Elliptic Systems with a Partially Sublinear Local Term

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Abstract. Let $1 < p < 2$. Under some assumptions on $V, K$, existence of infinitely many solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is proved for the Schrodinger-Poisson system

\[
\begin{cases}
-\Delta u + V(x)u + \phi u = K(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2 & \text{in } \mathbb{R}^3
\end{cases}
\]

as well as for the Klein-Gordon-Maxwell system

\[
\begin{cases}
-\Delta u + [V(x) - (\omega + e\phi)^2]u = K(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\
-\Delta \phi + e^2 u^2 \phi = -\epsilon \omega u^2 & \text{in } \mathbb{R}^3,
\end{cases}
\]

where $\omega, \epsilon > 0$. This is in sharp contrast to D’Aprile and Mugnai’s non-existence results.

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Key words: Schrodinger-Poisson system, Klein-Gordon-Maxwell system, infinitely many solutions.

1 Introduction and main results

In this paper, we study existence of infinitely many solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ to the Schrodinger-Poisson system

\[
\begin{cases}
-\Delta u + V(x)u + \phi u = K(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2 & \text{in } \mathbb{R}^3
\end{cases}
\]

for $1 < p < 2$.

This system has a wide background in physics. It is reduced from the Hartree-Fock equations by a mean field approximation ([9, 10]). It also describes the Klein-Gordon or
Schrödinger fields interacting with an electromagnetic field ([3]). The related Thomas-Fermi-von Weizsäcker model describes the ground states of nonrelativistic atoms and molecules in the quantum mechanics ([1]).

In [2], D’Aprile and Mugnai prove that if \( V \equiv 1 \equiv K \) then (1.1) has no nontrivial solution. In the present paper we prove that if \( V \) is a potential well and \( K \) is positive somewhere in \( \mathbb{R}^3 \) then (1.1) has infinitely many nontrivial solutions. To be more precise, as a special case of our main results, we will show that the system has infinitely many solutions provided that \( V, K \in C(\mathbb{R}^3, \mathbb{R}), \inf V > -\infty \), there is \( R > 0 \) such that \( V(x) > 0 \) for \( |x| \geq R \), \( \int_{|x|\geq R} V^{-1} < \infty \), \( K \) is bounded, and there exists \( x_0 \in \mathbb{R}^3 \) such that \( K(x_0) > 0 \). In fact, one of our main theorems states a much more general result for a more general system.

We will consider the more general system

\[
\begin{align*}
-\Delta u + V(x)u + \phi u &= f(x, u) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^3.
\end{align*}
\]

(1.2)

To state our main result, we need the following assumptions:

- (V) \( V \in C(\mathbb{R}^3, \mathbb{R}), \inf V > -\infty \), there is \( R > 0 \) such that \( V(x) > 0 \) for \( |x| \geq R \), \( \int_{|x|\geq R} V^{-1} < \infty \).

- (F) There exist positive numbers \( \delta \) and \( c \) and \( p \in (1, 2) \) such that \( f \in C(\mathbb{R}^3 \times [-\delta, \delta], \mathbb{R}) \), \( f(x, t) \) is odd in \( t \),

\[
|f(x, t)| \leq c|t|^{p-1} \quad \text{for } |t| \leq \delta,
\]

and there exist \( x_0 \in \mathbb{R}^3 \) and \( r > 0 \) such that

\[
\lim_{t \to 0} F(x, t)/t^2 = \infty
\]

uniformly in \( x \in B_r(x_0) := \{ x \in \mathbb{R}^3 \mid |x-x_0| < r \} \), where \( F(x, t) = \int_0^t f(x, s)ds \).

**Theorem 1.1.** Under (V) and (F), (1.2) has infinitely many nontrivial solutions in \( H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \).

Assumption (V) makes \( V \) look like a well-shaped potential. Note that the nonlinear term \( f(x, t) \) in assumption (F) is defined only for \( |t| \leq \delta \). Accordingly, the \( L^\infty(\mathbb{R}^3) \) norm of \( u \) in \( (u, \phi) \), the solution we will obtain, will have to be less than \( \delta \).

From (V) and (F), it is without loss of any generality to assume further in Theorem 1.1 that

\[
\inf V > 0 \quad \text{and} \quad \int_{\mathbb{R}^3} V^{-1} < \infty.
\]

(1.3)
This can be seen by adding $-\nu u$ to both sides of the first equation in (1.2), where $\nu$ is any number such that $\nu < \inf V$. The assumption that $\inf V \geq 1$ and $\int_{\mathbb{R}^3} V^{-1} < \infty$ was used in [6] in dealing with sublinear Schrödinger equations.

The following corollary is a direct consequence of Theorem 1.1.

**Corollary 1.1.** Under $(V)$, if $K \in C(\mathbb{R}^3, \mathbb{R})$, $K$ is bounded, and there exists $x_0 \in \mathbb{R}^3$ such that $K(x_0) > 0$, then (1.1) has infinitely many nontrivial solutions in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

Corollary 1.1 is in sharp contrast to the non-existence result for (1.1) in [2] which asserts that (1.1) has no nontrivial solution if $V \equiv 1 \equiv K$.

