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FINITE ELEMENT ANALYSIS OF OPTIMAL CONTROL PROBLEM GOVERNED BY STOKES EQUATIONS WITH L^2-NORM STATE-CONSTRAINTS*

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Abstract

An optimal control problem governed by the Stokes equations with L^2-norm state constraints is studied. Finite element approximation is constructed. The optimality conditions of both the exact and discretized problems are discussed, and the *a priori* error estimates of the optimal order accuracy in L^2-norm and H^1-norm are given. Some numerical experiments are presented to verify the theoretical results.

Mathematics subject classification: 49J20, 65N30
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1. Introduction

In many engineering applications, the control problems of various flow are very important. One can find lots of useful models for optimal control problems of flow motion with purposes of achieving some desired objectives in real-life applications. Many of those problems come from the fluids flow, aeronautical, chemical engineering, magnetic field and heat sources using radiation or the laser technology, see, for instance, [14, 15, 19, 21, 22, 31] and the references cited therein. There have been extensive research carried out on various theoretical aspects of optimal control problems governed by flow, for example, see [1, 15–18, 24], where control-constrained problems are studied. The state constrained control problems are also frequently met in practical applications, which have aroused many researchers' interests, for example, see [6, 7, 11, 35] for state constrained elliptic control problems. Besides the pointwise state constrained cases as in the above references [6, 11], the integral or the energy of the state are worth concerning in many control problems. For example, one probably wishes to constrain the concentration, the temperature in the average sense in some domain, or the kinetic energy of the flow, etc. In [7], Casas discussed the numerical approximation of optimal control problems governed by a second order semi-linear elliptic partial differential equation associated with finitely many state constraints and gave *a priori* error estimates in H^1-norm. In [25], Liu, Yang and Yuan studied the integral state-constrained control problems governed by an elliptic PDE, proposed a gradient projection algorithm and derived the *a priori* error estimates of the

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optimal order accuracy in L^2- and L^∞-norms. Furthermore, Yuan and Yang analyzed the finite element approximation of L^2-norm state-constrained elliptic control problems and constructed the Uzawa type iterative method in [35]. However, up to now, there has no systematical analysis in the literature for optimal control problems governed by the Stokes equations with state constraints. It is more complicated to study the finite element approximation of the flow control since one has to handle the mixed element.

The purpose of this article is to study the optimal control problems governed by the Stokes equations with L^2-norm constraints for the velocity, where the control is distributed in Ω without constraint. We construct the finite element approximation and analyze optimality conditions for both the exact and the discretized problems. We study *a priori* error estimates between the exact solution and its finite element approximation in L^2-norm and H^1-norm.

The outline of the article is as follows. In Section 2, we state the model problem and construct its finite element approximation. In Section 3, we derive the *a priori* error estimates for the finite element approximation. Finally, in Section 4, we give the Arrow-Hurwicz algorithm and perform some numerical experiments to verify the theoretical results given in Section 3.

2. Control Problem and Finite Element Approximation

Throughout the article, we use the standard definitions and notations of the Sobolev spaces as in [2]. Let Ω be a bounded and open connected domains in $\mathbb{R}^d$ for $d = 2$ or 3. Denote by $\mathbf{v} = (v_1, \cdots, v_d)$ the d-dimensional vector-valued function, $\mathbf{L}^p(\Omega) = \big(L^p(\Omega)\big)^d$, $\mathbf{H}^m(\Omega) = \big(H^m(\Omega)\big)^d$ and $\mathbf{W}^{m,p}(\Omega) = \big(W^{m,p}(\Omega)\big)^d$ the usual vector-valued Sobolev spaces with norms $\|\cdot\|_{m;\Omega} = \|\cdot\|_{H^m(\Omega)}$ and $\|\cdot\|_{m,p;\Omega} = \|\cdot\|_{W^{m,p}(\Omega)}$, respectively. We use $(\cdot,\cdot)_G$ to denote the inner product defined on the bounded and open set G, and if the $G = \Omega$ we omit the subscript, e.g., $(\cdot,\cdot)$. Introduce some function spaces

$$\mathbf{U} = \mathbf{L}^2(\Omega), \quad \mathbf{H} = \big(H_0^1(\Omega)\big)^d, \quad Q = \Big\{q \in L^2(\Omega);\ \int_\Omega q = 0\Big\},$$

which stand for the control space, the velocity sate space and the pressure state space, respectively.

2.1. Optimal control problem

We first state the model problem and its weak form. Let α be a positive constant and the objective functional $\mathcal{J} : \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$ be defined as:

$$\mathcal{J}(\mathbf{y}, \mathbf{u}) = \frac{1}{2}\int_\Omega |\mathbf{y} - \mathbf{y}_d|^2 + \frac{\alpha}{2}\int_\Omega |\mathbf{u} - \mathbf{u}_0|^2.$$

For a positive integer M, the constraint set is given by $\mathbf{K} = \bigcap\limits_{i=1}^{M} \mathbf{K}_i$, where

$$\mathbf{K}_i = \Big\{\mathbf{w} \in \mathbf{L}^2(\Omega);\ \|\mathbf{w}\|_{0;\Omega_i} \le \gamma_i\Big\}, \quad 1 \le i \le M, \tag{2.1}$$

and $\{\Omega_i\}_{i=1}^M$ are nonempty subsets of Ω such that $\Omega_j \bigcap \Omega_k = \emptyset$ for all $1 \le j < k \le M$, and the real number γ_i satisfies $\gamma_i > 0$ for all $1 \le i \le M$.

We investigate the following state-constrained optimal control problem:

$$\min_{\mathbf{y}(\mathbf{u}) \in \mathbf{K}} \mathcal{J}(\mathbf{y}(\mathbf{u}), \mathbf{u}) \tag{2.2}$$

subject to the Stokes equations:

$$\begin{cases} -\nu\Delta\mathbf{y}(\mathbf{u}) + \nabla p(\mathbf{u}) = \mathbf{f} + B\mathbf{u}, & \text{in } \Omega, \\ \nabla\cdot\mathbf{y}(\mathbf{u}) = 0, & \text{in } \Omega, \\ \mathbf{y}(\mathbf{u}) = \mathbf{0}, & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where the constant number $\nu > 0$ and B is a continuous linear operator from $\mathbf{L}^2(\Omega)$ to itself. To derive the weak form of the problem, define some bi-linear forms:

$$(\mathbf{w},\mathbf{z}) = \sum_{i=1}^{d}\int_{\Omega} w_i z_i, \quad a(\mathbf{w},\mathbf{z}) = \nu\sum_{i=1}^{d}(\nabla w_i,\nabla z_i), \quad b(\mathbf{z},q) = -(q,\nabla\cdot\mathbf{z}).$$

It is clear that the bi-linear form $a(\ \cdot\ ,\ \cdot\ )$ is continuous and elliptic in $\mathbf{H}$, i.e., there exist constants $a_l > 0$ and $a_u > 0$ such that

$$a_l\|\mathbf{z}\|_{1;\Omega}^2 \le a(\mathbf{z},\mathbf{z}), \quad \big|a(\mathbf{w},\mathbf{z})\big| \le a_u\|\mathbf{w}\|_{1;\Omega}\|\mathbf{z}\|_{1;\Omega}, \quad \forall\ \mathbf{w},\mathbf{z}\in\mathbf{H}. \tag{2.4}$$

On the other hand, it can be seen from [8, 12, 33] that the bi-linear form $b(\ \cdot\ ,\ \cdot\ )$ satisfies LBB-condition and the continuous condition, i.e., there exist constants $b_l > 0$ and $b_u > 0$ such that

$$b_l \le \inf_{q\in Q}\sup_{\mathbf{z}\in\mathbf{H}}\frac{b(\mathbf{z},q)}{\|\mathbf{z}\|_{1;\Omega}\|q\|_{0;\Omega}}, \quad b(\mathbf{z},q) \le b_u\|\mathbf{z}\|_{1;\Omega}\|q\|_{0;\Omega}, \quad \forall\ \mathbf{z}\in\mathbf{H},\ q\in Q. \tag{2.5}$$

Hence the weak form of the optimal control problem (2.2) reads:

$$(\mathscr{P}) \qquad \min_{\mathbf{y}(\mathbf{u})\in\mathbf{K}} \mathcal{J}\big(\mathbf{y}(\mathbf{u}),\mathbf{u}\big) \tag{2.6}$$

subject to

$$\begin{cases} a\big(\mathbf{y}(\mathbf{u}),\mathbf{w}\big) + b\big(\mathbf{w},p(\mathbf{u})\big) = \big(\mathbf{f}+B\mathbf{u},\mathbf{w}\big), & \forall\ \mathbf{w}\in\mathbf{H}, \\ b\big(\mathbf{y}(\mathbf{u}),q\big) = 0, & \forall\ q\in Q. \end{cases}$$

It is obvious that $\mathbf{K}_i$ is closed and convex in $\mathbf{L}^2$-topology for $1 \le i \le M$, and so is the set $\mathbf{K}$. The existence and uniqueness of the solution of problem (2.6) can be obtained by the usual way, see e.g. [24].

