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Existence of Renormalized Solutions of Nonlinear Elliptic Problems in Weighted Variable-Exponent Space

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Abstract. In this article, we study a general class of nonlinear degenerated elliptic problems associated with the differential inclusion $\beta(u)-div(a(x,Du)+F(u))\ni f$ in Ω where $f\in L^1(\Omega)$. A vector field $a(.,.)$ is a Carathéodory function. Using truncation techniques and the generalized monotonicity method in the framework of weighted variable exponent Sobolev spaces, we prove existence of renormalized solutions for general L^1-data.

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1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N\geq 1)$ with Lipschitz boundary if $N\geq 2$, where the variable exponent $p:\overline{\Omega}\to(1,\infty)$ is a continuous function, and ω be a weight function on Ω, i.e. each ω is a measurable a.e. positive on Ω. Let $W_0^{1,p(\cdot)}(\Omega,\omega)$ be the weighted variable exponent Sobolev space associated with the vector ω. We are interested in existence of renormalized solutions to the following nonlinear elliptic equation

$$(E,f)\begin{cases}\beta(u)-div(a(x,Du)+F(u))\ni f & \text{in} \quad \Omega\\ u=0 & \text{on} \quad \partial\Omega,\end{cases}$$

with a right-hand side f which is assumed to belong either to $L^\infty(\Omega)$ or to $L^1(\Omega)$ for Eq. (E,f). Furthermore, F and β are two functions satisfying the following assumptions:

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$(\mathbf{A}_0)$ $F: \mathbb{R} \to \mathbb{R}^N$ is locally lipschitz continuous and $\beta: \mathbb{R} \to 2^{\mathbb{R}}$ a set valued, maximal monotone mapping such that $0 \in \beta(0)$, Moreover, we assume that

$$\beta^0(l) \in L^1(\Omega), \tag{1.1}$$

for each $l \in \mathbb{R}$, where β^0 denotes the minimal selection of the graph of β. Namely $\beta_0(l)$ is the minimal in the norm element of $\beta(l)$

$$\beta_0(l) = \inf\{|r| \ / r \in \mathbb{R} \quad \text{and} \quad r \in \beta(l)\}.$$

Moreover, $a: \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions :

$(\mathbf{A}_1)$ There exists a positive constant λ such that $a(x,\xi)\cdot\xi \geq \lambda\omega(x)|\xi|^{p(x)}$ holds for all $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$.

$(\mathbf{A}_2)$ $|a_i(x,\xi)| \leq \alpha\omega^{1/p(x)}(x)[k(x)+\omega^{1/p'(x)}(x)|\xi|^{p(x)-1}]$ for almost every $x \in \Omega$, all $i = 1,....,N$, every $\xi \in \mathbb{R}^N$, where $k(x)$ is a nonnegative function in $L^{p'(\cdot)}(\Omega)$, $p'(x) := p(x)/(p(x)-1)$, and $\alpha > 0$.

$(\mathbf{A}_3)$ $(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta) \geq 0$ for almost every $x \in \Omega$ and every $\xi,\eta \in \mathbb{R}^N$.

We use in this paper the framework of renormalized solutions. This notion was introduced by Diperna and P.-L. Lions [7] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (E,f) by L. Boccardo *et al.* [5] when the right hand side is in $W^{-1,p'}(\Omega)$, by J.-M. Rakotoson [17] when the right hand side is in $L^1(\Omega)$, and finally by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [10] for the case of right hand side is general measure data. The equivalent notion of entropy solution has been introduced by Bénilan *et al.* in [4]. For results on existence of renormalized solutions of elliptic problems of type (E,f) with $a(,)$ satisfying a variable growth condition, we refer to [19], [12], [2] and [1]. One of the motivations for studying (E,f) comes from applications to electrorheological fluids (see [18] for more details) as an important class of non-Newtonian fluids.

For the convenience of the readers, we recall some definitions and basic properties of the weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega,\omega)$ and the weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$. Set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1\}.$$

For any $p \in C_+(\overline{\Omega})$, we define

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \quad p^- = \min_{x\in\overline{\Omega}} p(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega,\omega)$ that consists of all measurable real-valued functions u such that

$$L^{p(x)}(\Omega,\omega) = \left\{u : \Omega \to \mathbb{R}, \textit{measurable}, \int_\Omega |u(x)|^{p(x)}\omega(x)dx < \infty\right\}.$$