A system similar to (1.1) is the following system of coupled Klein-Gordon-Maxwell equations

$$
\begin{aligned}
-\Delta u + [V(x) - (\omega + e\phi)^2]u &= K(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi + e^2 u^2 \phi &= -e\omega u^2 \quad \text{in } \mathbb{R}^3,
\end{aligned}
$$

where $1 < p < 2$ and $\omega, e > 0$. The case $V \equiv m^2$ and $K \equiv 1$ is studied in [2], where one can find the physical meaning of the positive constants $m, e, \omega$ and the physical background of (1.4).

Our second main result is for the more general system

$$
\begin{aligned}
-\Delta u + [V(x) - (\omega + e\phi)^2]u &= f(x,u) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi + e^2 u^2 \phi &= -e\omega u^2 \quad \text{in } \mathbb{R}^3,
\end{aligned}
$$

**Theorem 1.2.** Under $(V)$ and $(F)$, (1.5) has infinitely many nontrivial solutions in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

As remarked above for Theorem 1.1, by adding $(-\nu + \omega^2)u$ to both sides of the first equation in (1.5) where $\nu$ is any number such that $\nu < \inf V$, we can assume in addition in Theorem 1.2 that

$$
\inf V > \omega^2 \quad \text{and} \quad \int_{\mathbb{R}^3} V^{-1} < \infty.
$$

The following corollary is a direct consequence of Theorem 1.2.

**Corollary 1.2.** Under $(V)$, if $K \in C(\mathbb{R}^3, \mathbb{R})$, $K$ is bounded, and there exists $x_0 \in \mathbb{R}^3$ such that $K(x_0) > 0$, then (1.4) has infinitely many nontrivial solutions in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

It is proved in [2] that if $V \equiv m^2$, $K \equiv 1$, and $m, e, \omega > 0$ then (1.4) has no nontrivial solution. So Corollary 1.2 provides a sharp contrast.

The assumption $\int_{\mathbb{R}^3} V^{-1} < \infty$, which is a crucial assumption in Theorems 1.1 and 1.2 according to D’Aprile and Mugnai’s non-existence results, will only provide compactness in our arguments. This assumption can be replaced with a similar assumption on $K$ when considering (1.1) and (1.4), as illustrated in the following two theorems.
Theorem 1.3. Assume that $V, K \in C(\mathbb{R}^3, \mathbb{R})$, $\inf V > 0$, $K \in L^{s,p} (\mathbb{R}^3)$ for some $2 \leq s \leq 6$, and there exists $x_0 \in \mathbb{R}^3$ such that $K(x_0) > 0$, then (1.1) has infinitely many nontrivial solutions in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

Theorem 1.4. Assume that $V, K \in C(\mathbb{R}^3, \mathbb{R})$, $\inf V > \omega^2$, $K \in L^{s,p} (\mathbb{R}^3)$ for some $2 \leq s \leq 6$, and there exists $x_0 \in \mathbb{R}^3$ such that $K(x_0) > 0$, then (1.4) has infinitely many nontrivial solutions in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

Some preparations will be given in Section 2. Theorem 1.1 will be proved in Section 3 and Theorem 1.2 will be proved in Section 4. We will prove Theorems 1.3 and 1.4 in Sections 5 and 6 respectively. In this paper, $C$ and $C_j$ are positive constants which may be variant even in the same line.

2 Preliminaries

Choose $\hat{f} \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ so that $\hat{f}$ is odd in $t \in \mathbb{R}$, $\hat{f}(x, t) = f(x, t)$ for $|t| < \delta$, and $\hat{f}(x, t) = 0$ for $|t| > 2\delta$. We also assume $\hat{f}$ to satisfy

$$|\hat{f}(x, t)| \leq c|t|^{p-1} \quad \text{for } |t| \leq 2\delta.$$ 

Consider the modified system

$$
\begin{cases}
-\Delta u + V(x)u + \phi u = \hat{f}(x, u) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3.
\end{cases}
$$

Any solution $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ of (2.1) satisfying $\|u\|_{L^\infty(\mathbb{R}^3)} < \delta$ is clearly a solution of (1.2). Therefore it suffices to find infinitely many solutions $(u_n, \phi_n)$ of (2.1) with $\|u_n\|_{L^\infty(\mathbb{R}^3)} \to 0$. The same remark holds for (1.5) and its modification

$$
\begin{cases}
-\Delta u + [V(x) - (\omega + e\phi)^2]u = \hat{f}(x, u) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi + e^2 u^2 = -e\omega u^2 \quad \text{in } \mathbb{R}^3.
\end{cases}
$$

We will work in the Banach space $E$ defined to be

$$E = \left\{ u \mid \int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) < \infty \right\},$$

in which the norm is

$$\|u\| = \left( \int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) \right)^{1/2}.$$ 

From $\inf V > 0$ and $\int_{\mathbb{R}^3} V^{-1} < \infty$, it can be deduced that

$$\text{mes}\left\{ \{ x \in B_r(y) : V(x) \leq M \} \right\} \to 0 \quad \text{as } |y| \to \infty,$$
for any \( M > 0 \) and \( r > 0 \). Then we may apply a result of [8] (or [4, Corollary 6.2]) to conclude that the embedding \( E \hookrightarrow L^2(\mathbb{R}^3) \) is compact. However, the assumption \( \inf V > 0 \) and \( \int_{\mathbb{R}^3} V^{-1} < \infty \) implies compactness for a larger interval of exponents.

