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ON HERMITIAN AND SKEW-HERMITIAN SPLITTING ITERATION METHODS FOR CONTINUOUS SYLVESTER EQUATIONS*

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Abstract

We present a *Hermitian and skew-Hermitian splitting* (**HSS**) iteration method for solving large sparse continuous Sylvester equations with non-Hermitian and positive definite/semi-definite matrices. The unconditional convergence of the HSS iteration method is proved and an upper bound on the convergence rate is derived. Moreover, to reduce the computing cost, we establish an inexact variant of the HSS iteration method and analyze its convergence property in detail. Numerical results show that the HSS iteration method and its inexact variant are efficient and robust solvers for this class of continuous Sylvester equations.

Mathematics subject classification: 15A24, 15A30, 15A69, 65F10, 65F30, 65F50, 65H10
Key words: Continuous Sylvester equation, HSS iteration method, Inexact iteration, Convergence.

1. Introduction

We consider iterative solutions of the continuous Sylvester equations of the form

$$AX + XB = F, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $F \in \mathbb{C}^{m\times n}$ are given complex matrices. Assume that

(A_1) A, B and F are large and sparse matrices;

(A_2) at least one of A and B is non-Hermitian; and

(A_3) both A and B are positive semi-definite, and at least one of them is positive definite.

Then from [14,29,31] we know that the continuous Sylvester equation (1.1) has a unique solution, as under the assumptions (A_1)-(A_3) there is no common eigenvalue between A and $-B$. Note that the continuous Lyapunov equation is a special case of the continuous Sylvester equation with $B = A^*$ and F Hermitian. Here and in the sequel, W^* is used to denote the conjugate transpose of the matrix $W \in \mathbb{C}^{m\times m}$, and we call W a positive definite or positive semi-definite matrix if so is its Hermitian part $\mathcal{H}(W) := \frac{1}{2}(W + W^*)$; note that a positive definite or positive semi-definite matrix is not necessarily Hermitian. We will also use $\mathcal{S}(W) := \frac{1}{2}(W - W^*)$ to denote the skew-Hermitian part of the matrix W. Obviously, it holds that $W = \mathcal{H}(W) + \mathcal{S}(W)$; see [2–6].

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The continuous Sylvester equation (1.1) has numerous applications in control and system theory [30,36,39], stability of linear systems [22], analysis of bilinear systems [32], power systems [25], linear algebra [16], signal processing [1], image restoration [11], filtering [21, 23], model order reduction [35], numerical methods for differential equations [8, 9], iterative methods for algebraic Riccati equations [7,18–20,30], matrix nearness problem [34, 41], finite element model updating [13,26], block-diagonalization of matrices [16,31] and so on. Many of these applications lead to stable Sylvester equations, i.e., Assumption (A_3) made in the above is satisfied.

The continuous Sylvester equation (1.1) is mathematically equivalent to the system of linear equations

$$\mathbf{A}x = f, \tag{1.2}$$

where $\mathbf{A} = I \otimes A + B^T \otimes I$, and the vectors x and f contain the concatenated columns of the matrices X and F, respectively, with $\otimes$ being the Kronecker product symbol and B^T representing the transpose of the matrix B. Of course, this is a numerically poor way to determine the solution X of the continuous Sylvester equation (1.1), as the system of linear equations (1.2) is costly to solve and can be ill-conditioned.

Standard methods for numerical solution of the continuous Sylvester equation (1.1) are the Bartels-Stewart and the Hessenberg-Schur methods [10, 15], which consist in transforming A and B into triangular or Hessenberg form by an orthogonal similarity transformation and then solving the resulting system of linear equations directly by a back-substitution process. These methods are classified as direct methods and are used, among others, by LAPACK and Matlab.

When the matrices A and B are large and sparse, iterative methods such as the Smith's method [37], the *alternating direction implicit* (**ADI**) method [11,24,33,40], the block *successive overrelaxation* (**BSOR**) method [38], the preconditioned conjugate gradient method [12], the matrix sign function method [27], and the matrix splitting methods [17] are often the methods of choice for efficiently and accurately solving the continuous Sylvester equation (1.1).

The Bartels-Stewart and the Hessenberg-Schur methods are applicable and effective for general continuous Sylvester equations of reasonably small sizes. For large and sparse continuous Sylvester equations, the afore-mentioned iterative methods are often superior to these direct methods, provided the matrices A and B are either Hermitian positive definite matrices or M-matrices. However, when the matrix A or B is not Hermitian, the convergence of these iterative methods may be theoretically not guaranteed, even if both matrices A and B are either asymptotically stable or N-stable (i.e., positive definite); this will be the case if the skew-Hermitian part of A or B is dominantly strong.