2.2. Optimality conditions

To get the optimality condition, we introduce the Lagrange functional $\mathcal{L}(\mathbf{u},\mathbf{s}) : \mathbf{U}\times\mathbb{R}^M \to \mathbb{R}$ associated with problem (2.6) such that

$$\mathcal{L}(\mathbf{u},\mathbf{s}) = \mathcal{J}(\mathbf{y}(\mathbf{u}),\mathbf{u}) + \sum_{i=1}^{M} s_i F_i(\mathbf{u}), \tag{2.7}$$

where

$$F_i(\mathbf{u}) = \frac{1}{2}\big(\|\mathbf{y}(\mathbf{u})\|_{0;\Omega_i}^2 - \gamma_i^2\big), \qquad \forall\ \mathbf{u}\in\mathbf{U}, \quad 1\le i\le M,$$

and s_i denotes the i-th component of real vector $\mathbf{s}$. Then the optimality conditions can be stated as the following lemma, for the details of the proof the readers may refer to [10, Clarke].

Lemma 2.1. *Let* $\mathbf{u}$ *be the solution of problem* (2.6)*, then there exists a real vector* $\mathbf{s} = (s_1,\cdots,s_M)$ *such that*

$$\begin{aligned} &(1) \quad s_i \ge 0, \quad s_i F_i(\mathbf{u}) = 0, \qquad 1\le i\le M, \\ &(2) \quad \frac{\partial\mathcal{L}}{\partial\mathbf{u}}\big(\mathbf{u},\mathbf{s}\big) = 0. \end{aligned} \tag{2.8}$$

Observing that the operator $\mathbf{y}(\cdot)$ is linear, we know that the second equation of (2.8) is equivalent to

$$\big(\mathbf{y}-\mathbf{y}_d,\mathbf{y}'(\mathbf{u})\circ\mathbf{v}\big)+\sum_{i=1}^{M}\big(s_i\mathbf{y},\mathbf{y}'(\mathbf{u})\circ\mathbf{v}\big)_{\Omega_i}+\alpha\big(\mathbf{u}-\mathbf{u}_0,\mathbf{v}\big)=0,$$

or

$$B^*\mathbf{y}^*+\alpha(\mathbf{u}-\mathbf{u}_0)=0,\quad \text{in } \Omega,$$

where the co-state $\mathbf{y}^*$ is defined by

$$\mathbf{y}^*=\big(\mathbf{y}'(\mathbf{u})\big)^*\left(\mathbf{y}-\mathbf{y}_d+\sum_{i=1}^{M}t_i\mathbf{y}\right)$$

associated with t_i being defined by

$$t_i=\begin{cases} s_i, & \text{if} \quad x\in\Omega_i,\\ 0, & \text{otherwise,}\end{cases}$$

for $1\le i\le M$. Define the piecewise constant function space $\mathbf{T}$ as:

$$\mathbf{T}=\Big\{\mathbf{t}=(t_1,\cdots,t_M);\ t_i\in\mathbb{R}\ \text{ in }\ \Omega_i;\ \ t_i=0\ \text{ in }\ \Omega\backslash\Omega_i;\quad 1\le i\le M\Big\}.$$

Therefore, we can obtain another form of the optimality conditions for problem (2.6).

Theorem 2.1. *The triplet* $(\mathbf{y},p,\mathbf{u})\in\mathbf{H}\times Q\times\mathbf{U}$ *is the solution of the problem* (2.6) *if and only if there exists a triplet* $(\mathbf{y}^*,p^*,\mathbf{t})\in\mathbf{H}\times Q\times\mathbf{T}$ *such that* $(\mathbf{y},p,\mathbf{u},\mathbf{y}^*,p^*,\mathbf{t})\in\mathbf{H}\times Q\times\mathbf{U}\times\mathbf{H}\times Q\times\mathbf{T}$ *satisfies*

$$(\mathscr{Q})\qquad \begin{cases} a(\mathbf{y},\mathbf{w})+b(\mathbf{w},p)=(\mathbf{f}+B\mathbf{u},\mathbf{w}), & \forall\ \mathbf{w}\in\mathbf{H},\\ b(\mathbf{y},q)=0, & \forall\ q\in Q,\\ a(\mathbf{y}^*,\mathbf{w})+b(\mathbf{w},p^*)=\left(\left(1+\sum_{i=1}^{M}t_i\right)\mathbf{y}-\mathbf{y}_d,\mathbf{w}\right), & \forall\ \mathbf{w}\in\mathbf{H},\\ b(\mathbf{y}^*,q)=0, & \forall\ q\in Q,\\ B^*\mathbf{y}^*+\alpha(\mathbf{u}-\mathbf{u}_0)=0, & \text{in}\ \ \Omega,\end{cases} \tag{2.9}$$

where the components of vector $\mathbf{t}$ *satisfy*

$$t_i=\begin{cases}\text{constant}\ge 0, & \|\mathbf{y}\|_{0,2;\Omega_i}=\gamma_i,\quad \text{in } \Omega_i,\\ 0, & \text{otherwise,}\end{cases} \tag{2.10}$$

for $1\le i\le M$.

2.3. Finite element approximation

We only consider n-simply elements, which are widely used in engineering applications. For the sake of simplicity, we assume that Ω is a polygon in $\mathbb{R}^2$ or polyhedron in $\mathbb{R}^3$. Let $\mathscr{T}^h=\bigcup T$ be a family of quasi-regular triangulations of Ω with maximum mesh size $h:=\max_{\tau\in\mathscr{T}^h}\{\text{diam}(T)\}$ and $\mathscr{T}_U^h=\bigcup T_U$ be a family of quasi-regular triangulations of Ω with maximum mesh size $h_U:=\max_{T_U\in\mathscr{T}_U^h}\big\{\text{diam}(T_U)\big\}$, in which each element has at most one face on $\partial\Omega$, and $\overline{T}$ and $\overline{T}'$ (or $\overline{T}_U$ and $\overline{T}'_U$) have either only one common vertex or a whole edge in 2-d case if $\overline{T}$ and $\overline{T}'\in\mathscr{T}^h$ (or $\overline{T}_U$ and $\overline{T}'_U\in\mathscr{T}_U^h$).

Associated with $\mathscr{T}^h$ are two finite element spaces $\mathbf{H}^h \subset \mathbf{H}$ and $Q^h \subset Q$ such that the finite element spaces $\mathbf{H}^h \times Q^h$ satisfies the discrete LBB-condition, i.e., there exists a constant $b_l' > 0$ such that

$$\inf_{q_h \in Q^h} \sup_{\mathbf{z}_h \in \mathbf{H}^h} \frac{b(\mathbf{z}_h, q_h)}{\|\mathbf{z}_h\|_{1;\Omega} \|q_h\|_{0;\Omega}} \geq b_l'. \tag{2.11}$$

And there exist two integers $m \geq 1$ and $n \geq 1$ such that

$$\begin{aligned} &\inf_{\mathbf{z}_h \in \mathbf{H}^h} \|\mathbf{z} - \mathbf{z}_h\|_{0;\Omega} + h\|\mathbf{z} - \mathbf{z}_h\|_{1;\Omega} \leq Ch^{m+1} \|\mathbf{z}\|_{m+1;\Omega}, \qquad && \forall\, \mathbf{z} \in \mathbf{H} \cap \mathbf{H}^{m+1}(\Omega), \\ &\inf_{q_h \in Q^h} \|q - q_h\|_{0;\Omega} \leq Ch^{n+1} \|q\|_{n+1;\Omega}, && \forall\, q \in Q \cap H^{n+1}(\Omega). \end{aligned} \tag{2.12}$$

The above assumptions are satisfied for Hood-Taylor ($\mathbf{P}_{n+1}$, P_n) finite elements when $m = n+1$ and for Mini ($\mathbf{P}_1 \bigoplus$Bubble, P_1) finite elements when $m = n = 1$, see, e.g., [5, 9, 12, 20],.

Associated with $\mathscr{T}_U^h$ is another finite dimensional subspace $\mathbf{U}^h := \{\mathbf{v}_h \in \mathbf{U} : \mathbf{v}_h|_{T_U}$ are polynomials of degree less than or equal to k ($0 \leq k \leq m$) for each $T_U \in \mathscr{T}_U^h\}$ such that

$$\inf_{\mathbf{v}_h \in \mathbf{U}^h} \|\mathbf{v} - \mathbf{v}_h\|_{0;\Omega} \leq Ch_U^{k+1} \|\mathbf{v}\|_{k+1;\Omega}, \qquad \forall\, \mathbf{v} \in \mathbf{U} \cap \mathbf{H}^{k+1}(\Omega). \tag{2.13}$$

Introduce the discretized constraint set $\mathbf{K}^h = \mathbf{H}^h \cap \mathbf{K}$. So the finite element approximation of problem (2.6) reads:

$$(\mathscr{P}^h) \qquad \min_{\mathbf{y}_h \in \mathbf{K}^h} \mathcal{J}(\mathbf{y}_h, \mathbf{u}_h) \tag{2.14}$$

subject to

$$\begin{cases} a(\mathbf{y}_h, \mathbf{w}_h) + b(\mathbf{w}_h, p_h) = (\mathbf{f} + B\mathbf{u}_h, \mathbf{w}_h), & \forall\, \mathbf{w}_h \in \mathbf{H}^h, \\ b(\mathbf{y}_h, q_h) = 0, & \forall\, q_h \in Q^h. \end{cases}$$

Similarly, we obtain the optimality conditions of problem (2.14), which is stated in the following theorem.