**Lemma 2.1.** If (1.3) is satisfied, then the embedding \( E \hookrightarrow L^s(\mathbb{R}^3) \) is compact for any \( 1 \leq s < 6 \).

**Proof.** It suffices to prove the result for \( s = 1 \). Assume \( u_n \rightharpoonup u \) weakly in \( E \). For any \( R > 0 \), write
\[
\int_{|x| > R} |u_n - u| \leq \left( \int_{|x| > R} V|u_n - u|^2 \right)^{1/2} \left( \int_{|x| > R} V^{-1} \right)^{1/2} = o_R(1),
\]
where \( o_R(1) \) is a quantity that converges to 0 as \( R \to \infty \) uniformly in \( n \). Then \( u_n \to u \) strongly in \( L^1(\mathbb{R}^3) \) since \( u_n \to u \) in \( L^1_{\text{loc}}(\mathbb{R}^3) \).

Since \( E \hookrightarrow L^{12/5}(\mathbb{R}^3) \), it follows from the Riesz representation theorem that for any \( u \in E \) there is a unique \( \phi = \phi_u \in D^{1,2}(\mathbb{R}^3) \) such that the second equation in (2.1) is solved. This \( \phi_u \) has an explicit representation
\[
\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy.
\]

We insert this \( \phi_u \) into the first equation in (2.1). Then (2.1) can be rewritten as
\[
-\Delta u + V(x)u + \phi_u u = \hat{f}(x,u) \quad \text{in} \quad \mathbb{R}^3.
\]
Solutions of (2.6) will be found via critical point theory. Set
\[
\hat{F}(x,t) = \int_0^t \hat{f}(x,s) ds.
\]

The functional associated with (2.6) is the functional \( J \) defined to be
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} \hat{F}(x,u)
\]
for \( u \in E \).

To convert (2.2) to a single equation, we recall the following result from [2].

**Lemma 2.2.** For any \( u \in E \), there is a unique \( \phi = \phi_u \in D^{1,2}(\mathbb{R}^3) \) which solves the second equation in (2.2). This \( \phi_u \) satisfies
\[
\frac{\omega}{e} \leq \phi_u \leq 0.
\]
Moreover, the map \( T : E \to D^{1,2}(\mathbb{R}^3) \) defined to be \( Tu = \phi_u \) is \( C^1 \) and for any \( u, v \in E \),
\[
(T'(u), v) = 2e(\Delta - e^2 u^2)^{-1}(\omega + e\phi_u)uv.
\]
Inserting $\tilde{\phi}_u$ into the second equation in (2.2) converts it into the single equation
\[-\Delta u + [V(x) - (\omega + e\tilde{\phi}_u)^2]u = \hat{f}(x,u) \quad \text{in } \mathbb{R}^3. \tag{2.9}\]

The functional associated with (2.9) is
\[
\tilde{J}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V u^2) - \frac{1}{2} \int_{\mathbb{R}^3} (\omega^2 + e\omega \tilde{\phi}_u)u^2 - \int_{\mathbb{R}^3} \hat{f}(x,u) \tag{2.10}\]

for $u \in E$. Since $|\hat{f}(x,t)| \leq c|t|^{p-1}$, we have
\[
\left| \int_{\mathbb{R}^3} \hat{f}(x,u) \right| \leq C \int_{\mathbb{R}^3} |u|^p \leq C \left( \int_{\mathbb{R}^3} V^{-\frac{1}{p-2}} \right)^{\frac{p-2}{2}} \left( \int_{\mathbb{R}^3} V u^2 \right)^{\frac{p}{2}} \leq C \|u\|^p. \tag{2.11}\]

Here we have used $\int_{\mathbb{R}^3} V^{-1} < \infty$ and the inequality
\[
V^{-\frac{1}{p-2}} \leq CV^{-1}, \tag{2.12}\]

which is a consequence of $1 < p < 2$ and the fact that $\inf V > 0$ in (2.6) and $\inf V > \omega^2$ in (2.9). Therefore $J$ and $\tilde{J}$ are coercive and bounded below. This property makes $J$ and $\tilde{J}$ into the framework of a classical critical point theorem due to Clark, which we state as follows.

**Theorem A.** Let $X$ be a Banach space, $\Phi \in C^1(X,\mathbb{R})$. Assume $\Phi$ satisfies the (PS) condition, is even and bounded below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exist a $k$-dimensional subspace $X_k$ of $X$ and $\rho_k > 0$ such that $\sup_{X_k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{u \in X : \|u\| = \rho\}$, then $\Phi$ has a sequence of negative critical values converging to 0.

Theorem A will be used to prove Theorems 1.3 and 1.4. However, Theorem A can not be applied to $J$ and $\tilde{J}$ to obtain the desired critical points since we need to have critical points with sufficiently small norms which Theorem A does not provide. Instead, we will use the following theorem which is proved in [7].

**Theorem B.** Under the same assumptions as in Theorem A, $\Phi$ has a sequence of nonzero critical points converging to 0.

We will use Theorem B to prove Theorems 1.1 and 1.2. To do this, we will verify that $J$ and $\tilde{J}$ satisfy the assumptions of Theorem A. Then, according to Theorem B, $J$ (and $\tilde{J}$ also) has infinitely many critical points $\{u_n\}$ with $\|u_n\| \to 0$. Then the Moser iteration technique will be used to show that $\|u_n\|_{L^\infty(\mathbb{R}^3)} \to 0$.