In this paper, we present an iterative method for solving the continuous Sylvester equation (1.1) by making use of the *Hermitian and skew-Hermitian* (**HS**) splittings of the matrices A and B. This *Hermitian and skew-Hermitian splitting* (**HSS**) iteration method is a matrix variant of the HSS iteration method firstly proposed in [6] for solving systems of linear equations, which are in spirit analogous to the ADI iteration methods [11, 24, 33, 40]. Via the HSS iteration method, the problem of solving a general continuous Sylvester equation is decomposed into a sequence of sub-problems about two coupled continuous Sylvester equations with respect to shifted Hermitian positive definite matrices and shifted skew-Hermitian matrices, respectively. When the matrices A and B are positive semi-definite, and at least one of them is positive definite, we prove that the HSS iteration converges unconditionally to the exact solution of the continuous Sylvester equation (1.1), with a bound on the convergence rate about the same as that of the conjugate gradient method when applied to a continuous Sylvester

equation with respect to the Hermitian matrices $\mathcal{H}(A)$ and $\mathcal{H}(B)$. Moreover, iteration parameters that minimize an upper bound of the contraction factor is obtained in terms of bounds of the largest and the smallest eigenvalues of the matrices $\mathcal{H}(A)$ and $\mathcal{H}(B)$. To further reduce the computing cost, we establish an *inexact Hermitian and skew-Hermitian splitting* (**IHSS**) iteration method, which uses certain efficient iterative methods to approximately solve the two specially structured continuous Sylvester equations involved in each step of the HSS iteration. The convergence of the IHSS iteration method is also analyzed in detail. Numerical experiments show that both HSS and IHSS iteration methods are efficient and robust solvers for the large sparse continuous Sylvester equations with non-Hermitian and positive definite matrices, and the latter is usually superior to the former in computing efficiency.

In the remainder of this paper, a matrix sequence $\{Y^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ is said to be convergent to a matrix $Y \in \mathbb{C}^{m\times n}$ if the corresponding vector sequence $\{y^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{mn}$ is convergent to the corresponding vector $y \in \mathbb{C}^{mn}$, where the vectors $y^{(k)}$ and y contain the concatenated columns of the matrices $Y^{(k)}$ and Y, respectively. If $\{Y^{(k)}\}_{k=0}^{\infty}$ is convergent, then its convergence factor and convergence rate are defined as those of $\{y^{(k)}\}_{k=0}^{\infty}$, correspondingly. In addition, we use $\lambda(W)$, $\|W\|_2$ and $\|W\|_F$ to denote the spectrum, the spectral norm, and the Frobenius norm of the matrix $W \in \mathbb{C}^{m\times m}$, respectively. Note that $\|\cdot\|_2$ is also used to represent the 2-norm of a vector.

2. The HSS Iteration Method

Evidently, the matrices A and B naturally admit the Hermitian and skew-Hermitian splittings

$$A = \mathcal{H}(A) + \mathcal{S}(A) \quad \text{and} \quad B = \mathcal{H}(B) + \mathcal{S}(B);$$

see [2, 3, 6]. Let α and β be given positive constants and I the identity matrix of suitable dimension. Then we have

$$\begin{aligned} A =& \big(\alpha I + \mathcal{H}(A)\big) + \big(\mathcal{S}(A) - \alpha I\big) \\ =& \big(\alpha I + \mathcal{S}(A)\big) + \big(\mathcal{H}(A) - \alpha I\big), \end{aligned}$$

and

$$\begin{aligned} B =& \big(\beta I + \mathcal{H}(B)\big) + \big(\mathcal{S}(B) - \beta I\big) \\ =& \big(\beta I + \mathcal{S}(B)\big) + \big(\mathcal{H}(B) - \beta I\big). \end{aligned}$$

It follows that the continuous Sylvester equation (1.1) can be equivalently written as the fixed-point matrix equations

$$\begin{cases} \big(\alpha I + \mathcal{H}(A)\big)X + X\big(\beta I + \mathcal{H}(B)\big) = \big(\alpha I - \mathcal{S}(A)\big)X + X\big(\beta I - \mathcal{S}(B)\big) + F, \\ \big(\alpha I + \mathcal{S}(A)\big)X + X\big(\beta I + \mathcal{S}(B)\big) = \big(\alpha I - \mathcal{H}(A)\big)X + X\big(\beta I - \mathcal{H}(B)\big) + F. \end{cases}$$

Under the assumptions (A_1)-(A_3), we easily know that there is no common eigenvalue between the matrices $\alpha I + \mathcal{H}(A)$ and $-(\beta I + \mathcal{H}(B))$, as well as between the matrices $\alpha I + \mathcal{S}(A)$ and $-(\beta I + \mathcal{S}(B))$, so that these two fixed-point matrix equations have unique solutions for all given right-hand side matrices.

Now, based on the above observations, we can establish the following Hermitian and skew-Hermitian splitting iteration method for solving the continuous Sylvester equation (1.1).

The HSS Iteration Method. Given an initial guess $X^{(0)} \in \mathbb{C}^{m\times n}$, compute $X^{(k+1)} \in \mathbb{C}^{m\times n}$ for $k = 0, 1, 2, \cdots$ using the following iteration scheme until $\{X^{(k)}\}_{k=0}^{\infty}$ satisfies the stopping criterion:

$$\begin{cases} \big(\alpha I + \mathcal{H}(A)\big)X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}\big(\beta I + \mathcal{H}(B)\big) \\ \qquad = \big(\alpha I - \mathcal{S}(A)\big)X^{(k)} + X^{(k)}\big(\beta I - \mathcal{S}(B)\big) + F, \\ \big(\alpha I + \mathcal{S}(A)\big)X^{(k+1)} + X^{(k+1)}\big(\beta I + \mathcal{S}(B)\big) \\ \qquad = \big(\alpha I - \mathcal{H}(A)\big)X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}\big(\beta I - \mathcal{H}(B)\big) + F, \end{cases}$$

where α and β are given positive constants.