Theorem 2.2. *The triplet* $(\mathbf{y}_h, p_h, \mathbf{u}_h) \in \mathbf{H}^h \times Q^h \times \mathbf{U}^h$ *is the solution of the problem* (2.14) *if and only if there exists a triplet* $(\mathbf{y}_h^*, p_h^*, \mathbf{t}_h) \in \mathbf{H}^h \times Q^h \times \mathbf{T}$ *such that* $(\mathbf{y}_h, p_h, \mathbf{u}_h, \mathbf{y}_h^*, p_h^*, \mathbf{t}_h) \in \mathbf{H}^h \times Q^h \times \mathbf{U}^h \times \mathbf{H}^h \times Q^h \times \mathbf{T}$ *satisfies the following optimality conditions:*

$$(\mathscr{Q}^h) \qquad \begin{cases} a(\mathbf{y}_h, \mathbf{w}_h) + b(\mathbf{w}_h, p_h) = (\mathbf{f} + B\mathbf{u}_h, \mathbf{w}_h), & \forall\, \mathbf{w}_h \in \mathbf{H}^h, \\ b(\mathbf{y}_h, q_h) = 0, & \forall\, q_h \in Q^h, \\ a(\mathbf{y}_h^*, \mathbf{w}_h) + b(\mathbf{w}_h, p_h^*) = \left(\left(1 + \sum_{i=1}^M t_{h,i} \right) \mathbf{y}_h - \mathbf{y}_d, \mathbf{w}_h \right), & \forall\, \mathbf{w}_h \in \mathbf{H}^h, \\ b(\mathbf{y}_h^*, q_h) = 0 & \forall\, q_h \in Q^h, \\ \alpha \mathbf{u}_h + \mathcal{P}_U^h B^*(\mathbf{y}_h^* - \alpha \mathbf{u}_0) = 0, & \text{in } \ \Omega, \end{cases} \tag{2.15}$$

where the components $\{t_{h,i}\}_{i=1}^M$ *of vector* $\mathbf{t}$ *satisfy*

$$t_{h,i} = \begin{cases} \text{constant} \geq 0 & \|\mathbf{y}_h\|_{0,2;\Omega_i} = \gamma_i, \quad \text{in } \Omega_i, \\ 0 & \text{otherwise}, \end{cases} \tag{2.16}$$

and $\mathcal{P}_U^h$ *is the* L^2*-projection operator from* $\mathbf{U}$ *to* $\mathbf{U}^h$ *such that*

$$(\mathcal{P}_U^h \boldsymbol{\varphi}, \mathbf{v}_h) = (\boldsymbol{\varphi}, \mathbf{v}_h), \qquad \forall\, \boldsymbol{\varphi} \in \mathbf{U},\ \mathbf{v}_h \in \mathbf{U}^h.$$

It is obvious that $\mathcal{P}_U^h$ is a linear operator since $\mathbf{U}^h$ is a linear subspace of the Banach space $\mathbf{U}$. Here the first order optimality conditions (2.15) are also sufficient since the state equations are linear and the object functional $\mathcal{J}(\,\cdot\,\cdot\,)$ is convex, so problem (2.14) is equivalent to problem (2.15).

3. A Priori Estimates

In this section, we analyze the convergent rates of the algorithm. We assume that the operator B is reversible from $\mathbf{L}^2(\Omega)$ to itself and from $\mathbf{H}^1(\Omega)$ to itself. It is easy to be proved there exits the constant independent of h and h_U such that

$$\|\mathbf{u}_h\|_{0;\Omega}+\|\mathbf{y}_h\|_{1;\Omega}+\|p_h\|_{0;\Omega}+\|\mathbf{y}_h^*\|_{1;\Omega}+\|p_h^*\|_{0;\Omega}+\max_{1\le i\le M}\|t_i\|_{L^\infty(\Omega)}\le C.$$

So there there exists a subsequence which weakly converges to one of solutions of the problem (2.9) as $h\to 0$. Since the solution of the problem (2.9) is unique, the sequence $(\mathbf{u}_h,\mathbf{y}_h,p_h,\mathbf{y}_h^*,p_h^*,\mathbf{t}_h)$ weakly converges to the exact solution $(\mathbf{u},\mathbf{y},p,\mathbf{y}^*,p^*,\mathbf{t})$. Now we are ready to study the *a priori* error estimates between the exact solution and the finite element solution. In this article, we consider finite element methods including Hood-Taylor element ($\mathbf{P}_{l+1},\ P_l;\ l\ge 1$) and Mini-element ($\mathbf{P}_{l+1}\bigoplus$ Bubble, $P_{l+1}; l=0$), so we assume the solution of the optimality conditions has the following regularity properties (see [8] for more details):

$$\mathbf{y},\ \mathbf{y}^*\in\mathbf{H}^{l+2}(\Omega),\quad p,\ p^*\in H^{l+1}(\Omega). \tag{3.1}$$

We first state the H^1-norm error estimates and the L^2-norm error estimates in the next two theorems, respectively. The constant number l is defined in above (3.1) and $k\le l+1$ is described in (2.13).

Theorem 3.1. *Let* $(\mathbf{y},p,\mathbf{y}^*,p^*,\mathbf{t},\mathbf{u})$ *and* $(\mathbf{y}_h,p_h,\mathbf{p}_h^*,p_h^*,\mathbf{t}_h,\mathbf{u}_h)$ *be the solutions of* (2.9) *and* (2.15)*, respectively. Then there hold the* $H^1\times H^0$*-norm error estimates for the velocity field and the pressure field as follows:*

$$\|\mathbf{y}-\mathbf{y}_h\|_{1;\Omega}+\|\mathbf{y}^*-\mathbf{y}_h^*\|_{1;\Omega}+\|p-p_h\|_{0;\Omega}+\|p^*-p_h^*\|_{0;\Omega}\le C\Big(h^{l+1}+h_U^{k+2}\Big), \tag{3.2}$$

and L^2*-norm error estimate for the control as follows:*

$$\|\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}\le C\Big(h^{l+2}+h_U^{k+1}\Big). \tag{3.3}$$

In the following theorem, we denote the infinite-norm of the real vector by $\|\cdot\|_\infty$.

Theorem 3.2. *Let* $(\mathbf{y},p,\mathbf{y}^*,p^*,\mathbf{t},\mathbf{u})$ *and* $(\mathbf{y}_h,p_h,\mathbf{p}_h^*,p_h^*,\mathbf{t}_h,\mathbf{u}_h)$ *be the solutions of* (2.9) *and* (2.15)*, respectively. Then there hold the following* L^2*-norm error estimates:*

$$\|\mathbf{y}-\mathbf{y}_h\|_{0;\Omega}+\|\mathbf{y}^*-\mathbf{y}_h^*\|_{0;\Omega}+||\mathbf{t}-\mathbf{t}_h||_\infty+\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}\le C\Big(h^{l+2}+h_U^{k+2}\Big). \tag{3.4}$$

The proofs of Theorem 3.1 and 3.2 follow from the following five lemmas. Introduce the following auxiliary equations:

$$\begin{cases} a(\mathbf{y}_h(\mathbf{u}),\mathbf{w}_h)+b(\mathbf{w}_h,p_h(\mathbf{u}))=(\mathbf{f}+B\mathbf{u},\mathbf{w}_h), & \forall\ \mathbf{w}_h\in\mathbf{H}^h,\\ b(\mathbf{y}_h(\mathbf{u}),q_h)=0, & \forall\ q_h\in Q^h,\\ a(\mathbf{y}_h^*(\mathbf{u}),\mathbf{w}_h)+b(\mathbf{w}_h,p_h^*(\mathbf{u}))=\Big(\Big(1+\sum_{i=1}^M t_i\Big)\,\mathbf{y}-\mathbf{y}_d,\mathbf{w}_h\Big), & \forall\ \mathbf{w}_h\in\mathbf{H}^h,\\ b(\mathbf{y}_h^*(\mathbf{u}),q_h)=0, & \forall\ q_h\in Q^h. \end{cases} \tag{3.5}$$

Firstly, we estimate the terms $\|\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h\|_{1;\Omega}+\|p_h(\mathbf{u})-p_h\|_{0;\Omega}$, $||\mathbf{t}-\mathbf{t}_h||_\infty$ and $\|\mathbf{y}_h^*(\mathbf{u})-\mathbf{y}_h^*\|_{1;\Omega}+\|p_h^*(\mathbf{u})-p_h^*\|_{0;\Omega}$.