## 3 Proof of Theorem 1.1

To prove Theorem 1.1, we will apply Theorem B to prove that $J$ has a sequence of critical points converging to 0 in $E$ and then we will prove that this sequence also converges to 0 in $L^\infty(\mathbb{R}^3)$.

We first verify the assumptions of Theorems A. Clearly, $J$ is a $C^1$ functional, $J$ is even and bounded below, and $J(0) = 0$. 


Lemma 3.1. J satisfies the (PS) condition. That is, any sequence \{u_n\} such that J(u_n) is bounded and J'(u_n) → 0 has a converging subsequence.

Proof. Since J(u_n) is bounded, it is clear that \{u_n\} is bounded. Therefore, we may extract a subsequence, still denoted by \{u_n\}, such that

\begin{align*}
  u_n &\to u \text{ weakly in } E, \quad (3.1a) \\
  u_n &\to u \text{ a.e. on } \mathbb{R}^3. \quad (3.1b)
\end{align*}

Choose \( a \geq \frac{6}{5} \) such that \( 1 < a(p-1) < 6 \). Since, from the embedding \( E \hookrightarrow L^a(p-1)(\mathbb{R}^3) \),

\[ \int_{\mathbb{R}^3} |\hat{f}(x,u_n)|^a \leq C \int_{\mathbb{R}^3} |u_n|^{a(p-1)} \leq C, \]

we may assume that \( \hat{f}(x,u_n) \to \hat{f}(x,u) \) weakly in \( L^a(\mathbb{R}^3) \). (3.2)

From \( -\Delta \phi_{u_n} = u_n^2 \) and \( -\Delta \phi_u = u^2 \), using the Hölder inequality, we have

\[ \int_{\mathbb{R}^3} |\nabla (\phi_{u_n} - \phi_u)|^2 = \int_{\mathbb{R}^3} (u_n^2 - u^2) (\phi_{u_n} - \phi_u) \]

\[ \leq \|u_n - u\|_{L^a(\mathbb{R}^3)} \|u_n + u\|_{L^{a(p-1)}(\mathbb{R}^3)} \|\phi_{u_n} - \phi_u\|_{L^6(\mathbb{R}^3)}. \]

Since, from Lemma 2.1, \( u_n \to u \) strongly in \( L^{\frac{6}{5}}(\mathbb{R}^3) \), we see that

\[ \phi_{u_n} \to \phi_u \text{ strongly in } D^{1,2}(\mathbb{R}^3) \text{ and } L^6(\mathbb{R}^3). \] (3.3)

For any \( v \in E \), taking limit as \( n \to \infty \) in

\[ \langle J'(u_n), v \rangle = \int_{\mathbb{R}^3} (\nabla u_n \nabla v + Vu_n v) + \int_{\mathbb{R}^3} \phi_{u_n} u_n v - \int_{\mathbb{R}^3} \hat{f}(x,u_n) v, \]

we use (3.1a)-(3.3) to see that

\[ \int_{\mathbb{R}^3} (\nabla u \nabla v + Vu v) + \int_{\mathbb{R}^3} \phi_u u v - \int_{\mathbb{R}^3} \hat{f}(x,u) v = 0. \]

Therefore,

\[ \langle J'(u_n), u_n - u \rangle \]

\[ = \|u_n - u\|^2 + \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) - \int_{\mathbb{R}^3} (\hat{f}(x,u_n) - \hat{f}(x,u))(u_n - u). \]

Then it is easy to see that \( \|u_n - u\| \to 0. \)
Lemma 3.2. For any $k \in \mathbb{N}$, there exist a $k$-dimensional subspace $E^k$ of $E$ and $\rho_k > 0$ such that

$$\sup_{E^k \cap S_{\rho_k}} J < 0. \tag{3.5}$$

Proof. Choose, for $j=1, \cdots, k$, $v_j \in C_0^\infty(\mathbb{R}^3)$ such that $v_j \neq 0$, $\text{supp} v_j \subset B_r(x_0)$, $\text{supp} v_i \cap \text{supp} v_j = \emptyset$ for $i \neq j$.

Let $E^k = \text{span}\{v_1, v_2, \cdots, v_k\}$. Choose positive numbers $\xi$ and $\tau$ such that

$$\xi \|u\|_{L^2(\mathbb{R}^3)}^2 \geq \|u\|^2, \quad \tau \|u\| \geq \|u\|_{L^\infty(\mathbb{R}^3)} \tag{3.6}$$

for any $u \in E^k$. From assumption (F), we find $\mu > 0$ such that

$$\hat{F}(x, t) \geq \xi t^2 \quad \text{for } x \in B_r(x_0) \text{ and } |t| < \mu.$$

Let $u \in E^k$ and $\|u\| = 1$. For $0 < \rho < \mu/\tau$, we have

$$J(\rho u) = \frac{\rho^2}{2} + \frac{\rho^4}{4} \int_{B_r(x_0)} \phi u^2 - \int_{B_r(x_0)} \hat{F}(x, \rho u)$$

$$\leq \frac{\rho^2}{2} + \frac{\rho^4}{4} \int_{B_r(x_0)} \phi u^2 - \xi \rho^2 \int_{B_r(x_0)} u^2$$

$$\leq -\frac{\rho^2}{2} + \frac{\rho^4}{4} \int_{B_r(x_0)} \phi u^2. \tag{3.7}$$

To obtain (3.5) it suffices to choose $\rho = \rho_k$ small enough.