Obviously, when either A or B is a zero matrix, and $X^{(\#)}$ and F reduce to column vectors, the HSS iteration method becomes the one for systems of linear equations; see [4,6]. In addition, when $B = A^*$ and F is Hermitian, it leads to an HSS iteration method for the continuous Lyapunov equations.

Because $\mathcal{H}(A)$, $\mathcal{H}(B)$ are Hermitian and $\mathcal{S}(A)$, $\mathcal{S}(B)$ are skew-Hermitian, the matrices $\alpha I + \mathcal{H}(A)$, $\alpha I + \mathcal{S}(A)$, $\alpha I + \mathcal{H}(B)$ and $\alpha I + \mathcal{S}(B)$ can be diagonalized by unitary matrices. This implies that it is possible to treat the two half-steps at each step of the HSS iteration by efficient direct algorithms.

By making use of Theorem 2.2 and Corollary 2.3 in [6], we can demonstrate the following convergence theorem about the HSS iteration method for solving the continuous Sylvester equation (1.1).

Theorem 2.1. *Assume that $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ are positive semi-definite matrices, and at least one of them is positive definite. Let $A = \mathcal{H}(A) + \mathcal{S}(A)$ and $B = \mathcal{H}(B) + \mathcal{S}(B)$ be the Hermitian and the skew-Hermitian parts of the matrices A and B, respectively, and α and β be positive constants. Denote by $\mathbf{A} = \mathbf{H} + \mathbf{S}$, with*

$$\mathbf{H} = I \otimes \mathcal{H}(A) + \mathcal{H}(B)^T \otimes I \quad \text{and} \quad \mathbf{S} = I \otimes \mathcal{S}(A) + \mathcal{S}(B)^T \otimes I, \tag{2.1}$$

and represent by

$$\mathbf{T}(\gamma) = (\gamma I + \mathbf{S})^{-1}(\gamma I - \mathbf{H})(\gamma I + \mathbf{H})^{-1}(\gamma I - \mathbf{S}), \tag{2.2}$$

and

$$\gamma = \alpha + \beta. \tag{2.3}$$

Then the HSS iteration method is convergent to the exact solution $X_\star \in \mathbb{C}^{m\times n}$ of the continuous Sylvester equation (1.1)*, and the convergence factor is given by the spectral radius $\rho(\mathbf{T}(\gamma))$ of the matrix $\mathbf{T}(\gamma)$, which is bounded as*

$$\rho\big(\mathbf{T}(\gamma)\big) \le \sigma(\gamma) := \max_{\lambda_i \in \lambda(\mathcal{H}(A))} \max_{\mu_j \in \lambda(\mathcal{H}(B))} \frac{\big|\gamma - (\lambda_i + \mu_j)\big|}{\big|\gamma + (\lambda_i + \mu_j)\big|}.$$

Moreover, if $\lambda_{\min}^{(\mathcal{H}(A))}$, $\lambda_{\min}^{(\mathcal{H}(B))}$ and $\lambda_{\max}^{(\mathcal{H}(A))}$, $\lambda_{\max}^{(\mathcal{H}(B))}$ are the lower and the upper bounds of the eigenvalues of the matrices $\mathcal{H}(A)$ and $\mathcal{H}(B)$, respectively, then

$$\tilde{\gamma} \equiv \arg\min_{\gamma} \left\{ \max_{\lambda_{\min} \le \lambda \le \lambda_{\max}} \left| \frac{\gamma - \lambda}{\gamma + \lambda} \right| \right\} = \sqrt{\lambda_{\min}\lambda_{\max}},$$

and

$$\sigma(\tilde{\gamma}) = \frac{\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}} = \frac{\sqrt{\kappa(\mathbf{H})} - 1}{\sqrt{\kappa(\mathbf{H})} + 1},$$

where

$$\lambda_{\min} = \lambda_{\min}^{(\mathcal{H}(A))} + \lambda_{\min}^{(\mathcal{H}(B))}, \qquad \lambda_{\max} = \lambda_{\max}^{(\mathcal{H}(A))} + \lambda_{\max}^{(\mathcal{H}(B))},$$

and $\kappa(\mathbf{H}) = \lambda_{\max}/\lambda_{\min}$ *is the spectral condition number of* $\mathbf{H}$.

Proof. By making use of the Kronecker product, we can rewrite the above-described HSS iteration in the following matrix-vector form:

$$\begin{cases} (I \otimes (\alpha I + \mathcal{H}(A)) + (\beta I + \mathcal{H}(B))^T \otimes I)x^{(k+\frac{1}{2})} \\ \qquad = (I \otimes (\alpha I - \mathcal{S}(A)) + (\beta I - \mathcal{S}(B))^T \otimes I)x^{(k)} + f, \\ (I \otimes (\alpha I + \mathcal{S}(A)) + (\beta I + \mathcal{S}(B))^T \otimes I)x^{(k+1)} \\ \qquad = (I \otimes (\alpha I - \mathcal{H}(A)) + (\beta I - \mathcal{H}(B))^T \otimes I)x^{(k+\frac{1}{2})} + f, \end{cases}$$

which can be arranged equivalently as

$$\begin{cases} (\gamma I + \mathbf{H})x^{(k+\frac{1}{2})} = (\gamma I - \mathbf{S})x^{(k)} + f, \\ (\gamma I + \mathbf{S})x^{(k+1)} = (\gamma I - \mathbf{H})x^{(k+\frac{1}{2})} + f. \end{cases} \tag{2.4}$$