Then, $L^{p(x)}(\Omega,\omega)$ endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega,\omega)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}\omega(x)dx\leq 1\right\}$$

becomes a normed space. When $\omega(x)\equiv 1$, we have $L^{p(x)}(\Omega,\omega)\equiv L^{p(x)}(\Omega)$ and we use the notation $|u|_{L^{p(x)}(\Omega)}$ instead of $|u|_{L^{p(x)}(\Omega,\omega)}$. The following Hölder type inequality is useful for the next sections. The weighted variable exponent Sobolev space $W^{1,p(x)}(\Omega,\omega)$ is defined by

$$W^{1,p(x)}(\Omega,\omega)=\{u\in L^{p(x)}(\Omega);|\nabla u|\in L^{p(x)}(\Omega,\omega)\},$$

where the norm is

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)}=|u|_{L^{p(x)}(\Omega)}+|\nabla u|_{L^{p(x)}(\Omega,\omega)} \tag{1.2}$$

or equivalently

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}+\omega(x)\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)}dx\leq 1\right\}$$

for all $u\in W^{1,p(x)}(\Omega,\omega)$.

It is significant that smooth functions are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. This feature was observed by Zhikov [21] in connection with the Lavrentiev phenomenon. However, if the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant C such that

$$|p(x)-p(y)|\leq\frac{C}{-\log|x-y|} \tag{1.3}$$

for every x,y with $|x-y|\leq\frac{1}{2}$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values,$W^{1,p(x)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega,\omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)}$ (see [13]). $W_0^{1,p(x)}(\Omega,\omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega,\omega)}$. Throughout the paper, we assume that $p\in C_+(\overline{\Omega})$ and ω is a measurable positive and a.e. finite function in Ω.

The plan of the paper is as follows. In Section 2, we give some preliminaries of the weighted variable exponent Lebesgue-Sobolev spaces which are given in [14] and we introduce the notions of weak and also renormalized solution for problem (E,f). Our first main result, existence of a renormalized solution to (E,f) for any L^∞ - data f, are collected in Section 3. Our second main result, existence of a renormalized solution to (E,f) for any L^1 - data f is collected in Section 4.

2 Preliminaries

2.1 Basic properties of the weighted variable exponent Sobolev spaces

In this section, we state some elementary properties for the (weighted) variable exponent Lebesgue-Sobolev spaces which will be used in the next sections. The basic properties of the variable exponent Lebesgue-Sobolev spaces, that is, when $\omega(x)\equiv 1$ can be found from [11,15].

Lemma 2.1. (See [11,15]). *(Generalised Hölder inequality).*

i) For any functions $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$, we have:
$|\int_\Omega uvdx|\leq (\frac{1}{p^-}+\frac{1}{p'^-})||u||_{p(\cdot)}||v||_{p'(\cdot)}\leq 2||u||_{p(\cdot)}||v||_{p'(\cdot)}$.

ii) For all $p,q\in C_+(\bar{\Omega})$ such that $p(x)\leq q(x)$ a.e in Ω, we have
$L^{q(\cdot)}\hookrightarrow L^{p(\cdot)}$ *and the embedding is continuous.*

Lemma 2.2. (See [14]). *Denote $\rho(u)=\int_\Omega \omega(x)|u(x)|^{p(x)}dx$ for all $u\in L^{p(x)}(\Omega,\omega)$. Then*

$$|u|_{L^{p(x)}(\Omega,\omega)}<1(=1;>1) \text{ if and only if } \rho(u)<1(=1;>1); \tag{2.1}$$

$$if\ |u|_{L^{p(x)}(\Omega,\omega)}>1 \text{ then } |u|^{p^-}_{L^{p(x)}(\Omega,\omega)}\leq\rho(u)\leq|u|^{p^+}_{L^{p(x)}(\Omega,\omega)}; \tag{2.2}$$

$$if\ |u|_{L^{p(x)}(\Omega,\omega)}<1 \text{ then } |u|^{p^+}_{L^{p(x)}(\Omega,\omega)}\leq\rho(u)\leq|u|^{p^-}_{L^{p(x)}(\Omega,\omega)}. \tag{2.3}$$

Remark 2.1. If we set

$$I(u)=\int_\Omega |u(x)|^{p(x)}+\omega(x)|\nabla u(x)|^{p(x)}dx$$

then following the same argument, we have

$$\min\left\{\|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)},\|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)}\right\}\leq I(u)\leq\max\left\{\|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)},\|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)}\right\}. \tag{2.4}$$

Throughout the paper, we assume that ω is a measurable positive and a.e. finite function in Ω satisfying that

(H$_1$) $\omega\in L^1_{loc(\Omega)}$ and $\omega^{-\frac{1}{(p(x)-1)}}\in L^1_{loc}(\Omega)$;

(H$_2$) $\omega^{-s(x)}\in L^1(\Omega)$ with $s(x)\in\left(\frac{N}{p(x)},\infty\right)\cap\left[\frac{1}{p(x)-1},\infty\right)$.