Proof of Theorem 1.1. According to Lemmas 3.1 and 3.2, all the assumptions in Theorem A are satisfied. By Theorem B, $J$ has a sequence of critical points $\{u_n\}$ converging to 0 in $E$. It suffices to prove that $\|u_n\|_{L^\infty(\mathbb{R}^3)} \to 0$.

Since $u_n$ solves equation (2.6), we have

$$-\Delta u_n + V(x) u_n + \phi_{u_n} u_n = \hat{f}(x, u_n) \quad \text{in } \mathbb{R}^3 \tag{3.8}$$

in the weak sense. We estimate the $L^\infty$ norm of $u_n$ as in [7]. Let $\alpha > 0$, $M > 0$ and set $u_n^M(x) = \max\{-M, \min\{u_n(x), M\}\}$. Multiplying both sides of (3.8) with $|u_n^M|^\alpha u_n^M$ implies

$$\frac{4}{(\alpha+2)^2} \int_{\mathbb{R}^3} |\nabla |u_n^M|^\frac{\alpha+1}{2}|^2 \leq C \int_{\mathbb{R}^3} |u_n^M|^\alpha,$$ 

which together with the Sobolev inequality yields

$$\|u_n^M\|_{L^\infty(\mathbb{R}^3)} \leq (C_1(\alpha+2)^2) \|u_n^M\|_{L^{\frac{\alpha+1}{\alpha+2}}(\mathbb{R}^3)}, \tag{3.10}$$
for some $C_1 \geq 1$ independent of $n$ and $a$. Set $\alpha_0 = 5$ and $\alpha_k = 3(\alpha_{k-1} + 2) - 1$, that is $\alpha_k = \frac{5}{2}(3^k+1-1)$, for $k = 1, 2, \cdots$. From the last inequality, an iterating process leads to

$$\|u_n^M\|_{L^{\alpha_k+1}(\mathbb{R}^3)} \leq \exp\left(\sum_{i=0}^{k} \frac{2\ln(C_1(\alpha_i+2))}{\alpha_i+2}\right) \|u_n^M\|_{L^{\nu_k}(\mathbb{R}^3)}^\nu,$$

(3.11)

where $\nu_k = \prod_{i=0}^{k} (\alpha_i+1)/(\alpha_i+2)$. Sending $M$ to infinity and then $k$ to infinity, as a consequence, we have

$$\|u_n\|_{L^\infty(\mathbb{R}^3)} \leq \exp\left(\sum_{i=0}^{\infty} \frac{2\ln(C_1(\alpha_i+2))}{\alpha_i+2}\right) \|u_n\|_{L^\nu(\mathbb{R}^3)}^\nu,$$

(3.12)

where $\nu = \prod_{i=0}^{\infty} (\alpha_i+1)/(\alpha_i+2)$ is a number in $(0,1)$ and $\exp\left(\sum_{i=0}^{\infty} \frac{2\ln(C_1(\alpha_i+2))}{\alpha_i+2}\right)$ is a positive number. Therefore, $\|u_n\|_{L^\infty(\mathbb{R}^3)} \to 0$ as $n \to \infty$. □

4 Proof of Theorem 1.2

To prove Theorem 1.2, we will apply Theorem B to prove that $\tilde{J}$ has a sequence of critical points converging to 0 in $E$ and then we will prove that this sequence also converges to 0 in $L^\infty(\mathbb{R}^3)$.

Clearly, $\tilde{J}$ is a $C^1$ functional, $\tilde{J}$ is even and bounded below, and $\tilde{J}(0) = 0$.

Lemma 4.1. $\tilde{J}$ satisfies the (PS) condition.

Proof. Suppose that $\{u_n\}$ is a sequence such that $\tilde{J}(u_n)$ is bounded and $\tilde{J}'(u_n) \to 0$. Since $\tilde{J}$ is coercive and $\tilde{J}(u_n)$ is bounded, $\{u_n\}$ is bounded. Therefore, we may extract a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \to u \text{ weakly in } E.$$

(4.1)

Choose $a \geq 6/5$ such that $1 < a(p-1) < 6$. Then

$$\tilde{J}(x,u_n) \to \tilde{J}(x,u) \text{ weakly in } L^a(\mathbb{R}^3).$$

(4.2)

From the equations

$$-\Delta \phi_{u_n} + e^2 u_n^2 \phi_{u_n} = -e\omega u^2_n,$$

$$-\Delta \phi_u + e^2 u^2 \phi_u = -e\omega u^2,$$
we have
\[
\int_{\mathbb{R}^3} |\nabla (\tilde{\phi}_u - \tilde{\phi}_n)|^2 \\
\leq -e^2 \int_{\mathbb{R}^3} (u_n^2 - u^2) \tilde{\phi}_u (\tilde{\phi}_u - \tilde{\phi}_n) - e\omega \int_{\mathbb{R}^3} (u_n^2 - u^2) (\tilde{\phi}_u - \tilde{\phi}_n) \\
\leq e^2 \|u_n - u\|_{L^3(\mathbb{R}^3)} \|u_n + u\|_{L^3(\mathbb{R}^3)} \|\tilde{\phi}_u - \tilde{\phi}_n\|_{L^3(\mathbb{R}^3)} \\
+ e\omega \|u_n - u\|_{L^2(\mathbb{R}^3)} \|u_n + u\|_{L^{3/2}(\mathbb{R}^3)} \|\tilde{\phi}_u - \tilde{\phi}_n\|_{L^3(\mathbb{R}^3)}.
\]