Evidently, the iteration scheme (2.4) is the HSS iteration method for solving the system of linear equations (1.2), with $\mathbf{A} = \mathbf{H} + \mathbf{S}$; see [4, 6]. After concrete operations, the HSS iteration (2.4) can be neatly expressed as a stationary fixed-point iteration as follows:

$$x^{(k+1)} = \mathbf{T}(\gamma)x^{(k)} + \mathbf{G}(\gamma)f,$$

where $\mathbf{T}(\gamma)$ is the iteration matrix defined in (2.2), with $\mathbf{H}$, $\mathbf{S}$ and γ being given in (2.1) and (2.3), respectively, and

$$\mathbf{G}(\gamma) = 2\gamma(\gamma I + \mathbf{S})^{-1}(\gamma I + \mathbf{H})^{-1}.$$

We can easily verify that $\mathbf{H}$ is a Hermitian matrix, $\mathbf{S}$ is a skew-Hermitian matrix, and γ is a positive constant. Moreover, when either $A \in \mathbb{C}^{m\times m}$ or $B \in \mathbb{C}^{n\times n}$ is positive definite, the matrix $\mathbf{H}$ is Hermitian positive definite. Hence, by making use of Theorem 2.2 in [6] we know that the HSS iteration method (2.4) converges unconditionally to the exact solution $x_\star \in \mathbb{C}^{mn}$ of the system of linear equations $\mathbf{A}x = f$, with the convergence factor being $\rho(\mathbf{T}(\gamma))$, and it holds that

$$\rho(\mathbf{T}(\gamma)) \le \sigma(\gamma) < 1, \quad \forall \gamma > 0.$$

This straightforwardly shows that the HSS iteration method for the continuous Sylvester equation (1.1) also converges unconditionally to the exact solution $X_\star \in \mathbb{C}^{m\times n}$, with the convergence factor $\rho(\mathbf{T}(\gamma))$ being bounded by $\sigma(\gamma)$.

In addition, by making use of Corollary 2.3 in [6], we know that the iteration parameter $\tilde{\gamma}$ that minimizes the upper bound $\sigma(\gamma)$ is given by

$$\tilde{\gamma} = \sqrt{\lambda_{\min}\lambda_{\max}},$$

and the corresponding upper bound of the convergence factor is given by

$$\sigma(\tilde{\gamma}) = \frac{\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}} = \frac{\sqrt{\kappa(\mathbf{H})} - 1}{\sqrt{\kappa(\mathbf{H})} + 1}.$$

This completes the proof. □

Theorem 2.1 shows that the HSS iteration converges unconditionally to the exact solution of the continuous Sylvester equation (1.1), with the upper bound on the rate of convergence being bounded by $\sigma(\gamma)$, which depends only on the spectrums of the Hermitian parts $\mathcal{H}(A)$ and $\mathcal{H}(B)$, but does not depend on the the spectrums of the skew-Hermitian parts $\mathcal{S}(A)$ and $\mathcal{S}(B)$, on the spectrums of the matrices A and B, or on the eigenvectors of the matrices $\mathcal{H}(A)$, $\mathcal{H}(B)$, $\mathcal{S}(A)$, $\mathcal{S}(B)$, A and B.

Besides, when the parameter $\tilde{\gamma}$ is employed, the upper bound of the convergence rate of the HSS iteration is about the same as that of the conjugate gradient method when applied to the continuous Sylvester equation with the Hermitian matrices $\mathcal{H}(A)$ and $\mathcal{H}(B)$, and it does become the same when, in particular, the matrices A and B are Hermitian. It should be mentioned that, when the matrices A and B are normal, we have

$$\mathcal{H}(A)\mathcal{S}(A) = \mathcal{S}(A)\mathcal{H}(A) \quad \text{and} \quad \mathcal{H}(B)\mathcal{S}(B) = \mathcal{S}(B)\mathcal{H}(B),$$

and therefore, $\rho(\mathbf{T}(\gamma)) = \sigma(\gamma)$. The parameter $\tilde{\gamma}$ then minimizes both $\rho(\mathbf{T}(\gamma))$ and $\sigma(\gamma)$.

The actual iteration parameters α and β can be chosen as $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$ such that $\tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}$. For example, we may take $\tilde{\alpha} = \tilde{\beta} = \frac{1}{2}\tilde{\gamma}$. Because

$$\sigma(\gamma) \leq \max\left\{ \max_{\lambda_i \in \lambda(\mathcal{H}(A))} \frac{|\alpha - \lambda_i|}{|\alpha + \lambda_i|}, \quad \max_{\mu_j \in \lambda(\mathcal{H}(B))} \frac{|\beta - \mu_j|}{|\beta + \mu_j|} \right\},$$

we may also take

$$\tilde{\alpha} := \arg\min_{\alpha} \left\{ \max_{\lambda_{\min}^{(\mathcal{H}(A))} \leq \lambda \leq \lambda_{\max}^{(\mathcal{H}(A))}} \left| \frac{\alpha - \lambda}{\alpha + \lambda} \right| \right\} = \sqrt{\lambda_{\min}^{(\mathcal{H}(A))} \lambda_{\max}^{(\mathcal{H}(A))}},$$

$$\tilde{\beta} := \arg\min_{\beta} \left\{ \max_{\lambda_{\min}^{(\mathcal{H}(B))} \leq \mu \leq \lambda_{\max}^{(\mathcal{H}(B))}} \left| \frac{\beta - \mu}{\beta + \mu} \right| \right\} = \sqrt{\lambda_{\min}^{(\mathcal{H}(B))} \lambda_{\max}^{(\mathcal{H}(B))}}.$$