Lemma 3.1. *Let $\big(\mathbf{y}_h(\mathbf{u}),p_h(\mathbf{u})\big)$ and $\big(\mathbf{y}_h,p_h\big)$ be the solutions of Eqs.* (3.5) *and* (2.15)*, respectively. Then there holds the following inequality*

$$\|\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h\|_{1;\Omega}+\|p_h(\mathbf{u})-p_h\|_{0;\Omega}\le C\Big(\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}+h_U\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}\|_{0;\Omega}\Big). \tag{3.6}$$

Proof. From Eqs. (2.15) and (3.5), we have

$$\begin{cases} a\big(\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h,\mathbf{w}_h\big)+b(\mathbf{w}_h,p_h(\mathbf{u})-p_h)=\big(B(\mathbf{u}-\mathbf{u}_h),\mathbf{w}_h\big), & \forall\ \mathbf{w}_h\in\mathbf{H}^h,\\ b(\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h,q_h)=0, & \forall\ q_h\in Q^h. \end{cases} \tag{3.7}$$

Observing that

$$\begin{aligned}|(\mathbf{u}-\mathbf{u}_h,B^*\mathbf{w}_h)&\le|(\mathbf{u}-\mathcal{P}_U^h\mathbf{u},B^*\mathbf{w}_h-\mathcal{P}_U^hB^*\mathbf{w}_h)+|(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h,B^*\mathbf{w}_h)|\\ &\le C\Big(\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}+h_U\|\mathbf{u}-\mathcal{P}_U^h\mathbf{u}\|_{0;\Omega}\Big)\|B^*\mathbf{w}_h\|_{1;\Omega}\\ &\le C\Big(\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}+h_U\|\mathbf{u}-\mathcal{P}_U^h\mathbf{u}\|_{0;\Omega}\Big)\|\mathbf{w}_h\|_{1;\Omega},\end{aligned}$$

and by taking $\mathbf{w}_h=\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h$ and using (2.4), we have

$$\|\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h\|_{1;\Omega}\le C\Big(\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}+h_U\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}\|_{0;\Omega}\Big).$$

With LBB-condition (2.11), we can obtain

$$\begin{aligned}b_l'\|p_h(\mathbf{u})-p_h\|_{0;\Omega}&\le\sup_{\mathbf{w}_h\in\mathbf{H}^h}\frac{1}{\|\mathbf{w}_h\|_{1;\Omega}}\Big(|(\mathbf{u}-\mathbf{u}_h,B^*\mathbf{w}_h)|+|a(\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h,\mathbf{w}_h)|\Big)\\ &\le C\Big(\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}+h_U\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}\|_{0;\Omega}\Big).\end{aligned}$$

Combining the above two inequalities, we have (3.6). Hence Lemma 3.1 is proved. □

Lemma 3.2. *Let $\mathbf{t}$ and $\mathbf{t}_h$ be the solutions of Eqs.* (2.9) *and* (2.15)*, respectively. There exists $h_0>0$ such that for $0<h,h_U\le h_0$,*

$$\begin{aligned}\|\mathbf{t}-\mathbf{t}_h\|_\infty\le C\Big(&\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}+\|\mathbf{y}-\mathbf{y}_h(\mathbf{u})\|_{0;\Omega}\\ &+(h+h_U)\|\nabla(\mathbf{y}^*-\mathbf{y}_h^*)\|_{0;\Omega}+h_U\|\mathbf{u}-\mathcal{P}_U^h\mathbf{u}\|_{0;\Omega}\Big).\end{aligned} \tag{3.8}$$

Proof. For $1\le i\le M$, since $\lim\limits_{h\to0}t_{h,i}=t_i$ and $t_i=0$ if $\|\mathbf{y}\|_{0;\Omega_i}<\gamma_i$, we only need to check the case $\|\mathbf{y}\|_{0;\Omega_i}=\gamma_i$. Let $\phi_i\in C_0^\infty(\Omega_i)$ such that $0\le\phi_i\le1$ and $\|\phi_i\mathbf{y}\|_{0;\Omega_i}\ge\gamma_i/2$. Noting that

$$\begin{aligned}&a(\mathbf{y}^*-\mathbf{y}_h^*,\mathbf{w}_h)=\sum_{i=1}^M(t_i\mathbf{y}-t_{h,i}\mathbf{y}_h,\mathbf{w}_h)_{\Omega_i}+(\mathbf{y}-\mathbf{y}_h,\mathbf{w}_h)\\ =&\frac12\sum_{i=1}^M(t_i-t_{h,i})(\mathbf{y}+\mathbf{y}_h,\mathbf{w}_h)_{\Omega_i}+\frac12\sum_{i=1}^M(t_i+t_{h,i})(\mathbf{y}-\mathbf{y}_h,\mathbf{w}_h)_{\Omega_i}+(\mathbf{y}-\mathbf{y}_h,\mathbf{w}_h)\end{aligned}$$

and taking $\mathbf{w}_h = \mathcal{I}_h(\phi_i \mathbf{y}_h)$, we have

$$\begin{aligned}&a(\mathbf{y}^* - \mathbf{y}_h^*, \mathcal{I}_h(\phi_i \mathbf{y}_h))\\ =&\frac{1}{2}(t_i - t_{h,i})(\mathbf{y} + \mathbf{y}_h, \mathcal{I}_h(\phi_i \mathbf{y}_h))_{\Omega_i} + (t_i + t_{h,i})(\mathbf{y} - \mathbf{y}_h, \mathcal{I}_h(\phi_i \mathbf{y}_h))_{\Omega_i} + (\mathbf{y} - \mathbf{y}_h, \mathcal{I}_h(\phi_i \mathbf{y}_h))\end{aligned}$$

such that

$$\begin{aligned}&(t_i - t_{h,i})(\mathbf{y}, \phi_i \mathbf{y})_{\Omega_i}\\ =&a(\mathbf{y}^* - \mathbf{y}_h^*, \mathcal{I}_h(\phi_i \mathbf{y}_h)) - (t_i + t_{h,i})(\mathbf{y} - \mathbf{y}_h, \mathcal{I}_h(\phi_i \mathbf{y}_h))_{\Omega_i} - (\mathbf{y} - \mathbf{y}_h, \mathcal{I}_h(\phi_i \mathbf{y}_h))\\ &- (t_i - t_{h,i})(\mathbf{y}, (\mathcal{I}_h - \mathcal{I})(\phi_i \mathbf{y}))_{\Omega_i} - (t_i - t_{h,i})(\mathbf{y}, \mathcal{I}_h(\phi_i(\mathbf{y}_h - \mathbf{y})))_{\Omega_i}\\ &- \frac{1}{2}(t_i - t_{h,i})(\mathbf{y}_h - \mathbf{y}, \mathcal{I}_h(\phi_i \mathbf{y}_h))_{\Omega_i}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&a(\mathbf{y}^* - \mathbf{y}_h^*, \mathcal{I}_h(\phi_i \mathbf{y}_h))\\ =&a(\mathbf{y}^* - \mathbf{y}_h^*, (\mathcal{I}_h - \mathcal{I})(\phi_i \mathbf{y}_h)) + \nu(\mathbf{y}^* - \mathbf{y}_h^*, \mathbf{y}_h \Delta\phi_i) - 2\nu((\mathbf{y}^* - \mathbf{y}_h^*)\nabla\phi_i, \nabla \mathbf{y}_h)\\ &+ \nu(\nabla(\phi_i(\mathbf{y}^* - \mathbf{y}_h^*)), \nabla(\mathbf{y}_h - \mathbf{y})) + \nu(\nabla(\phi_i(\mathbf{y}^* - \mathbf{y}_h^*)), \nabla \mathbf{y})\end{aligned}$$

and

$$\begin{aligned}\nu(\nabla(\phi_i(\mathbf{y}^* - \mathbf{y}_h^*)), \nabla \mathbf{y}) =&(\mathbf{f} + B\mathbf{u}, \phi_i(\mathbf{y}^* - \mathbf{y}_h^*)) + (p, \nabla \cdot (\phi_i(\mathbf{y}^* - \mathbf{y}_h^*)))\\ =&(\phi_i(\mathbf{f} + B\mathbf{u} - \nabla p), \mathbf{y}^* - \mathbf{y}_h^*).\end{aligned}$$