Since, from Lemma 2.1, $u_n \to u$ strongly in $L^{12/5}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$, we see that $\tilde{\phi}_u \to \tilde{\phi}_u$ strongly in $D^{1,2}(\mathbb{R}^3)$ and $L^6(\mathbb{R}^3)$. (4.3)

From Lemma 2.2, for $v \in E$, we see that
\[
\langle J'(u_n), v \rangle = \int_{\mathbb{R}^3} (\nabla u_n \nabla v + Vu_n v) - \int_{\mathbb{R}^3} (\omega + e\tilde{\phi}_u)^2 u_n v - \int_{\mathbb{R}^3} \hat{f}(x, u_n) v.
\]

Letting $n \to \infty$ and using (4.1)-(4.3) to conclude
\[
\int_{\mathbb{R}^3} (\nabla u \nabla v + Vu v) - \int_{\mathbb{R}^3} (\omega + e\tilde{\phi}_u)^2 u v - \int_{\mathbb{R}^3} \hat{f}(x, u) v = 0. 
\]

Therefore,
\[
\langle J'(u_n), u_n - u \rangle = \|u_n - u\|^2 - \int_{\mathbb{R}^3} [(\omega + e\tilde{\phi}_u)^2 u_n - (\omega + e\tilde{\phi}_u)^2 u] |u_n - u| \\
- \int_{\mathbb{R}^3} (\hat{f}(x, u_n) - \hat{f}(x, u)) |u_n - u|.
\]

Then it is easy to see that $\|u_n - u\| \to 0$. □

**Lemma 4.2.** For any $k \in \mathbb{N}$, there exist a $k$-dimensional subspace $E^k$ of $E$ and $\rho_k > 0$ such that
\[
\sup_{E^k \cap S_{\rho_k}} \hat{f} < 0.
\]

**Proof.** Since, by Lemma 2.2,
\[
-\frac{1}{2} \int_{\mathbb{R}^3} (\omega^2 + e\omega \tilde{\phi}_u) u^2 \leq 0,
\]
we have
\[
J(u) \leq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) - \int_{\mathbb{R}^3} \hat{f}(x, u).
\]

The proof is then the same as that of Lemma 3.2. □

**Proof of Theorem 1.2.** We use Lemmas 4.1 and 4.2 and the same argument as in the proof of Theorem 1.1. □
5 Proof of Theorem 1.3

The functional associated with (1.1) is the functional $I$ defined to be
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} K(x)|u|^p \tag{5.1}
\]
for $u \in E$.

In order to prove Theorem 1.3, we will apply Theorem A to prove that $I$ has a sequence of negative critical values converging to 0.

We first verify the assumptions of Theorems A. Clearly, $I$ is a $C^1$ functional, $I$ is even, and $I(0) = 0$. 

Lemma 5.1. $I$ is coercive and bounded below.

Proof. By the Hölder inequality and $\phi_u \geq 0$, we have
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} K(x)|u|^p \\
\geq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) - \left( \int_{\mathbb{R}^3} |K(x)|^{\frac{1}{s-p}} \right)^{\frac{s-p}{s}} \left( \int_{\mathbb{R}^3} |u|^s \right)^{\frac{p}{s}}. \tag{5.2}
\]
Since $K(x) \in L^{\frac{6}{s-p}}(\mathbb{R}^3)$ and $2 \leq s \leq 6$, using the Sobolev imbedding $E \hookrightarrow L^s(\mathbb{R}^3)$ we obtain
\[
I(u) \geq \frac{1}{2} \|u\|^2 - C\|u\|^p. \tag{5.3}
\]
The conclusion follows since $1 < p < 2$. \qed

Lemma 5.2. $I$ satisfies the (PS) condition.

Proof. Let $\{u_n\}$ be a sequence such that $I(u_n)$ is bounded and $I'(u_n) \to 0$. Since $I(u_n)$ is bounded, it follows from Lemma 5.5 that $\{u_n\}$ is bounded. Therefore, passing to a subsequence, we assume that
\[
u_n \to u \text{ weakly in } E, \tag{5.4a}
u_n \to u \text{ in } L^a_{\text{loc}}(\mathbb{R}^3), \quad 1 \leq a < 6, \tag{5.4b}
u_n \to u \text{ a.e. on } \mathbb{R}^3. \tag{5.4c}
\]
By $I'(u_n) \to 0$ and $u_n \to u$ weakly in $E$, we have
\[
o(1) = \langle I'(u_n) - I'(u), u_n - u \rangle \\
= \|u_n - u\|^2 + \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) \\
- \int_{\mathbb{R}^3} K(x) (|u_n|^{p-2} y_n - |u|^{p-2} y)(u_n - u) \\
\Delta \|u_n - u\|^2 + I_1 + I_2. \tag{5.5}
\]
We write $I_1$ as

$$I_1 = D(u_n^2, u_n^2) + D(u^2, u^2) - D(u_n^2, u_n u) - D(u^2, u_n u),$$

where

$$D(f, g) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x)g(y)}{|x-y|} dxdy.$$  \hspace{1cm} (5.6a)