In this case, it holds that

$$\begin{aligned} \sigma(\tilde{\gamma}) &\leq \max\left\{ \frac{\sqrt{\lambda_{\max}^{(\mathcal{H}(A))}} - \sqrt{\lambda_{\min}^{(\mathcal{H}(A))}}}{\sqrt{\lambda_{\max}^{(\mathcal{H}(A))}} + \sqrt{\lambda_{\min}^{(\mathcal{H}(A))}}}, \quad \frac{\sqrt{\lambda_{\max}^{(\mathcal{H}(B))}} - \sqrt{\lambda_{\min}^{(\mathcal{H}(B))}}}{\sqrt{\lambda_{\max}^{(\mathcal{H}(B))}} + \sqrt{\lambda_{\min}^{(\mathcal{H}(B))}}} \right\} \\ &= \max\left\{ \frac{\sqrt{\kappa(\mathcal{H}(A))} - 1}{\sqrt{\kappa(\mathcal{H}(A))} + 1}, \quad \frac{\sqrt{\kappa(\mathcal{H}(B))} - 1}{\sqrt{\kappa(\mathcal{H}(B))} + 1} \right\}. \end{aligned}$$

We remark again that the HSS iteration method is unconditional convergent to the exact solution of the continuous Sylvester equation (1.1) for non-Hermitian positive definite matrices A and B. This is different from the existing iteration methods which are only guaranteed to be convergent for the case that A and B are M-matrices or Hermitian positive definite matrices.

Of course, we could directly apply the HSS iteration method to the expanded standard linear system (1.2) to obtain an approximate solution to the continuous Sylvester equation (1.1). However, it turns out that this approach is not so appropriate and efficient due to the following reasons:

(a) it increases the dimension of the target linear system from $\max\{m, n\}$ to $m \times n$, i.e., for the continuous Sylvester equation (1.1) we only need to treat with matrices of orders m or n, but for the expanded linear system (1.2) we need to treat with matrix of order $m \times n$; which is a considerable increase, especially when m or n is very large;

(b) the matrix $\mathbf{A}$ may not inherit some useful properties of the matrices A and B, which could be very important and heavily affecting their sparse factorizations, etc.;

(c) a solution matrix X, reconstructed from a solution vector x obtained from solving the expanded linear system (1.2), may loss certain important and useful properties possessed by the original solution matrix $X_\star$ of the continuous Sylvester equation (1.1);

(d) the above-mentioned three reasons may equally occur for the two sub-systems of linear equations involved in the two-half step iterates of the HSS iteration methods for solving the continuous Sylvester equation (1.1) and the expanded linear system (1.2).

Therefore, the HSS iteration method established and discussed here should be an appropriate and efficient approach for iteratively solving the continuous Sylvester equation (1.1).

3. The Inexact HSS Iteration Methods

The HSS iteration method is a two-step iteration scheme for solving the continuous Sylvester equation (1.1). At the first half-step, we need to solve continuous Sylvester equations of the form

$$(\alpha I + \mathcal{H}(A))X + X(\beta I + \mathcal{H}(B)) = F_S, \tag{3.1}$$

and at the second half-step, we need to solve continuous Sylvester equations of the form

$$(\alpha I + \mathcal{S}(A))X + X(\beta I + \mathcal{S}(B)) = F_H, \tag{3.2}$$

where F_H and F_S are prescribed m-by-n complex matrices. This may be costly in actual implementations, especially when the sizes of the matrices involved are very large, though these matrices may have special properties and structures. For example, the matrices in (3.1) are shifted Hermitian positive definite or positive semi-definite matrices and, thus, are Hermitian positive definite; and the matrices in (3.2) are shifted skew-Hermitian matrices and, thus, are positive definite but not Hermitian. To further improve the computational efficiency of the HSS iteration method, we can solve the two sub-problems (3.1) and (3.2) inexactly by utilizing certain effective iteration methods, e.g., the (block) Gauss-Seidel, the (block) SOR, the ADI, the conjugate gradient or the Krylov subspace methods; see [11, 12, 24, 33, 38, 40]. This naturally results in the following inexact HSS iteration method for solving the continuous Sylvester equation (1.1).

Given an initial guess $X^{(0)} \in \mathbb{C}^{m\times n}$, for $k = 0, 1, 2, \cdots$ until $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ satisfies the stopping criterion, solve $X^{(k+\frac{1}{2})} \in \mathbb{C}^{m\times n}$ approximately from

$$\begin{aligned}(\alpha I+\mathcal{H}(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\beta I + \mathcal{H}(B))\\ \approx(\alpha I - \mathcal{S}(A))X^{(k)} + X^{(k)}(\beta I - \mathcal{S}(B)) + F\end{aligned}$$

by employing an inner iteration (e.g., the Gauss-Seidel method) with $X^{(k)}$ as the initial guess; then solve $X^{(k+1)} \in \mathbb{C}^{m\times n}$ approximately from

$$(\alpha I+\mathcal{S}(A))X^{(k+1)} + X^{(k+1)}(\beta I + \mathcal{S}(B))$$
$$\approx(\alpha I - \mathcal{H}(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\beta I - \mathcal{H}(B)) + F$$

by employing an inner iteration (e.g., the Gauss-Seidel method) with $X^{(k+\frac{1}{2})}$ as the initial guess, where α and β are given positive constants.