The reasons that we assume (**H$_1$**) and (**H$_2$**) can be found in [14].

Remark 2.2. ([14])

(i) If ω is a positive measurable and finite function, then $L^{p(x)}(\Omega,\omega)$ is a reflexive Banach space.

(ii) Moreover, if $(\mathbf{H}_1)$ holds, then $W^{1,p(x)}(\Omega,\omega)$ is a separable and reflexive Banach space.

For $p,s\in C_+(\overline{\Omega})$, denote $p_s(x)=\frac{p(x)s(x)}{s(x)+1}<p(x)$,
where $s(x)$ is given in $(\mathbf{H}_2)$. Assume that, we fix the variable exponent restrictions

$$\begin{cases} p_s^*(x)=\frac{p(x)s(x)N}{(s(x)+1)N-p(x)s(x)} & \text{if } N>p_s(x),\\ p_s^*(x) \quad \text{arbitrary} & \text{if} \quad N\le p_s(x) \end{cases}$$

for almost all $x\in\Omega$. These definitions play a key role in our paper. We shall frequently make use of the following (compact) imbedding theorem for the weighted variable exponent Lebesgue-Sobolev space in the next sections.

Lemma 2.3. ([14]) *Let $p,s\in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition* (1.3) *and let* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *be satisfied. If $r\in C_+(\overline{\Omega})$ and $1<r(x)\le p_s^*(x)$, then we obtain the continuous imbedding*

$$W^{1,p(x)}(\Omega,\omega)\hookrightarrow L^{r(x)}(\Omega).$$

Moreover, we have the compact imbedding

$$W^{1,p(x)}(\Omega,\omega)\hookrightarrow L^{r(x)}(\Omega)$$

provided that $1<r(x)<p_s^*(x)$ for all $x\in\overline{\Omega}$.

From Lemma 2.3, we have Poincaré-type inequality immediately.

Corollary 2.1. ([14]) Let $p\in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (1.3). If $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ hold, then the estimate

$$\|u\|_{L^{p(x)}(\Omega)}\le C\|\nabla u\|_{L^{p(x)}(\Omega,\omega)}$$

holds for every $u\in C_0^\infty(\Omega)$ with a positive constant C independent of u.

Throughout this paper, let $p\in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (1.3) and $X:=W_0^{1,p(x)}(\Omega,\omega)$ be the weighted variable exponent Sobolev space that consists of all real valued functions u from $W^{1,p(x)}(\Omega,\omega)$ which vanish on the boundary $\partial\Omega$, endowed with the norm

$$\|u\|_X=\inf\left\{\lambda>0:\int_\Omega\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)}\omega(x)dx\le 1\right\},$$

which is equivalent to the norm (1.2) due to Corollary 2.1. The following proposition gives the characterization of the dual space $(W_0^{k,p(x)}(\Omega,\omega))^*$, which is analogous to ([15],

Theorem 3.16). We recall that the dual space of weighted Sobolev spaces $W_0^{1,p(x)}(\Omega,\omega)$ is equivalent to $W^{-1,p'(x)}(\Omega,\omega)$, where $\omega^* = \omega^{1-p'(x)}$.
The following notations will be used throughout the paper: for $k \geq 0$, the truncation at heigth k is defined by

$$T_k(r) := \begin{cases} -k, & if\, r \leq -k, \\ r, & if\, |r| < k, \\ k, & if\, r \geq k, \end{cases}$$

and let $h_l : \mathbb{R} \to \mathbb{R}$ be defined by

$$h_l(r) := \min\Big((l+1-|r|)^+, 1 \Big) \quad \text{for each} \quad r \in \mathbb{R}.$$

For $\delta > 0$, we define

$$H_\delta^+(r) := \begin{cases} 0 & if\, r < 0 \\ \frac{1}{\delta} r & if\, 0 \leq r \leq \delta \\ 1 & if\, r > \delta \end{cases} \tag{2.5}$$

$$\text{and} \quad H_\delta(r) = \begin{cases} -1 & if\, r < -\delta \\ \frac{1}{\delta} r & if\, -\delta \leq r \leq \delta \\ 1 & if\, r > \delta \end{cases}.$$

Remark 2.3. The Lipschitz character of F and Stokes formula together with the boundary condition ($u = 0$ on $\partial\Omega$) of problem give $\int_\Omega F(u) DT_k(u) dx = 0$ (see [19]).