Using [5, Theorem 9.8], we have

$$D(u_n^2, u_n u) \leq (D(u_n^2, u_n^2))^\frac{1}{2} (D(u_n u, u_n u))^\frac{1}{2} \leq \frac{1}{2} D(u_n^2, u_n^2) + \frac{1}{2} D(u_n u, u_n u),$$

$$D(u^2, u_n u) \leq (D(u^2, u^2))^\frac{1}{2} (D(u_n u, u_n u))^\frac{1}{2} \leq \frac{1}{2} D(u^2, u^2) + \frac{1}{2} D(u_n u, u_n u).$$

Therefore,

$$I_1 \geq \frac{1}{2} D(u_n^2, u_n^2) + \frac{1}{2} D(u^2, u^2) - D(u_n u, u_n u).$$  \hspace{1cm} (5.7)

The Hölder inequality implies

$$D(u_n u, u_n u) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n(x)u(x)u_n(y)u(y)}{|x-y|} dxdy$$

$$\leq (D(u_n^2, u_n^2))^\frac{1}{2} (D(u^2, u^2))^\frac{1}{2}. \hspace{1cm} (5.8)$$

Combining the last two inequalities we conclude that

$$I_1 \geq 0. \hspace{1cm} (5.9)$$

For $I_2$, by the Hölder inequality, for any $R > 0$, we have

$$|I_2| \leq \left( \int_{B_R} |K(x)|^\frac{1}{p-1} \right)^{\frac{1}{p-1}} \left( \int_{B_R} \left( \frac{|u_n|^{p-2}u_n - |u|^{p-2}u}{|u_n - u|} \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}$$

$$+ \left( \int_{\mathbb{R}^3 \setminus B_R} |K(x)|^\frac{1}{p-1} \right)^{\frac{1}{p-1}} \left( \int_{\mathbb{R}^3 \setminus B_R} \left( \frac{|u_n|^{p-2}u_n - |u|^{p-2}u}{|u_n - u|} \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}.$$

Given $\varepsilon > 0$, we fix $R > 0$ such that

$$\left( \int_{\mathbb{R}^3 \setminus B_R} |K(x)|^\frac{1}{p-1} \right)^{\frac{1}{p-1}} < \varepsilon. \hspace{1cm} (5.10)$$

Since $\{u_n\}$ is bounded in $L^4(\mathbb{R}^3)$ and $u_n \to u$ in $L^4_{\text{loc}}(\mathbb{R}^3)$, we see that

$$\left( \int_{\mathbb{R}^3} \left( \frac{|u_n|^{p-2}u_n - |u|^{p-2}u}{|u_n - u|} \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \leq C, \hspace{1cm} (5.11a)$$

$$\int_{B_R} \left( \frac{|u_n|^{p-2}u_n - |u|^{p-2}u}{|u_n - u|} \right)^{\frac{p}{p-1}} = o(1). \hspace{1cm} (5.11b)$$
Therefore,
\[
|I_2| \leq o(1) + C\varepsilon. \tag{5.12}
\]
Letting \( n \to \infty \) and \( \varepsilon \to 0 \), we see that
\[
I_2 = o(1). \tag{5.13}
\]
From (5.5), (5.9), and (5.13), we conclude that \( \|u_n - u\| \to 0 \).

**Lemma 5.3.** For any \( k \in \mathbb{N} \), there exist a \( k \)-dimensional subspace \( E_k \) of \( E \) and \( \rho_k > 0 \) such that
\[
\sup_{E_k \cap S_{\rho_k}} I < 0. \tag{5.14}
\]

**Proof.** Choose \( \delta, r > 0 \) such that \( K(x) > \delta \) in \( B_r(x_0) \). Define \( E_k \) as in the proof of Lemma 3.2. Let \( u \in E_k \) and \( \|u\| = 1 \). For \( \rho > 0 \), we have
\[
I(\rho u) \leq \frac{\rho^2}{2} + \frac{\rho^4}{4} \int_{B_r(x_0)} \phi u^2 - \delta \rho^p \int_{B_r(x_0)} |u|^p. \tag{5.15}
\]
Since \( 1 < p < 2 \), it suffices to choose \( \rho = \rho_k \) small enough.

**Proof of Theorem 1.3.** Use the above three lemmas and Theorem A.

6 Proof of Theorem 1.4

The functional associated with (1.4) is the functional \( \tilde{I} \) defined to be
\[
\tilde{I}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) - \frac{1}{2} \int_{\mathbb{R}^3} (\omega^2 + \epsilon \omega \tilde{\varphi} u^2) - \int_{\mathbb{R}^3} K(x)|u|^p
\]
for \( u \in E \).

Similar to the proof of Theorem 1.3, we will apply Theorem A to prove that \( \tilde{I} \) has a sequence of negative critical values converging to 0.

Clearly, \( \tilde{I} \) is a \( C^1 \) functional, \( \tilde{I} \) is even, and \( \tilde{I}(0) = 0 \).