So we get

$$\begin{aligned}\frac{\gamma_i}{2}|t_i - t_{h,i}| \le& (1 + t_i + t_{h,i})\|\mathbf{y} - \mathbf{y}_h\|_{\Omega_i}\|\mathcal{I}_h(\phi_i \mathbf{y}_h)\|_{L^2(\Omega_i)}\\ &+ |t_i - t_{h,i}|\gamma_i\big(\|(\mathcal{I}_h - \mathcal{I})(\phi_i \mathbf{y})\|_{0;\Omega_i} + \|\mathcal{I}_h(\phi_i(\mathbf{y}_h - \mathbf{y}))\|_{0;\Omega_i}\big)\\ &+ \frac{1}{2}|t_i - t_{h,i}|\|\mathbf{y}_h - \mathbf{y}\|_{0;\Omega_i}\|\mathcal{I}_h(\phi_i \mathbf{y}_h)\|_{0;\Omega_i}\\ &+ \nu\|\nabla(\mathbf{y}^* - \mathbf{y}_h^*)\|_{0;\Omega_i}\Big(\|\nabla(\mathcal{I}_h - \mathcal{I})(\phi_i \mathbf{y}_h)\|_{0;\Omega_i} + \|\nabla\phi_i\|_{0,\infty;\Omega_i}\|\nabla(\mathbf{y}_h - \mathbf{y})\|_{0,2;\Omega_i}\Big)\\ &+ \nu\|\mathbf{y}^* - \mathbf{y}_h^*\|_{0;\Omega_i}\Big(\|\mathbf{y}_h\|_{0,\Omega}\|\Delta\phi_i\|_{0,\infty;\Omega_i} + 2\|\nabla\phi_i\|_{0,\infty;\Omega_i}\|\nabla \mathbf{y}_h\|_{0,2;\Omega_i}\Big)\\ &+ \|\mathbf{y}^* - \mathbf{y}_h^*\|_{0;\Omega_i}\|\mathbf{f} + B\mathbf{u} - \nabla p\|_{0;\Omega_i}\end{aligned}$$

such that

$$|t_i - t_{h,i}| \le C\Big(h|t_i - t_{h,i}| + \|\mathbf{y} - \mathbf{y}_h\|_{0;\Omega_i} + \|B^*(\mathbf{y}^* - \mathbf{y}_h^*)\|_{0;\Omega_i} + h\|\nabla(\mathbf{y}^* - \mathbf{y}_h^*)\|_{0;\Omega_i}\Big). \tag{3.9}$$

Applying $B^*(\mathbf{y}^* - \mathbf{y}_h^*) = (\mathcal{I} - \mathcal{P}_U^h)(B^*(\mathbf{y}^* - \mathbf{y}_h^*)) + \alpha(\mathcal{P}_U^h \mathbf{u} - \mathbf{u}_h)$ and Lemma 3.3 and the standard finite element error estimates (as listed later in (3.15)) to (3.9), we have

$$\begin{aligned}&|t_i - t_{h,i}|\\ \le& C\Big(\|\mathbf{u}_h - \mathcal{P}_U^h \mathbf{u}\|_{0;\Omega} + \|\mathbf{y} - \mathbf{y}_h(\mathbf{u})\|_{0;\Omega} + \big(h + h_U\big)\|\nabla(\mathbf{y}^* - \mathbf{y}_h^*)\|_{0;\Omega} + h_U\|\mathbf{u} - \mathcal{P}_U^h \mathbf{u}\|_{0;\Omega}\Big)\end{aligned}$$

for sufficiently small h. Then Lemma 3.2 is proved. □

Secondly, we estimate the term $\|\mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h\|_{1;\Omega}$ and $\|p_h^*(\mathbf{u}) - p_h\|_{0;\Omega}$.

Lemma 3.3. *Let* $\big(\mathbf{y}_h^*(\mathbf{u}), p_h^*(\mathbf{u})\big)$ *and* $\big(\mathbf{y}_h^*, p_h^*\big)$ *be the solutions of Eqs.* (3.5) *and* (2.15), *respectively. Then there holds the following inequality*

$$\|\mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h^*\|_{1;\Omega} + \|p_h^*(\mathbf{u}) - p_h^*\|_{0;\Omega} \leq C\Big(\|\mathcal{P}_U^h\mathbf{u} - \mathbf{u}_h\|_{0;\Omega} + h_U\|\mathcal{P}_U^h\mathbf{u} - \mathbf{u}\|_{0;\Omega} + \|\mathbf{y} - \mathbf{y}_h(\mathbf{u})\|_{0;\Omega} + (h + h_U)\|\nabla(\mathbf{y}^* - \mathbf{y}_h^*(\mathbf{u}))\|_{0;\Omega}\Big). \tag{3.10}$$

Proof. From Eqs. (2.15) and (3.5), we have

$$\begin{cases} a(\mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h^*, \mathbf{w}_h) + b(\mathbf{w}_h, p_h^*(\mathbf{u}) - p_h^*) = \sum\limits_{i=1}^{M}(t_i\mathbf{y} - t_{h,i}\mathbf{y}_h, \mathbf{w}_h)_{\Omega_i} + (\mathbf{y} - \mathbf{y}_h, \mathbf{w}_h), & \forall\, \mathbf{w}_h \in \mathbf{H}^h, \\ b(\mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h^*, q_h) = 0, & \forall\, q_h \in Q^h. \end{cases}$$

By taking $\mathbf{w}_h = \mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h^*$, we get

$$a(\mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h^*, \mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h^*) = \sum_{i=1}^{M}(t_i\mathbf{y} - t_{h,i}\mathbf{y}_h, \mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h^*)_{\Omega_i} + (\mathbf{y} - \mathbf{y}_h, \mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h^*),$$

such that

$$\|\mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h^*\|_{0;\Omega}^2 \leq C\Big(\|\mathbf{t} - \mathbf{t}_h\|_\infty^2 + \|\mathbf{y} - \mathbf{y}_h\|_{0;\Omega}^2\Big). \tag{3.11}$$

Applying (3.8) to (3.11) and letting h and h_U suitably small lead to (3.10) for $\mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h^*$.

Further, by LBB-condition (2.11), we can obtain

$$\begin{aligned} & b_l'\|p_h^*(\mathbf{u}) - p_h^*\|_{0;\Omega} \\ & \leq \sup_{\mathbf{w}_h \in \mathbf{H}^h} \frac{1}{\|\mathbf{w}_h\|_{1;\Omega}}\left(\sum_{i=1}^{M}|(t_i\mathbf{y} - t_{h,i}\mathbf{y}_h, \mathbf{w}_h)_{\Omega_i}| + |(\mathbf{y} - \mathbf{y}_h, \mathbf{w}_h)| + |a(\mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h^*, \mathbf{w}_h)|\right) \\ & \leq C\Big(\|\mathbf{t} - \mathbf{t}_h\|_\infty + \|\mathbf{y} - \mathbf{y}_h\|_{0;\Omega} + \|\nabla(\mathbf{y}_h^*(\mathbf{u}) - \mathbf{y}_h^*)\|_{0;\Omega}\Big). \end{aligned}$$

This leads (3.10) for $\mathbf{p}_h^*(\mathbf{u}) - \mathbf{p}_h^*$. Hence Lemma 3.3 is proved. □

Thirdly, we estimate the term $\|\mathcal{P}^h\mathbf{u} - \mathbf{u}_h\|_{0;\Omega}$.

Lemma 3.4. *Let* $\mathbf{u}$ *and* $\mathbf{u}_h$ *be the solutions of Eqs.* (2.9) *and* (2.15), *respectively. Then there holds the estimate:*

$$\|\mathcal{P}_U^h\mathbf{u} - \mathbf{u}_h\|_{0;\Omega} \leq C\Big(\|\mathbf{y} - \mathbf{y}_h(\mathbf{u})\|_{0;\Omega} + \|\mathbf{y}^* - \mathbf{y}_h^*(\mathbf{u})\|_{0;\Omega} + h_U\|\mathbf{u} - \mathcal{P}_U^h\mathbf{u}\|_{0;\Omega}\Big). \tag{3.12}$$

Proof. It follows from (2.10), (2.16), $\|\mathbf{y}\|_{0;\Omega_i} \leq \gamma_i$ and $\|\mathbf{y}_h\|_{0;\Omega_i} \leq \gamma_i$ that

$$(t_i\mathbf{y} - t_{h,i}\mathbf{y}_h, \mathbf{y}_h(\mathbf{u}) - \mathbf{y})_{\Omega_i} - (t_i\mathbf{y} - t_{h,i}\mathbf{y}_h, \mathbf{y}_h(\mathbf{u}) - \mathbf{y}_h)_{\Omega_i} = (t_i\mathbf{y}, \mathbf{y}_h - \mathbf{y}) + t_{h,i}(\mathbf{y}_h, \mathbf{y} - \mathbf{y}_h)_{\Omega_i} \leq 0,$$

which implies that

$$-(t_i\mathbf{y} - t_{h,i}\mathbf{y}_h, \mathbf{y}_h(\mathbf{u}) - \mathbf{y}_h)_{\Omega_i} \leq -(t_i\mathbf{y} - t_{h,i}\mathbf{y}_h, \mathbf{y}_h(\mathbf{u}) - \mathbf{y})_{\Omega_i}.$$