2.2 Notions of solutions

2.2.1 Weak solutions

Definition 2.1. *A weak solution to (E,f) is a pair of functions $(u,b) \in W_0^{1,p(\cdot)}(\Omega,\omega) \times L^1(\Omega)$ satisfying $F(u) \in (L^1_{\text{loc}}(\Omega))^N$, $b \in \beta(u)$ almost everywhere in Ω and*

$$b - \operatorname{div}(a(x,Du) + F(u)) = f \quad in\ D'(\Omega). \tag{2.6}$$

2.2.2 Renormalized solutions

Definition 2.2. *A renormalized solution to (E,f) is a pair of functions (u,b) satisfying the following conditions:*

$(R1)$ *$u : \Omega \to \mathbb{R}$ is measurable, $b \in L^1(\Omega)$, $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$*

for a.e. $x \in \Omega$.

$(R2)$ *For each* $k>0, T_k(u)\in W_0^{1,p(\cdot)}(\Omega,\omega)$ *and*

$$\int_\Omega b\cdot h(u)\varphi+\int_\Omega (a(x,Du)+F(u))\cdot D(h(u)\varphi)=\int_\Omega fh(u)\varphi \tag{2.7}$$

holds for all $h\in C_c^1(\mathbb{R})$ *and all* $\varphi\in W_0^{1,p(\cdot)}(\Omega,\omega)\cap L^\infty(\Omega)$.

$(R3)$ $\int_{\{k<|u|<k+1\}} a(x,Du)\cdot Du \longrightarrow 0 \quad as \quad k\longrightarrow\infty.$

Remark 2.4. For $p\in(1,\infty)$, $\tau_0^{p(\cdot)}(\Omega)$ is defined as the set of measurable functions $u:\Omega\to\mathbb{R}$ such that for $k>0$ the truncated functions $T_k(u)\in W_0^{1,p(\cdot)}(\Omega,\omega)$ and for every $u\in\tau_0^{p(\cdot)}(\Omega)$ there exists a unique measurable function $v:\Omega\to\mathbb{R}^{\mathbb{N}}$ such that $\nabla T_K(u)=v\chi_{\{|u|<k\}}$ for a.e. $x\in\Omega$, [see [20], [4] for more details].

Remark 2.5. Note that if (u,b) is a renormalized solution to (E,f) such that $u\in W_0^{1,p(\cdot)}(\Omega,\omega)$, then (u,b) in general is not a weak solution in the sense of Definition 2.1, since we did not assume a growth condition on F and therefore $F(u)$ in general may fail to be locally integrable. If (u,b) is a renormalized solution of (E,f) such that $u\in L^\infty(\Omega)$, it is a direct consequence of Definition 2.1 that u is in $W_0^{1,p(\cdot)}(\Omega,\omega)$ and since (2.7) holds with the formal choice $h\equiv 1$, (u,b) is a weak solution.

Indeed, let us choose $\varphi\in D(\Omega)$ and plug $h_l(u)\varphi$ as a test function in (2.7). Since $u\in L^\infty(\Omega)$, we can pass to the limit with $l\to\infty$ and find that u solves (E,f) in the sense of distributions.

3 Case where $f\in L^\infty(\Omega)$ -data

3.1 Resultat d'existence

In this subsection we will state existence of renormalized solutions to (E,f) in Theorem 3.1. In the next subsections we will present the proof.

Theorem 3.1. Under assumptions $(H_1)-(H_2)$, $(A_0)-(A_3)$ and $f\in L^\infty(\Omega)$. There, exists at least one renormalized solution (u,b) to (E,f).

3.2 Proof of Theorem 3.1

3.2.1 Approximate problem

First we approximate (E,f) for $f\in L^\infty(\Omega)$ by problems for which existence can be proved by standard variational arguments. For $0<\varepsilon\leq 1$, let $\beta_\varepsilon:\mathbb{R}\longrightarrow\mathbb{R}$ be the Yosida approximation (see [6]) of β . We introduce the operators

$$A_{1,\varepsilon}: W_0^{1,p(\cdot)}(\Omega,\omega)\longrightarrow W^{-1,p'(\cdot)}(\Omega,\omega^*)$$
$$u\longrightarrow \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u))+\varepsilon\arctan(u)-div\,a(x,Du),$$

$A_{2,\varepsilon}: W_0^{1,p(\cdot)}(\Omega,\omega) \longrightarrow \quad W^{-1,p'(\cdot)}(\Omega,\omega^*), u \longrightarrow -div\, F(T_{\frac{1}{\varepsilon}}(u))$.