**Lemma 6.1.** \( \tilde{I} \) is coercive and bounded below.

**Proof.** By the Hölder inequality and the fact that \( \tilde{\varphi} u \leq 0 \), we have
\[
\tilde{I}(u) \geq \frac{1}{2} \|u\|^2 - \frac{\omega^2}{2} \int_{\mathbb{R}^3} u^2 - \left( \int_{\mathbb{R}^3} |K(x)| \frac{1}{p'} \right) \frac{\|u\|^p}{\|u\|^p}
\]
which together with the assumption \( \inf V > \omega^2 \) and the Sobolev inequality yields
\[
\tilde{I}(u) \geq C_1 \|u\|^2 - C_2 \|u\|^p.
\]
Thus \( \tilde{I} \) is coercive and bounded below, since \( 1 < p < 2 \).
Lemma 6.2.  $I$ satisfies the (PS) condition.

Proof. Suppose that $\{u_n\}$ is a sequence such that $I(u_n)$ is bounded and $I'(u_n) \to 0$. By Lemma 6.1, $\{u_n\}$ is bounded. Therefore, we may extract a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \to u \quad \text{weakly in } E,$$  \hfill (6.1a)
$$u_n \to u \quad \text{in } L^6_{\text{loc}}(\mathbb{R}^3), \quad 1 \leq a < 6,$$  \hfill (6.1b)
$$u_n \to u \quad \text{a.e. on } \mathbb{R}^3.$$  \hfill (6.1c)

From the equation

$$-\Delta \phi_{u_n} + e^2 u_n^2 \phi_{u_n} = -e\omega u_n^2,$$  \hfill (6.2)

we have

$$\int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2 + e^2 \int_{\mathbb{R}^3} u_n^2 \phi_{u_n}^2 \leq e\omega \|u_n\|^2_{L^6(\mathbb{R}^3)} \|\phi_{u_n}\|_{L^6(\mathbb{R}^3)}.$$  \hfill (6.3)

We then deduce that $\{\phi_{u_n}\}$ is bounded in $L^6(\mathbb{R}^3)$. According to the proof of Lemma 5.2, we see that

$$o(1) = (I'(u_n) - I'(u), u_n - u) = \|u_n - u\|^2 - \int_{\mathbb{R}^3} (\omega + e\phi_{u_n})^2 (u_n - u)^2$$
$$- \int_{\mathbb{R}^3} [(\omega + e\phi_{u_n})^2 - (\omega + e\phi_{u})^2] u (u_n - u) + o(1).$$

Write the second integral on the right side as

$$\int_{\mathbb{R}^3} [(\omega + e\phi_{u_n})^2 - (\omega + e\phi_{u})^2] u (u_n - u)$$
$$= 2\omega e \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_{u}) u (u_n - u) + e^2 \int_{\mathbb{R}^3} [(\phi_{u_n})^2 - (\phi_{u})^2] u (u_n - u)$$
$$\leq 2\omega e I_3 + e^2 I_4. \quad (6.4)$$

For any $R > 0$, the Hölder inequality implies

$$|I_3| \leq \|\phi_{u_n} - \phi_{u}\|_{L^6(\mathbb{R}^3)} \|u\|_{L^{12}(\mathbb{R}^3)} \|u_n - u\|_{L^{12}(\mathbb{R}^3)},$$
$$+ \|\phi_{u_n} - \phi_{u}\|_{L^6(\mathbb{R}^3)} \|\phi_{u} + \phi_{u_n}\|_{L^6(\mathbb{R}^3)} \|u\|_{L^3(\mathbb{R}^3)} \|u_n - u\|_{L^3(\mathbb{R}^3)},$$

and

$$|I_4| \leq \|\phi_{u_n} - \phi_{u}\|_{L^6(\mathbb{R}^3)} \|\phi_{u_n} + \phi_{u}\|_{L^6(\mathbb{R}^3)} \|u\|_{L^3(\mathbb{R}^3)} \|u_n - u\|_{L^3(\mathbb{R}^3)}.$$
Since \( \{ \tilde{\phi}_{u_n} \} \) is bounded in \( L^6(\mathbb{R}^3) \), \( u_n \to u \) in \( L^{12}_{\text{loc}}(\mathbb{R}^3) \cap L^3_{\text{loc}}(\mathbb{R}^3) \), and \( u \in L^{12}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3) \), it is clear that \( I_3 = o(1) \) and \( I_4 = o(1) \). It then can be deduced that

\[
\|u_n - u\|^2 - \int_{\mathbb{R}^3} (\omega + e\tilde{\phi}_{u_n})^2(u_n - u)^2 = o(1). \tag{6.5}
\]

Since \( 0 \leq \omega + e\tilde{\phi}_{u_n} \leq \omega \) and \( \inf V > \omega^2 \), we conclude that \( \|u_n - u\|^2 = o(1) \).

**Lemma 6.3.** For any \( k \in \mathbb{N} \), there exist a \( k \)-dimensional subspace \( E^k \) of \( E \) and \( \rho_k > 0 \) such that

\[
\sup_{E^k \cap S_{\rho_k}} \tilde{I} < 0. \tag{6.6}
\]

**Proof.** Note that

\[
\tilde{I}(u) \leq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) - \int_{\mathbb{R}^3} K(x)|u|^p. \tag{6.7}
\]

The proof of Lemma 5.3 works here.

**Proof of Theorem 1.4.** This is a consequence of the above three lemmas and Theorem A.

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**References**