Then from equations (2.9), (2.15) and (3.5) we have

$$
\begin{aligned}
&\alpha\big(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h,\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\big)=\alpha\big(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h\big)=-(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h,B^*\mathbf{y}^*-\mathcal{P}_U^hB^*\mathbf{y}_h^*\big)\\
=&-\big(B(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h),\mathbf{y}^*-\mathbf{y}_h^*(\mathbf{u})\big)-\big(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h,B^*\mathbf{y}_h^*(\mathbf{u})+\alpha(\mathbf{u}_h-\mathbf{u}_0)\big)\\
&=-\big(B(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h),\mathbf{y}^*-\mathbf{y}_h^*(\mathbf{u})\big)-\big(B(\mathbf{u}-\mathbf{u}_h),\mathbf{y}_h^*(\mathbf{u})-\mathbf{y}_h^*\big)-\big(B(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}),\mathbf{y}_h^*(\mathbf{u})-\mathbf{y}_h^*\big)\\
=&-\big(B(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h),\mathbf{y}^*-\mathbf{y}_h^*(\mathbf{u})\big)-a\big(\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h,\mathbf{y}_h^*(\mathbf{u})-\mathbf{y}_h^*\big)-\big(B(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}),\mathbf{y}_h^*(\mathbf{u})-\mathbf{y}_h^*\big)\\
=&-\big(B(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h),\mathbf{y}^*-\mathbf{y}_h^*(\mathbf{u})\big)-\left(\left(1+\sum_{i=1}^M t_i\right)\mathbf{y}-\left(1+\sum_{i=1}^M t_{h,i}\right)\mathbf{y}_h,\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h\right)\\
&\quad-\big(B(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}),\mathbf{y}_h^*(\mathbf{u})-\mathbf{y}_h^*\big)\\
\le&-\big(B(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h),\mathbf{y}^*-\mathbf{y}_h^*(\mathbf{u})\big)-\big(\mathbf{y}-\mathbf{y}_h,\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h\big)-\left(\sum_{i=1}^M t_i\mathbf{y}-\sum_{i=1}^M t_{h,i}\mathbf{y}_h,\mathbf{y}_h(\mathbf{u})-\mathbf{y}\right)\\
&\quad-\big(B(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}),\mathbf{y}_h^*(\mathbf{u})-\mathbf{y}_h^*\big).
\end{aligned}
$$

So for $0<\epsilon\ll 1$, there holds

$$
\begin{aligned}
&\alpha\big(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h,\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\big)_\Omega+(\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h,\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h)\\
&\le-\big(B(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h),\mathbf{y}^*-\mathbf{y}_h^*(\mathbf{u})\big)-\big(\mathbf{y}-\mathbf{y}_h(u),\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h\big)\\
&\qquad-\big(\sum_{i=1}^M t_i(\mathbf{y}-\mathbf{y}_h)+\sum_{i=1}^M(t_i-t_{h,i})\mathbf{y}_h,\mathbf{y}_h(\mathbf{u})-\mathbf{y}\big)-\big(B(\mathcal{P}_U^h\mathbf{u}-\mathbf{u}),\mathbf{y}_h^*(\mathbf{u})-\mathbf{y}_h^*\big)\\
&\le\epsilon\Big(\alpha\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}^2+\|\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h\|_{0;\Omega}^2+\|\nabla(\mathbf{y}_h^*(\mathbf{u})-\mathbf{y}_h^*)\|_{0;\Omega}^2+\|\mathbf{t}-\mathbf{t}_h\|_\infty^2\Big)\\
&\qquad+C\epsilon^{-1}\Big(\|\mathbf{y}-\mathbf{y}_h(\mathbf{u})\|_{0;\Omega}^2+\|\mathbf{y}^*-\mathbf{y}_h^*(\mathbf{u})\|_{0;\Omega}^2+h_U^2\|\mathbf{u}-\mathcal{P}_U^h\mathbf{u}\|_{0;\Omega}^2\Big).
\end{aligned}\tag{3.13}
$$

Applying (3.6)–(3.10) in (3.13), we obtain (3.12). Thus Lemma 3.4 is proved. □

Next, combining the results in Lemma 3.1-3.4, we have the following conclusion:

Lemma 3.5. *There holds the estimate*

$$
\begin{aligned}
&\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}+\|\mathbf{y}_h(\mathbf{u})-\mathbf{y}_h\|_{1;\Omega}+\|\mathbf{y}_h^*(\mathbf{u})-\mathbf{y}_h^*\|_{1;\Omega}\\
&\qquad+\|p_h(\mathbf{u})-p_h\|_{0;\Omega}+\|p_h^*(\mathbf{u})-p_h^*\|_{0;\Omega}+\|\mathbf{t}-\mathbf{t}_h\|_\infty\\
&\le C\Big(\|\mathbf{y}-\mathbf{y}_h(\mathbf{u})\|_{0;\Omega}+\|\mathbf{y}^*-\mathbf{y}_h^*(\mathbf{u})\|_{0;\Omega}+h_U\|\mathbf{u}-\mathcal{P}_U^h\mathbf{u}\|_{0;\Omega}\Big).
\end{aligned}\tag{3.14}
$$

Finally, we give the proof of Theorems 3.1 and 3.2.

Proof. Using the standard finite element analysis for the Stokes equations (for the proof the readers may refer to [12]), we have the following H^1-norm estimates:

$$
\begin{aligned}
&\|\mathbf{y}-\mathbf{y}_h(\mathbf{u})\|_{1;\Omega}+||p-p_h(\mathbf{u})||_{0;\Omega}\le Ch^{l+1}\Big(\|\mathbf{y}\|_{l+2;\Omega}+\|p\|_{l+1;\Omega}\Big),\\
&\|\mathbf{y}^*-\mathbf{y}_h^*(\mathbf{u})\|_{1;\Omega}+\|p^*-p_h^*(\mathbf{u})\|_{0;\Omega}\le Ch^{l+1}\Big(\|\mathbf{y}^*\|_{l+2;\Omega}+\|p^*\|_{l+1;\Omega}\Big),
\end{aligned}\tag{3.15}
$$

and the L^2-norm estimates:

$$
\begin{aligned}
&\|\mathbf{y}-\mathbf{y}_h(\mathbf{u})\|_{0;\Omega}\le Ch^{l+2}\Big(\|\mathbf{y}\|_{l+2;\Omega}+\|p\|_{l+1;\Omega}\Big),\\
&\|\mathbf{y}^*-\mathbf{y}_h^*(\mathbf{u})\|_{0;\Omega}\le Ch^{l+2}\Big(\|\mathbf{y}^*\|_{l+2;\Omega}+\|p^*\|_{l+1;\Omega}\Big).
\end{aligned}\tag{3.16}
$$

Combining these estimates with the results in Lemma 3.5, we derive (3.2)–(3.4). □

4. Numerical Experiments

An augmented Lagrangian method was proposed to solve the state and control constrained optimal control problems by Bergounioux and Kunisch in [3], and they also constructed another method named a primal-dual strategy to solve these problems, which can be seen in [4]. In [25], Liu, Yang and Yuan proposed a project gradient algorithm to deal with the integral state constraint problem. To solve problem (2.6), we use the Arrow-Hurwicz algorithm in our following experiments, which has been studied in [9], [13] and [35]. The Arrow-Hurwicz algorithm is described briefly as follows.

`Arrow-Hurwicz algorithm`

Step 1. `Set` $k=0$ `and fix a step length` $\rho>0$.
`Select initial approximations` $\mathbf{t}^0$ `and` $\mathbf{u}_h^0$.

Step 2. `Set` $l=0$ `and` $\mathbf{u}_h^{k,0}=\mathbf{u}_h^k$.

Step 3. `Solve equations:`

$$\begin{cases} a\big(\mathbf{y}_h^{k,l},\mathbf{w}_h\big)+b\big(p_h^{k,l},\mathbf{w}_h\big)=\big(\mathbf{f}+B\mathbf{u}_h^{k,l},\mathbf{w}_h\big) & \forall\ \mathbf{w}_h\in\mathbf{H}^h,\\ b\big(\mathbf{y}_h^{k,l},q_h\big)=0, & \forall\ q_h\in Q^h,\\ a\big(\mathbf{y}_h^{*k,l},\mathbf{w}_h\big)+b\big(p_h^{*k,l},\mathbf{w}_h\big)-\big((1+\sum_{i=1}^{M}t_i^k)\mathbf{y}_h^{k,l},\mathbf{w}_h\big)=\big(-\mathbf{y}_d,\mathbf{w}_h\big) & \forall\ \mathbf{w}_h\in\mathbf{H}^h,\\ b\big(\mathbf{y}_h^{*k,l},q_h\big)=0, & \forall\ q_h\in Q^h. \end{cases}$$

Step 4. `Let` $\mathbf{u}_h^{k,l+1}=\mathbf{u}_h^{k,l}-\mathcal{P}_U^h\big(B^*\mathbf{y}_h^{*k,l}+\alpha(\mathbf{u}_h^{k,l}-\mathbf{u}_0)\big)$.
`If` $\|\mathbf{u}_h^{k,l+1}-\mathbf{u}_h^{k,l}\|_{0;\Omega}>\mathtt{Tol}_U$, set $l=l+1$ and then go to **Step 3**.