Because of (A2) and (A3), $A_{1,\varepsilon}$ is well-defined and monotone (see [16], p.157). Since $\beta_\varepsilon \circ T_{\frac{1}{\varepsilon}}$ and arctan are bounded and continuous and thanks to the growth condition (A2) on a, it follows that $A_{1,\varepsilon}$, is hemicontinuous (see [16], p.157). From the continuity and boundedness of $F \circ T_{\frac{1}{\varepsilon}}$ it follows that $A_{2,\varepsilon}$ is strongly continuous. Therefore the operator $A_\varepsilon := A_{1,\varepsilon} + A_{2,\varepsilon}$ is pseudomonotone. Using the monotonicity of β_ε, the Gauss-Green Theorem for Sobolev functions and the boundary condition on the convection term $\int_\Omega F(T_{\frac{1}{\varepsilon}}(u)) \cdot Du$, we show by similar arguments as in [14] that A_ε is coercive and bounded. Then it follows from ([16], Theorem 2.7) that A_ε is surjective, i.e., for each $0<\varepsilon\leq 1$ and $f \in W^{-1,p'(\cdot)}(\Omega,\omega^*)$ there exists at least one solution $u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega,\omega)$ to the problem

$$(E_\varepsilon,f)\begin{cases} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon\arctan(u_\varepsilon) - div\Big(a(x,Du_\varepsilon) + F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\Big) = f & \text{in} \quad \Omega \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

such that

$$\int_\Omega \Big(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon\arctan(u_\varepsilon)\Big)\varphi + \int_\Omega \Big(a(x,Du_\varepsilon) + F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\Big) \cdot D\varphi = <f,\varphi> \tag{3.1}$$

holds for all $\varphi \in W_0^{1,p(\cdot)}(\Omega,\omega)$, where $<.,.>$ denotes the duality pairing between $W_0^{1,p(\cdot)}(\Omega,\omega)$ and $W^{-1,p'(\cdot)}(\Omega,\omega^*)$.

In the next remark, we establish uniqueness of solutions u_ε of (E_ε,f) with right-hand sides $f \in L^\infty(\Omega)$ through a comparison principle that will play an important role in the approximation of renormalized solutions to (E,f) with $f \in L^1(\Omega)$.

Remark 3.1. For $0<\varepsilon\leq 1$ fixed and $f,\tilde{f} \in L^\infty(\Omega)$ let $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p(\cdot)}(\Omega,\omega)$ be solutions of (E_ε,f) and $(E_\varepsilon,\tilde{f})$ respectively. Then, the following comparison principle holds:

$$\varepsilon\int_\Omega (\arctan(u_\varepsilon) - \arctan(\tilde{u}_\varepsilon))^+ \leq \int_\Omega (f-\tilde{f}) sign_0^+(u_\varepsilon - \tilde{u}_\varepsilon). \tag{3.2}$$

Proof. We can copy the proof in [1], Remark 4.2 for the case of a constant exponent with slight modifications such as exchanging the space $W_0^{1,p}(\Omega,\omega)$ by $W_0^{1,p(\cdot)}(\Omega,\omega)$. □

Remark 3.2. Let $f,\tilde{f} \in L^\infty(\Omega)$ be such that $f \leq \tilde{f}$ almost everywhere in Ω, $\varepsilon>0$ and $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p(\cdot)}(\Omega,\omega)$ solutions to $(E_\varepsilon,f),(E_\varepsilon,\tilde{f})$ respectively. Then it is an immediate consequence of Remark 3.1 that $u_\varepsilon \leq \tilde{u}_\varepsilon$ almost everywhere in Ω. Furthermore, from the monotonicity of $\beta_\varepsilon \circ T_{\frac{1}{\varepsilon}}$ it follows that also $\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \leq \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon))$ almost everywhere in Ω.

3.2.2 A priori estimates

Lemma 3.1. *For $0<\varepsilon\leq 1$ and $f \in L^\infty(\Omega)$ let $u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega,\omega)$ be a solution of (E_ε,f). Then,* *i) There exists a constant* $C_1 = C_1(\|f\|_\infty, \lambda, p(\cdot), N) > 0$*, not depending on* ε*, such that*

$$\|Du_\varepsilon\|_{L^{p(\cdot)}(\Omega,\omega)} \leq C_1. \tag{3.3}$$

ii)

$$\|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\|_\infty \leq \|f\|_\infty. \tag{3.4}$$

iii) For all l*,* $k > 0$*, we have*

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x,Du_\varepsilon)\cdot Du_\varepsilon \leq k \int_{\{|u_\varepsilon|>l\}} |f|. \tag{3.5}$$