To simplify numerical implementation and convergence analysis, we may rewrite the above IHSS iteration method as the following equivalent scheme.

The IHSS Iteration Method. Given an initial guess $X^{(0)} \in \mathbb{C}^{m\times n}$, for $k = 0, 1, 2, \cdots$ until $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ converges,
1. approximate the solution of

$$(\alpha I + \mathcal{H}(A))Z^{(k)} + Z^{(k)}(\beta I + \mathcal{H}(B)) = R^{(k)},$$

with $R^{(k)} = F - AX^{(k)} - X^{(k)}B$, by iterating until $Z^{(k)}$ is such that the residual

$$P^{(k)} = R^{(k)} - \Big(\big(\alpha I + \mathcal{H}(A)\big)Z^{(k)} + Z^{(k)}\big(\beta I + \mathcal{H}(B)\big)\Big)$$

satisfies

$$\|P^{(k)}\|_F \le \varepsilon_k \|R^{(k)}\|_F,$$

and then compute $X^{(k+\frac{1}{2})} = X^{(k)} + Z^{(k)}$;
2. approximate the solution of

$$(\alpha I + \mathcal{S}(A))Z^{(k+\frac{1}{2})} + Z^{(k+\frac{1}{2})}(\beta I + \mathcal{S}(B)) = R^{(k+\frac{1}{2})},$$

with $R^{(k+\frac{1}{2})} = F - AX^{(k+\frac{1}{2})} - X^{(k+\frac{1}{2})}B$, by iterating until $Z^{(k+\frac{1}{2})}$ is such that the residual

$$Q^{(k+\frac{1}{2})} = R^{(k+\frac{1}{2})} - \Big(\big(\alpha I + \mathcal{S}(A)\big)Z^{(k+\frac{1}{2})} + Z^{(k+\frac{1}{2})}\big(\beta I + \mathcal{S}(B)\big)\Big)$$

satisfies

$$\big\|Q^{(k+\frac{1}{2})}\big\|_F \le \eta_k \big\|R^{(k+\frac{1}{2})}\big\|_F,$$

and then compute $X^{(k+1)} = X^{(k+\frac{1}{2})} + Z^{(k+\frac{1}{2})}$. Here, $\{\varepsilon_k\}$ and $\{\eta_k\}$ are prescribed tolerances used to control the accuracies of the inner iterations.

By making use of Theorems 3.1 and 3.2 in [6], we can demonstrate the following convergence result about the above IHSS iteration method.

Theorem 3.1. *Let the conditions of Theorem* 2.1 *be satisfied. If* $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ *is an iteration sequence generated by the IHSS iteration method and if* $X_\star \in \mathbb{C}^{m\times n}$ *is the exact solution of the continuous Sylvester equation* (1.1)*, then it holds that*

$$\|X^{(k+1)} - X_\star\|_S \le (\sigma(\gamma) + \theta\varrho\eta_k)(1 + \theta\varepsilon_k)\|X^{(k)} - X_\star\|_S, \quad k = 0, 1, 2, \cdots.$$

where the norm $\|\cdot\|_S$ *is defined as* $\|Y\|_S = \|(\alpha I + \mathcal{S}(A))Y + Y(\beta I + \mathcal{S}(B))\|_F$ *for any matrix* $Y \in \mathbb{C}^{m\times n}$*, and the constants* ϱ *and* θ *are given by*

$$\varrho = \|(\gamma I + \mathbf{S})(\gamma I + \mathbf{H})^{-1}\|_2 \quad \textit{and} \quad \theta = \|\mathbf{A}(\gamma I + \mathbf{S})^{-1}\|_2, \tag{3.3}$$

with the matrices $\mathbf{H}$ *and* $\mathbf{S}$ *being defined in* (2.1) *and the constant* γ *being defined in* (2.3)*. In particular, if*

$$(\sigma(\gamma) + \theta\varrho\eta_{\max})(1 + \theta\varepsilon_{\max}) < 1, \tag{3.4}$$

then the iteration sequence $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ *converges to* $X_\star \in \mathbb{C}^{m\times n}$*, where* $\varepsilon_{\max} = \max_k\{\varepsilon_k\}$ *and* $\eta_{\max} = \max_k\{\eta_k\}$*.*

Proof. By making use of the Kronecker product and the notations introduced in Theorem 2.1, we can rewrite the above-described IHSS iteration in the following matrix-vector form:

$$\begin{cases} (\gamma I + \mathbf{H})z^{(k)} = r^{(k)}, & x^{(k+\frac{1}{2})} = x^{(k)} + z^{(k)}, \\ (\gamma I + \mathbf{S})z^{(k+\frac{1}{2})} = r^{(k+\frac{1}{2})}, & x^{(k+1)} = x^{(k+\frac{1}{2})} + z^{(k+\frac{1}{2})}, \end{cases} \tag{3.5}$$

with $r^{(k)} = f - \mathbf{A}x^{(k)}$ and $r^{(k+\frac{1}{2})} = f - \mathbf{A}x^{(k+\frac{1}{2})}$, where $z^{(k)}$ is such that the residual

$$p^{(k)} = r^{(k)} - (\gamma I + \mathbf{H})z^{(k)}$$

satisfies $\|p^{(k)}\|_2 \le \varepsilon_k\|r^{(k)}\|_2$, and $z^{(k+\frac{1}{2})}$ is such that the residual

$$q^{(k+\frac{1}{2})} = r^{(k+\frac{1}{2})} - (\gamma I + \mathbf{S})z^{(k+\frac{1}{2})}$$

satisfies $\|q^{(k+\frac{1}{2})}\|_2 \le \eta_k\|r^{(k+\frac{1}{2})}\|_2$.

