 1,328 |
| 0,1550 | 0,0561 | 0,990 | 0,0552 | 1,203 |
| 0,1042 | 0,0257 | 0,986 | 0,0353 | 1,128 |
| 0,0785 | 0,0147 | 0,975 | 0,0259 | 1,080 |
| 0,0525 | 0,0067 | 0,965 | 0,0170 | 1,050 |

In Fig. 5.1 we investigate the decrease of the errors

$$\eta_u^\gamma = \Big( \int_0^T |u(t) - u_h^\gamma(t)|^2 dt \Big)^{\frac{1}{2}} \quad \text{and} \quad \eta_y^\gamma = \Big( \sum_{n=1}^N \tau_n \big\| y(\cdot, t_n) - Y^{\gamma,n} \big\|^2 \Big)^{\frac{1}{2}}$$

in dependence of $\gamma$, where $u_h^\gamma$ denotes the solution of $(TP)_h^\gamma$ with corresponding discrete state $Y^\gamma$. One clearly sees that increasing the value of $\gamma$ over a certain $h-$dependent threshold yields no further error reduction. Furthermore, our numerical experiments suggest the parameter coupling $\frac{1}{\sqrt{\gamma}} \sim h$. A corresponding analysis in the case of elliptic optimal control problems with state constraints can be found in [13].

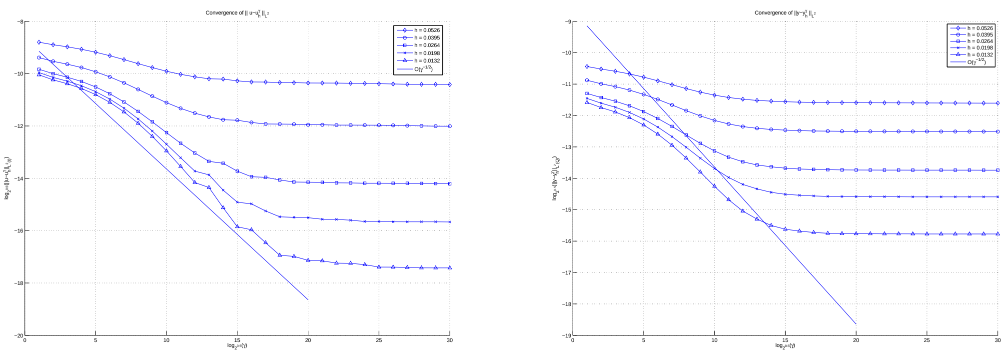

Fig. 5.1. Error behaviour for $\eta_u^\gamma$ (left), and $\eta_y^\gamma$ (right).

**Acknowledgments.** The authors thank Christian Kahle for his support in providing the numerical experiment. Furthermore, they gratefully acknowledge the support of the DFG Priority Program 1253 entitled Optimization With Partial Differential Equations.

## References

[1] J.F. Bonnans, P. Jaisson, Optimal control of a parabolic equation with time-dependent state constraints, Rapport de Recherche INRIA RR 6784 (2008).

[2] E. Casas, Control of an elliptic problem with pointwise state constraints, *SIAM J. Control. Optim.*, **24** (1986), 1309–1322.

[3] E. Casas, Boundary control of semilinear elliptic equations with pointwise state constraints, *SIAM J. Control. Optim.*, **31** (1993), 993–1006.

[4] E. Casas, Error Estimates for the Numerical Approximation of Semilinear Elliptic Control Problems with Finitely Many State Constraints, *ESAIM, Contr. Opt. Ca.*, **8** (2002), 345–374.

[5] D. Clever, J. Lang, Optimal Control of Radiative Heat Transfer in Glass Cooling with Restrictions on the Temperature Gradient, Preprint SPP1253-20-01 (2008).

[6] K. Deckelnick, M. Hinze, Convergence of a finite element approximation to a state constrained elliptic control problem, *SIAM J. Numer. Anal.*, **45** (2007), 1937–1953.

[7] K. Deckelnick, A. Günther, M. Hinze, Finite element approximation of elliptic control problems with constraints on the gradient, *Numer. Math.*, **111** (2009), 335–350.

[8] J.C. de los Reyes, P. Merino, J. Rehberg, F. Tröltzsch, Optimality conditions for state constrained pde control problems with time-dependent controls, Preprint, TU Berlin (2008).

[9] K. Eriksson, C. Johnson, Adaptive finite element methods for parabolic problems II: Optimal error estimates in $L_\infty L_2$ and $L_\infty L_\infty$, *SIAM J. Numer. Anal.*, **32** (1995), 706–740.

[10] M. Gunzburger, E. Ozugurlu, J. Turner, H. Zhang, Controlling transport phenomena in the Czochralski crystal growth process, *J. Cryst. Growth*, **234** (2002).

[11] M. Heinkenschloss, R.W.H. Hoppe, V. Schulz, Numerical Techniques for Optimization Problems with PDE constraints, Report No. 4/2009, Mathematisches Forschungsinstitut Oberwolfach (2009).

[12] M. Hintermüller, K., K. Kunisch, Path following methods for a class of constrained minimization methods in function spaces, *SIAM J. Optimiz.*, 17 (2006), 159-187.

[13] M. Hintermüller, M. Hinze, A note on an optimal parameter adjustment in a Moreau-Yosida-based approach to state constrained elliptic control problems, *SIAM J. Numer. Anal.*, 47 (2009), 1666-1683.

[14] M. Hinze, A variational discretization concept in control constrained optimization: the linear-quadratic case, *J. Comput. Optim. Appl.*, **30** (2005), 45–63.

[15] M. Hinze, R. Pinnau, M. Ulbrich, S. Ulbrich, Optimization with pde constraints, Mathematical Modelling: Theory and Applications, Volume **23**, Springer, 2009.

[16] M. Hinze, S. Ziegenbalg, Optimal control of the free boundary in a two-phase Stefan problem, *J. Comput. Phys.*, **223** (2007), 657–684.

[17] M. Hinze, S. Ziegenbalg, Solidification of a GaAs melt - optimal control of the phase interface, J. Crystal Growth DOI:10.1016/j.jcrysgro.2009.02.031 (2009).

[18] W. Liu, W. Gong, N. Yan, A new finite element approximation of a state constrained optimal control problem, *J. Comput. Math.*, 27 (2009), 97-114.

[19] C. Meyer, Error estimates for the finite element approximation of an elliptic control problem with pointwise constraints on the state and the control, *Control Cybern.*, **37** (2008), 51-85.

[20] I. Neitzel, F. Tröltzsch, On regularization methods for the numerical solution of parabolic control problems with pointwise state constraints, ESAIM: Control, Optimisation and Calculus of Variations, DOI 10.1051/cocv:2008038 (2008).

[21] R.H. Nochetto, C. Verdi, Convergence past singularities for a fully discrete approximation of curvature-driven interfaces, *SIAM J. Numer. Anal.*, **34** (1997), 490–512.

[22] R. Pinnau, Analysis of Optimal Boundary Control for Radiative Heat Transfer Modelled by the $SP_1$–System, *CMS*, **5** (2007), 951–969.

[23] L. Yuan, D. Yang, A Posteriori Error Estimate of Optimal Control Problem of PDEs with Integral Constraint for State, *J. Comput. Math.*, 27 (2009), 525-542.