Proof. i) Taking u_ε as a test function in (3.1), we obtain

$$\int_\Omega \Big(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon \arctan(u_\varepsilon)\Big) u_\varepsilon dx + \int_\Omega a(x,Du_\varepsilon)\cdot Du_\varepsilon dx \\ + \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot Du_\varepsilon dx = \int_\Omega f u_\varepsilon dx.$$

As the first term on the left-hand side is nonnegative and the integral over the convection term vanishes by (A_1), we have:

$$\begin{aligned} \lambda \int_\Omega |Du_\varepsilon|^{p(\cdot)} \omega(x) dx &\leq \int_\Omega a(x,Du_\varepsilon)\cdot Du_\varepsilon dx \\ &\leq \int_\Omega f u_\varepsilon dx = \int_\Omega f u_\varepsilon \omega^{1/p(x)} \omega^{-1/p(x)} dx \\ &\leq C(p(\cdot),N)\|f\|_\infty \|Du_\varepsilon\|_{L^{p(\cdot)}(\Omega,\omega)}, \end{aligned} \tag{3.6}$$

where $C(p(\cdot),N) > 0$ is a constant coming from the Hölder and Poincaré inequalities. From (2.4) and (3.6) it follows that either

$$\|Du_\varepsilon\|_{L^{p(\cdot)}(\Omega,\omega)} \leq \Big(\frac{1}{\lambda} C(p(\cdot),N)\|f\|_\infty\Big)^{\frac{1}{p^- - 1}}$$

or

$$\|Du_\varepsilon\|_{L^{p(\cdot)}(\Omega,\omega)} \leq \Big(\frac{1}{\lambda} C(p(\cdot),N)\|f\|_\infty\Big)^{\frac{1}{p^+ - 1}}.$$

Setting $C_1 := \max\bigg(\Big(\frac{1}{\lambda}C(p(\cdot),N)\|f\|_\infty\Big)^{\frac{1}{p^+-1}}, \Big(\frac{1}{\lambda}C(p(\cdot),N)\|f\|_\infty\Big)^{\frac{1}{p^--1}}\bigg)$, we get i).

ii) Taking $\frac{1}{\delta}[T_{k+\delta}(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))]$ as a test function in (3.1), passing to the limit with $\delta \to 0$ and choosing $k > \|f\|_\infty$, we obtain (ii).

iii) For $k,l>0$ fixed, we take $T_k(u_\varepsilon - T_l(u_\varepsilon))$ as a test function in (3.1). Using $\int_\Omega a(x,Du_\varepsilon)\cdot DT_k(u_\varepsilon - T_l(u_\varepsilon))dx = \int_{\{l<|u_\varepsilon|<l+k\}} a(x,Du_\varepsilon)\cdot Du_\varepsilon dx$, and as the first term and the second on the left-hand side is nonnegative and the convection term vanishes, we get

$$\begin{aligned}\int_{\{l<|u_\varepsilon|<l+k\}} a(x,Du_\varepsilon)\cdot Du_\varepsilon dx &\leq \int_\Omega f T_k(u_\varepsilon - T_l(u_\varepsilon))dx \\ &\leq k\int_{\{|u_\varepsilon|>l\}} |f| dx.\end{aligned}$$

$\square$

Remark 3.3. For $k>0$, from Lemma 3.1 (iii), we deduce

$$|\{|u_\varepsilon|\geq l\}| \leq l^{-(p_s^*)^-} C(p(\cdot),p^-,\lambda,C_1) \tag{3.7}$$

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x,Du_\varepsilon)\cdot Du_\varepsilon \leq k\|f\|_\infty |\{|u_\varepsilon|>l\}| \leq C_2(k) l^{-(p_s^*)^-}. \tag{3.8}$$

Indeed, we have the following continuous embedding

$$W_0^{1,p(x)}(\Omega,\omega) \hookrightarrow L^{p_s^*(x)}(\Omega) \hookrightarrow L^{(p_s^*)^-}(\Omega),$$

where $(p_s^*(x))^- := \frac{p^- s - N}{(s^-+1)N - p^- s^-}$.