Evidently, the iteration scheme (3.5) is the inexact HSS iteration method for solving the system of linear equations (1.2), with $\mathbf{A} = \mathbf{H}+\mathbf{S}$; see [4,6]. Hence, by making use of Theorem 3.2 in [6] we can obtain the estimate

$$|||x^{(k+1)} - x_\star||| \le (\sigma(\gamma) + \theta\varrho\eta_k)(1 + \theta\varepsilon_k)|||x^{(k)} - x_\star|||, \quad k = 0, 1, 2, \cdots, \tag{3.6}$$

where the norm $|||\cdot|||$ is defined as follows: for a vector $y \in \mathbb{C}^{mn}$, $|||y||| = \|(\gamma I + \mathbf{S})y\|_2$; and for a matrix $Y \in \mathbb{C}^{mn\times mn}$, $|||Y||| = \|(\gamma I + \mathbf{S})Y(\gamma I + \mathbf{S})^{-1}\|_2$ is the correspondingly induced matrix norm. Note that

$$|||y||| = \|(\gamma I + \mathbf{S})y\|_2 = \|(\alpha I + \mathcal{S}(A))Y + Y(\beta I + \mathcal{S}(B))\|_F = \|Y\|_S.$$

Hence, we can equivalently rewrite the estimate (3.6) as

$$\|X^{(k+1)} - X_\star\|_S \le (\sigma(\gamma) + \theta\varrho\eta_k)(1 + \theta\varepsilon_k)\|X^{(k)} - X_\star\|_S, \quad k = 0, 1, 2, \cdots.$$

This is exactly the estimate what we were deriving. □

We remark that if the continuous Sylvester equations of the forms (3.1) and (3.2) can be solved exactly in some applications, the corresponding quantities $\{\varepsilon_k\}$ and $\{\eta_k\}$ and, hence, $\varepsilon_{\max}$ and $\eta_{\max}$, can be set to be zero. It then follows that the convergence rate of the IHSS iteration reduces to the same as that of the HSS iteration. In general, Theorem 3.1 shows that in order to guarantee the convergence of the IHSS iteration, it is not necessary for $\{\varepsilon_k\}$ and $\{\eta_k\}$ to approach to zero as k is increasing. All we need is that the condition (3.4) is satisfied. Therefore, in actual applications, we need to choose the inner iteration tolerances $\{\varepsilon_k\}$ and $\{\eta_k\}$ so that the computational work of IHSS iteration method is minimized and the original convergence rate of the HSS iteration is asymptotically recovered. A theoretical guarantee for the latter demand is described in the following theorem.

Theorem 3.2. *Let the conditions of Theorem* 2.1 *be satisfied. Suppose that both* $\{\tau_1(k)\}$ *and* $\{\tau_2(k)\}$ *are nondecreasing and positive sequences satisfying* $\tau_1(k) \geq 1$, $\tau_2(k) \geq 1$ *and* $\lim\limits_{k\to\infty} \sup \tau_1(k) = \lim\limits_{k\to\infty} \sup \tau_2(k) = +\infty$, *and that both* δ_1 *and* δ_2 *are real constants in the interval* $(0, 1)$ *satisfying*

$$\varepsilon_k \leq c_1 \delta_1^{\tau_1(k)} \quad \text{and} \quad \eta_k \leq c_2 \delta_2^{\tau_2(k)}, \qquad k = 0, 1, 2, \cdots, \tag{3.7}$$

where c_1 *and* c_2 *are nonnegative constants. Then it holds that*

$$\|X^{(k+1)} - X_\star\|_S \leq \left(\sqrt{\sigma(\gamma)} + \omega\theta\delta^{\tau(k)}\right)^2 \|X^{(k)} - X_\star\|_S,$$

where ϱ *and* θ *are defined in* (3.3), $\tau(k)$ *and* δ *are defined by*

$$\tau(k) = \min\{\tau_1(k), \tau_2(k)\} \quad \text{and} \quad \delta = \max\{\delta_1, \delta_2\},$$

and

$$\omega = \max\left\{\sqrt{c_1 c_2 \varrho}, \quad \frac{1}{2\sqrt{\sigma(\gamma)}}(c_1\sigma(\gamma) + c_2\varrho)\right\}.$$

In particular, we have

$$\lim_{k\to\infty} \sup \frac{\|X^{(k+1)} - X_\star\|_S}{\|X^{(k)} - X_\star\|_S} \leq \sigma(\gamma),$$

i.e., the convergence rate of the IHSS iteration method is asymptotically the same as that of the HSS iteration method.

Proof. The conclusion is straightforward according to Theorem 3.4 in [6]. □

Of course, besides (3.7) there may be other rules for which $\{\varepsilon_k\}$ and $\{\eta_k\}$ approach to zero and the asymptotic convergence factor of the IHSS iteration tends to that of the HSS iteration.