Step 5. `For all` $i=1,...,M$, let

$$t_i^{k+1}=\begin{cases} \max\Big\{0,t_i^k+\rho\big(\|y_h^{k,l}\|_{0;\Omega_i}-\gamma_i\big)\Big\} & \texttt{in }\ \Omega_i\\ 0 & \texttt{elsewhere}. \end{cases}$$

Step 6. `Stop if` $\|\mathbf{t}^{k+1}-\mathbf{t}^k\|_\infty<\mathtt{Tol}_T$ and output

$$\mathbf{u}_h=\mathbf{u}_h^{k,l},\quad \mathbf{y}_h=\mathbf{y}_h^{k,l},\quad p_h=p_h^{k,l}.$$

. `Otherwise let` $\mathbf{u}_h^{k+1}=\mathbf{u}_h^{k,l+1}$, `set` $k:=k+1$. `Then go to` **step 2**.

For the proof of the convergence of the above algorithm, the readers may refer to [9, 13] or [35].

In this section, we perform some numerical experiments to verify the theoretical results derived in Section 3. In these numerical experiments, we use the `C++` software package: `AFEpack`, the readers may read [30] or browse `http://www.acm.caltech.edu/rli/AFEPack` for more details.

Let $\Omega=(0,1)\times(0,1)$ and $\mathbf{K}=\big\{\mathbf{w};\ \|\mathbf{w}\|_{0;\Omega}\le 1\big\}$, we investigate the problem as follows:

$$\min_{\mathbf{y}\in\mathbf{K}}\Big\{\frac{1}{2}\int_\Omega|\mathbf{y}-\mathbf{y}_d|^2+\frac{1}{2}\int_\Omega|\mathbf{u}-\mathbf{u}_0|^2\Big\},\tag{4.1}$$

subject to

$$\begin{cases}-\dfrac{1}{10}\Delta\mathbf{y}+\nabla p=\mathbf{f}+\mathbf{u}, & \text{in }\Omega,\\ \nabla\cdot\mathbf{y}=0, & \text{in }\Omega,\\ \mathbf{y}=\mathbf{0}, & \text{on }\partial\Omega,\end{cases}\tag{4.2}$$

which means that $d=2$, $\alpha=1$, $\nu=0.1$, $M=1$ and $\gamma=1$ in problem (2.6). In the following experiments, we adopt the exact solution as:

$$\begin{cases}y_1=1000x_1^2(x_1-1)^2(2x_2-1)(x_2^2-x_2)/C_0,\\ y_2=-1000(2x_1-1)(x_1^2-x_1)x_2^2(x_2-1)^2/C_0,\\ p=1000(x_1x_2-0.25),\quad u_1=u_2=100\sin(4\pi x_1)\sin(4\pi x_2),\\ \mathbf{y}^*=-0.1\mathbf{y},\quad p^*=-100(x_1x_2-0.25),\quad t=(C_0-1),\end{cases}\tag{4.3}$$

where $C_0=3.8880789567826111$ such that $\|\mathbf{y}\|_{0;\Omega}=1$. The right-hand term is given by $\mathbf{f}=-0.1\Delta\mathbf{y}+\nabla p-\mathbf{u}$ and $\mathbf{y}_d=0.1\Delta\mathbf{y}^*-\nabla p^*+C_0\mathbf{y}$, and $\mathbf{u}_0=\mathbf{y}^*+\mathbf{u}$.

We perform two groups of numerical experiments, in which we compute all the variables on one mesh in Experiment 1 and on multi-mesh in Experiment 2, respectively. For abbreviation, we denote the L^2-norm, H^1-norm and the $\mathbf{L}^2$-norm, $\mathbf{H}^1$-norm defined in the domain Ω by $\|\cdot\|_0$, $\|\cdot\|_1$ in below, respectively.

4.1. Numerical experiment 1: on uniform mesh

In the first experiment, we check the convergence rates to verify the *a priori error* estimates given in Section 3. We solve the problem in three cases. In these cases we use Hood-Taylor elements to approximate the Stokes equations, and the piecewise quadratic, linear, constant elements to approximate the control, respectively. Namely, we want to confirm the convergence rates with respect to $k=2,1$ and 0 in Theorems 3.1 and 3.2.

Table 4.1: Numerical results of Example 1 ($l=1,\ k=2$).

Variable - element	$\mathbf{y}_h,\mathbf{y}_h^*$ - $\mathbf{P}_2$ element, p_h,p_h^* - P_1 element, $\mathbf{u}_h$ - $\mathbf{P}_2$ element			
mesh	mesh1	mesh2	mesh3	mesh4
h	0.05	0.025	0.0125	0.00625
DOFs: states	8794	34016	134446	534566
$\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_0$	7.38e-05	1.35e-05	8.53e-07	4.45e-08
$\|\mathbf{y}-\mathbf{y}_h\|_0$	2.89e-03	2.47e-04	3.04e-05	2.76e-06
$\|\mathbf{y}^*-\mathbf{y}_h^*\|_0$	7.38e-05	1.35e-05	8.53e-07	4.45e-08
$\lvert t-t_h\rvert$	3.40e-04	7.18e-05	4.53e-06	2.36e-07
$\|\mathbf{u}-\mathbf{u}_h\|_0$	1.41e-01	1.91e-02	2.44e-03	3.08e-04
$\|\mathbf{y}-\mathbf{y}_h\|_1$	3.94e-01	6.68e-02	1.38e-02	2.52e-03
$\|\mathbf{y}^*-\mathbf{y}_h^*\|_1$	1.55e-03	1.53e-04	1.94e-05	3.28e-06
$\|p-p_h\|_0$	2.32e-01	5.78e-02	1.44e-02	3.60e-03
$\|p^*-p_h^*\|_0$	2.39e-04	5.80e-05	1.44e-05	3.60e-06

Example 1. Firstly, we use the piecewise quadratic elements to approximate the control, i.e., $l = 1$ and $k = 2$. The numerical results are listed in Table 4.1.

From Table 4.1, it is easy to calculate the convergence rates, which are listed in Table 4.2.

$$\begin{aligned}&\|\mathbf{y}-\mathbf{y}_h\|_0+\|\mathbf{y}^*-\mathbf{y}_h^*\|_0+\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_0+|t-t_h|=\mathcal{O}(h^{l+2}+h^{k+2})=\mathcal{O}(h^3),\\&\|\mathbf{y}-\mathbf{y}_h\|_1+\|\mathbf{y}^*-\mathbf{y}_h^*\|_1\le\mathcal{O}(h^{l+1}+h^{k+2})=\mathcal{O}(h^2),\end{aligned}$$

which are consistent with the *a priori* error estimates given in Section 3.

Table 4.2: Convergence rates of numerical Example 1.

Mesh	$\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_0$	$\|\mathbf{y}-\mathbf{y}_h\|_0$	$\|\mathbf{y}^*-\mathbf{y}_h^*\|_0$	$\|t-t_h\|$	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$\|\mathbf{y}-\mathbf{y}_h\|_1$	$\|\mathbf{y}^*-\mathbf{y}_h^*\|_1$
$1\to 2$	2.44	3.54	2.44	2.24	2.88	2.56	3.33
$2\to 3$	3.98	3.02	3.98	3.98	2.96	2.27	2.98
$3\to 4$	4.26	3.45	4.26	4.25	2.98	2.45	2.56

Example 2. Next, we approximate the control $\mathbf{u}$ by the piecewise linear elements. The numerical results provided in Table 4.3, from which the convergence rates are obtained and listed in Table 4.4.

Table 4.3: Numerical results of Example 2 ($l = 1,\ k = 1$).

Variable - element	$\mathbf{y}_h, \mathbf{y}_h^*$ - $\mathbf{P}_2$ element, p_h, p_h^* - P_1 element, $\mathbf{u}_h$ - $\mathbf{P}_1$ element			
mesh	mesh1	mesh2	mesh3	mesh4
h	0.05	0.025	0.0125	0.00625
$\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_0$	7.21e-04	7.09e-05	5.64e-06	5.07e-07
$\|\mathbf{y}-\mathbf{y}_h\|_0$	6.14e-03	4.41e-04	3.96e-05	3.38e-06
$\|\mathbf{y}^*-\mathbf{y}_h^*\|_0$	7.25e-04	7.09e-05	5.64e-06	5.07e-07
$\|t-t_h\|$	5.75e-03	2.57e-04	2.19e-05	2.06e-06
$\|\mathbf{u}-\mathbf{u}_h\|_0$	1.70e+0	4.01e-01	9.91e-02	2.46e-02
$\|\mathbf{y}-\mathbf{y}_h\|_1$	4.21e-01	6.82e-02	1.39e-02	2.53e-03
$\|\mathbf{y}^*-\mathbf{y}_h^*\|_1$	1.19e-02	9.05e-04	6.85e-05	6.36e-06
$\|p-p_h\|_0$	2.33e-01	5.78e-02	1.44e-02	3.60e-03
$\|p^*-p_h^*\|_0$	4.81e-04	6.58e-05	1.46e-05	3.60e-06

Table 4.4: Convergence rates of numerical Example 2.