Let $l>0$ large enough, we have: it follows from

$$\|T_l(u)\|_{L^{(p_s^*)^-}(\Omega)} \leq C\|DT_l(u)\|_{L^{p(x)}(\Omega,\omega)} \leq C\Big(\int_\Omega \omega(x)|DT_l(u)|^{p(x)}dx\Big)^\nu,$$

where

$$\nu = \begin{cases} \frac{1}{p^-} & \text{if} \quad \|DT_l(u)\|_{L^{p(x)}(\Omega,\omega)} \geq 1 \\ \frac{1}{p^+} & \text{if} \quad \|DT_k(u)\|_{L^{p(x)}(\Omega,\omega)} \leq 1. \end{cases}$$

Noting that $\{|u_\varepsilon|\geq l\} = \{|T_l(u_\varepsilon)|\geq l\}$, we have

$$\begin{aligned}|\{|u_\varepsilon|\geq l\}| &\leq \left(\frac{\|T_l(u)\|_{L^{(p_s^*)^-}(\Omega)}}{l}\right)^{(p_s^*)^-} \\ &\leq l^{-(p_s^*)^-}\left(C\Big(\int_\Omega \omega(x)|DT_l(u)|^{p(x)}dx\Big)^\nu\right)^{(p_s^*)^-}.\end{aligned} \tag{3.9}$$

Combining (3.3), (3.6)and (3.9), setting

$$C(p(\cdot),(p_s^*)^-,\lambda,C_1) = C^{(p_s^*)^-}\left(\frac{C(p(\cdot),N)\|f\|_\infty}{\lambda}C_1\right)^{\nu(p_s^*)^-} > 0,$$

we obtain

$$|\{|u_\varepsilon|\geq l\}|\leq C(p(\cdot),(p_s^*)^-,\lambda,C_1)l^{-(p_s^*)^-}. \tag{3.10}$$

So we have

$$\lim_{l\to+\infty}|\{|u_\varepsilon|\geq l\}|=0.$$

Hence (3.10) provides (3.8) with $C_2(k):=C(p(\cdot),(p_s^*)^-,\lambda,C_1)k\|f\|_\infty$.

3.2.3 Basic convergence results

Lemma 3.2. *For $0<\varepsilon\leq 1$ and $f\in L^\infty(\Omega)$, let $u_\varepsilon\in W_0^{1,p(\cdot)}(\Omega,\omega)$ be the solution of (E_ε,f). There exist $u\in W_0^{1,p(\cdot)}(\Omega,\omega)$ and $b\in L^\infty(\Omega)$ such that for a not relabeled subsequence of $(u_\varepsilon)_{0<\varepsilon\leq 1}$ as $\varepsilon\downarrow 0$:*

$$u_\varepsilon \rightharpoonup u \quad in \quad L^{p(\cdot)}(\Omega,\omega) \quad and\ a.e.\ in \quad \Omega \tag{3.11}$$

$$Du_\varepsilon \rightharpoonup Du \quad in\ (L^{p(\cdot)}(\Omega,\omega))^N \tag{3.12}$$

$$and \quad \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \rightharpoonup b \quad weakly\text{-}^*\ in \quad L^\infty(\Omega). \tag{3.13}$$

Moreover, for any

$$k>0,$$

$$DT_k(u_\varepsilon) \rightharpoonup DT_k(u) \quad in \quad (L^{p(\cdot)}(\Omega,\omega))^N \tag{3.14}$$

$$a(x,DT_k(u_\varepsilon)) \rightharpoonup a(x,DT_k(u)) \quad in \quad (L^{p'(\cdot)}(\Omega,\omega^*))^N. \tag{3.15}$$

Proof. Since (3.11)-(3.14) follow directly from Lemma 3.1 and Remark 3.3.

It is left to prove (3.15). For this end by $(A2)$ and (3.3) it follows that given any subsequence of $(a(x,DT_k(u_\varepsilon))_\varepsilon)$, there exists a subsequence, still denoted by $(a(x,DT_k(u_\varepsilon))_\varepsilon)$, such that $a(x,DT_k(u_\varepsilon))\rightharpoonup\Phi_k$ in $(L^{p'(\cdot)}(\Omega,\omega^*))^N$.

We will we prove that $\Phi_k=a(x,DT_k(u))$ a.e. of Ω. The proof consists in three assertions.

Assertion i: For every function $h\in W^{1,\infty}(\mathbb{R})$, $h\geq 0$ with $supp(h)$ compact, we will prove that,

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot D[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))]dx\leq 0. \tag{3.16}$$

Taking $h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))$ as test function in (3.1), we have

$$\begin{aligned}\int_\Omega\Big(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)+\varepsilon\arctan(u_\varepsilon)\Big)h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\\ +\int_\Omega\Big(a(x,Du_\varepsilon)+F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\Big)\cdot D[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))]\\ =\int_\Omega fh(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u)).\end{aligned} \tag{3.17}$$

Using $|h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))|\leq 2k\|h\|_\infty$, by Lebesgue's dominated convergence theorem, we find that

$$\lim_{\varepsilon\to 0}\int_\Omega fh(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))=0.$$

and $\lim_{\varepsilon\to 0}\int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot D[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))]=0$.