4. Numerical Results

In this section, we use a few numerical results to show the effectiveness of both HSS and IHSS iterations for solving the continuous Sylvester equation (1.1).

In actual computations, all iterations are started from the zero matrix, performed in MATLAB with machine precision 10^{-16}, and terminated when the current iterate satisfies $\|R^{(k)}\|_F / \|R^{(0)}\|_F \leq 10^{-6}$, where

$$R^{(k)} = F - AX^{(k)} - X^{(k)}B$$

is the residual of the k-th HSS iterate. In the IHSS iteration, we set $\varepsilon_k = \eta_k = 0.01$, $k = 0, 1, 2, \cdots$, and use the Smith's method [37] as the inner iteration scheme.

4.1. Results for the HSS iteration

We consider the continuous Sylvester equation (1.1) with $m = n$ and the matrices

$$A = B = M + 2rN + \frac{100}{(n+1)^2} I,$$

where $M, N \in \mathbb{R}^{n\times n}$ are the tridiagonal matrices given by

$$M = \text{tridiag}(-1, 2, -1) \quad \text{and} \quad N = \text{tridiag}(0.5, 0, -0.5).$$

This class of problems may arise in the preconditioned Krylov subspace iteration methods used for solving the systems of linear equations resulting from the finite difference or the Sinc-Galerkin discretization of various differential equations and boundary value problems; see [2, 3, 6, 8, 9].

This continuous Sylvester equation is solved by the HSS and the SOR iteration methods. The number of iteration steps (denoted as **IT**) and the computing time in seconds (denoted as **CPU**) are listed in Table 4.1, while the experimentally found optimal values of the iteration parameters α_{exp} (with $\beta_{\text{exp}} = \alpha_{\text{exp}}$) and ω_{exp} used for the HSS and the SOR iterations are given in Table 4.2. From Table 4.1 we observe that the HSS iteration method considerably outperforms the SOR iteration method in both iteration step and CPU time.

We can also solve the continuous Sylvester equation (1.1) through employing the HSS iteration method in [6] to the standard system of linear equations (1.2). The numerical results are listed in Table 4.3. By comparing the results in Tables 4.1 and 4.3, we see that the matrix form of the HSS iteration method is more effective than its vector form, as it requires much less computing times. In addition, the former can solve much larger problems than the latter.

Table 4.1: IT and CPU for HSS and SOR.

	HSS						SOR					
	$r = 0.01$		$r = 0.1$		$r = 1.0$		$r = 0.01$		$r = 0.1$		$r = 1.0$	
n	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU
8	10	0.001	9	0.001	10	0.001	11	0.005	11	0.005	8	0.004
16	17	0.005	14	0.004	13	0.004	19	0.036	20	0.038	16	0.031
32	27	0.030	28	0.032	24	0.027	37	0.296	38	0.303	32	0.256
64	44	0.235	57	0.322	40	0.237	74	2.520	77	2.624	64	2.197
128	93	2.771	100	3.389	62	2.007	152	24.515	156	25.056	128	20.641
256	203	44.665	156	33.893	95	20.486	310	244.407	304	236.687	256	205.225

Table 4.2: The Optimal Values α_{exp} for HSS and ω_{exp} for SOR.

	α_{exp}			ω_{exp}		
n	$r = 0.01$	$r = 0.1$	$r = 1.0$	$r = 0.01$	$r = 0.1$	$r = 1.0$
8	2.00	2.00	2.00	1.19	1.18	1.00
16	1.00	0.80	1.20	1.34	1.35	1.00
32	0.40	0.40	0.95	1.56	1.54	1.00
64	0.17	0.23	0.81	1.74	1.69	1.00
128	0.09	0.13	0.62	1.86	1.77	1.00
256	0.05	0.09	0.51	1.92	1.81	1.00

Table 4.3: IT and CPU for HSS Applied to Solving (1.2).

	$r = 0.01$		$r = 0.1$		$r = 1.0$	
n	IT	CPU	IT	CPU	IT	CPU
8	10	0.008	9	0.027	10	0.008
16	17	0.084	14	0.069	13	0.067
32	27	1.812	28	1.888	24	1.598
64	44	57.025	57	73.929	40	52.367

4.2. Results for IHSS iteration

We consider the continuous Sylvester equation (1.1) with $m = n$ and the matrices

$$\begin{cases} A = \mathrm{diag}(1, 2, \cdots, n) + rL^T, \\ B = 2^{-t}I + \mathrm{diag}(1, 2, \cdots, n) + rL^T + 2^{-t}L, \end{cases}$$

with L the strictly lower triangular matrix having ones in the lower triangle part. Here, t is a problem parameter to be specified in actual computations. This example is a modification of Example 4.3 in [28].

This continuous Sylvester equation (1.1) is solved by the HSS and the IHSS iteration methods, and the corresponding results are listed in Table 4.4. From Table 4.4 we observe that the IHSS iteration method considerably outperforms the HSS iteration method in both iteration step and CPU time.

Table 4.4: IT and CPU for HSS and IHSS.

		HSS		IHSS	
$\alpha_{\exp} + \beta_{\exp}$	n	IT	CPU	IT	CPU
7	8	16	0.017	15	0.018
10	16	21	0.090	18	0.009
15	32	27	0.496	21	0.034
25	64	35	2.851	26	0.162
40	128	43	17.212	32	1.271
68	256	51	111.301	38	11.060

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