Mesh	$\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_0$	$\|\mathbf{y}-\mathbf{y}_h\|_0$	$\|\mathbf{y}^*-\mathbf{y}_h^*\|_0$	$\|t-t_h\|$	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$\|\mathbf{y}-\mathbf{y}_h\|_1$	$\|\mathbf{y}^*-\mathbf{y}_h^*\|_1$
$1\to 2$	3.34	3.80	3.35	4.48	2.08	2.62	3.72
$2\to 3$	3.65	3.47	3.65	3.55	2.01	2.29	3.72
$3\to 4$	3.47	3.54	3.47	3.41	2.00	2.45	3.43

By the above two examples, it can be seen that $\|\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}=\mathcal{O}(h^{k+1})$, where $k = 2$ in Example 1 and $k = 1$ in Example 2, respectively. At the same time,

$$\|\mathbf{y}-\mathbf{y}_h\|_{0;\Omega}+\|\mathbf{y}^*-\mathbf{y}_h^*\|_{0;\Omega}+\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}+|t-t_h|=\mathcal{O}(h^3)$$

whatever $\|\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}=\mathcal{O}(h^3)$ or $\|\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}=\mathcal{O}(h^2)$, which are consistent with Theorem 3.2.

In these numerical results, it is interesting to see that the convergence order between t and t_h is better than the theoretical results obtained in Section 3. That may lead to good approximations of $\|\mathbf{y}^*-\mathbf{y}_h^*\|_{0;\Omega}+\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}$ in the above two examples. It is not clear that whether there exist some super-convergence or this is caused by some other reasons. However, the approximation of $\mathbf{t}$ determines the approximation of $\mathbf{y}^*$ directly, because $\mathbf{t}$ is the right-hand side term of of the equation for the co-state equation of $\mathbf{y}^*$. To further examine that, one can see the next example, where we use the piecewise constant elements to discretize $\mathbf{u}_h$.

For abbreviation, the convergence rates of $\|p-p_h\|_{0;\Omega}$ and $\|p^*-p_h^*\|_{0;\Omega}$ are omitted in all tables, it is obvious that they are the same as the orders of $\|\mathbf{y}-\mathbf{y}_h\|_{1;\Omega}$ and $\|\mathbf{y}^*-\mathbf{y}_h^*\|_{1;\Omega}$, respectively.

Example 3. Finally, let us use the piecewise constant elements to approximate the control $\mathbf{u}$. The numerical results are given in Table 4.5. From Table 4.5, it is easy to obtain the convergence rates, see Table 4.6.

Table 4.5: Numerical results of Example 3 ($l=1,\ k=0$).

variable - element	$\mathbf{y}_h,\mathbf{y}_h^*$ - $\mathbf{P}_2$ element, p_h,p_h^* - P_1 element, $\mathbf{u}_h$ - $\mathbf{P}_0$ element			
mesh	mesh1	mesh2	mesh3	mesh4
h	0.05	0.025	0.0125	0.00625
$\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_0$	7.81e-03	2.11e-03	4.91e-04	1.20e-04
$\|\mathbf{y}-\mathbf{y}_h\|_0$	4.38e-02	1.08e-02	2.69e-03	6.73e-04
$\|\mathbf{y}^*-\mathbf{y}_h^*\|_0$	7.88e-03	2.12e-03	4.92e-04	1.20e-04
$\|t-t_h\|$	2.95e-02	7.91e-03	1.63e-03	3.88e-04
$\|\mathbf{u}-\mathbf{u}_h\|_0$	1.27e+1	6.42e+0	3.21e+0	1.60e+0
$\|\mathbf{y}-\mathbf{y}_h\|_1$	8.93e-01	2.06e-01	5.04e-02	1.23e-02
$\|\mathbf{y}^*-\mathbf{y}_h^*\|_1$	1.06e-01	2.70e-02	6.57e-03	1.63e-03
$\|p-p_h\|_0$	2.71e-01	6.68e-02	1.66e-02	4.16e-03
$\|p^*-p_h^*\|_0$	3.89e-03	9.71e-04	2.39e-04	5.94e-05

Table 4.6: Convergence rates of numerical Example 3.

mesh	$\|\mathcal{P}_U^h\mathbf{u}-\mathbf{u}_h\|_0$	$\|\mathbf{y}-\mathbf{y}_h\|_0$	$\|\mathbf{y}^*-\mathbf{y}_h^*\|_0$	$\|t-t_h\|$	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$\|\mathbf{y}-\mathbf{y}_h\|_1$	$\|\mathbf{y}^*-\mathbf{y}_h^*\|_1$
$1\to 2$	1.88	2.01	1.89	1.89	0.99	2.11	1.97
$2\to 3$	2.10	2.00	2.10	2.27	0.99	2.03	2.03
$3\to 4$	2.02	2.00	2.02	2.07	0.99	2.02	2.00

From the above three examples, it can be seen that $\|\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}=\mathcal{O}(h^{k+1})$ with respect to $k=2,1,0$, respectively. At the same time,

$$\|\mathbf{y}-\mathbf{y}_h\|_{1;\Omega}+\|p-p_h\|_{0;\Omega}+\|\mathbf{y}^*-\mathbf{y}_h^*\|_{1;\Omega}+\|p^*-p_h^*\|_{0;\Omega}=\mathcal{O}(h^2)$$

whatever $\|\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}=\mathcal{O}(h^3), \|\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}=\mathcal{O}(h^2)$ or $\|\mathbf{u}-\mathbf{u}_h\|_{0;\Omega}=\mathcal{O}(h)$, which coincides with the theoretical results in Section 3.

4.2. Numerical experiment 2: on multi-mesh

In this experiment we consider the case of using the multi-mesh, on which we use a coarse mesh to approximate the state and co-state, and another dense mesh to approximate the control. This strategy can save much computational work since most of the calculation is to solve the state equations and co-state equations repeatedly. In fact, a precise optimal control arouses more of our interests in applications, so there is no point to over-compute the state variables.

Example 4. For the sake of comparison, we compute the same problem as the above, and use the piecewise quadric elements to approximate the control as in Example 1, but adopt two sets of meshes. In the meantime, we use the same set of mesh as in Example 1 to approximate the control, but another set of coarser mesh to approximate the states.

The numerical results are listed in Table 4.7. It is seen from Table 4.7 that the number of DOFs of the state-mesh reduces substantially in comparison with the data corresponding to Example 1 (Table 4.1). At the same time, the accuracy of the control is kept (when $h \leq 0.025$), so it is clear that much computational work is saved.

Table 4.7: Numerical results of Example 4 ($l = 1,\ k = 2$ on multi-mesh).

variable - element	$\mathbf{y}_h, \mathbf{y}_h^*$ - $\mathbf{P}_2$ element, p_h, p_h^* - P_1 element, $\mathbf{u}_h$ - $\mathbf{P}_2$ element			
mesh	multi-mesh1	multi-mesh2	multi-mesh3	multi-mesh4
h	0.2	0.1	0.05	0.025
DOFs: states	664	2330	8794	34016
h_U	0.05	0.025	0.0125	0.00625
DOFs: control	2065	7713	30145	118721
$\lVert\mathcal{P}_U^h\mathbf{u} - \mathbf{u}_h\rVert_0$	3.63e-01	3.95e-03	7.23e-05	1.15e-05
$\lVert\mathbf{y} - \mathbf{y}_h\rVert_0$	3.28e-01	3.35e-02	2.89e-03	2.47e-04
$\lVert\mathbf{y}^* - \mathbf{y}_h^*\rVert_0$	3.63e-01	3.95e-03	7.23e-05	1.15e-05
$\lvert t - t_h\rvert$	2.24e+0	2.25e-02	3.31e-04	6.31e-05
$\lVert\mathbf{u} - \mathbf{u}_h\rVert_0$	3.91e-01	2.00e-02	2.42e-03	3.08e-04
$\lVert\mathbf{y} - \mathbf{y}_h\rVert_1$	1.14e+1	2.27e+0	3.94e-01	6.68e-02
$\lVert\mathbf{y}^* - \mathbf{y}_h^*\rVert_1$	2.74e+0	3.85e-02	1.54e-03	1.45e-04
$\lVert p - p_h\rVert_0$	3.79e+0	9.57e-01	2.32e-01	5.78e-02
$\lVert p^* - p_h^*\rVert_0$	9.51e-02	1.86e-03	2.39e-04	5.80e-05

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