By using the same arguments in [2], we can prove that

$$\limsup_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))]dx\geq 0.$$

Passage to limit in (3.17) and using the above results, we obtain (3.16).

Assertion ii: We prove that for every $k>0$,

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot[DT_k(u_\varepsilon)-DT_k(u)]dx\leq 0. \tag{3.18}$$

Indeed: See [1].

Assertion iii: In this step, we prove by monotonicity arguments that for $k>0$, $\Phi_k=a(x,DT_k(u))$ for almost every $x\in\Omega$. Let $\varphi\in D(\Omega)$ and $\tilde{\alpha}\in\mathbb{R}$. Using (3.18), we have

$$\begin{aligned}
&\tilde{\alpha}\lim_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot D\varphi dx\\
\geq&\limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot[DT_k(u_\varepsilon)-DT_k(u)+D(\tilde{\alpha}\varphi)]dx\\
\geq&\limsup_{\varepsilon\to 0}\int_\Omega a(x,D(T_k(u)-\tilde{\alpha}\varphi))\cdot[DT_k(u_\varepsilon)-DT_k(u)+D(\tilde{\alpha}\varphi)]dx\\
\geq&\limsup_{\varepsilon\to 0}\int_\Omega a(x,D(T_k(u)-\tilde{\alpha}\varphi))\cdot D(\tilde{\alpha}\varphi)]dx\\
\geq&\tilde{\alpha}\int_\Omega a(x,D(T_k(u)-\tilde{\alpha}\varphi))\cdot D\varphi dx.
\end{aligned}$$

Dividing by $\tilde{\alpha}>0$ and by $\tilde{\alpha}<0$, pasing the limit with $\tilde{\alpha}\to 0$, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot D\varphi dx=\int_\Omega a(x,DT_k(u))\cdot D\varphi dx.$$

This means that $\forall k>0$, $\int_\Omega \Phi_k\cdot D\varphi dx=\int_\Omega a(x,DT_k(u))\cdot D\varphi dx$ and so

$$\Phi_k=a(x,DT_k(u)) \quad \text{in} \quad D^{'}(\Omega)$$

for all $k>0$. Hence $\Phi_k=a(x,DT_k(u))$ a.e. in Ω and so $a(x,DT_k(u_\varepsilon))\rightharpoonup a(x,DT_k(u))$ weakly in $(L^{p'(\cdot)}(\Omega,\omega^*))^N$. $\square$

Remark 3.4. As immediate consequence of (3.18) and (A3), we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))-a(x,DT_k(u))\cdot(DT_k(u_\varepsilon)-DT_k(u))=0. \tag{3.19}$$

Lemma 3.3. *The limit u of the approximate solution u_ε of (E_ε,f) satisfies*

$$\lim_{l\to\infty}\int_{\{l<|u|<l+1\}} a(x,Du)\cdot Dudx=0. \tag{3.20}$$

Proof. To this end, observe that for any fixed $l\geq 0$, one has

$$\begin{aligned}&\int_{\{l<|u_\varepsilon|<l+1\}} a(x,Du_\varepsilon)\cdot Du_\varepsilon dx=\int_\Omega a(x,Du_\varepsilon)\cdot(DT_{l+1}(u_\varepsilon)-DT_l(u_\varepsilon))dx\\ =&\int_\Omega a(x,DT_{l+1}(u_\varepsilon))\cdot DT_{l+1}(u_\varepsilon)dx-\int_\Omega a(x,DT_l(u_\varepsilon))\cdot DT_l(u_\varepsilon))dx.\end{aligned}$$

According (3.19) is at liberty to passe to the limit as $\varepsilon\to 0$ for fixed $l\geq 0$ and to obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\{l<|u_\varepsilon|<l+1\}} a(x,Du_\varepsilon)\cdot Du_\varepsilon dx\\ =&\int_\Omega a(x,DT_{l+1}(u))\cdot DT_{l+1}(u)dx-\int_\Omega a(x,DT_l(u))\cdot DT_l(u))dx\\ =&\int_{\{l<|u|<l+1\}} a(x,Du)\cdot Dudx.\end{aligned} \tag{3.21}$$

Taking the limit as $l\to+\infty$ in (3.21) and using the estimate (3.8) show that u satisfies (R3) and the proof of the lemma is complete. $\square$

3.3 Concluding the proof of Theorem 3.1

Let $h\in\mathcal{C}_c^1(\mathbb{R})$ and $\phi\in W_0^{1,p(\cdot)}(\Omega,\omega)\cap L^\infty(\Omega)$ be arbitrary. Taking $h_l(u_\varepsilon)h(u)\phi$ as a test function in (3.1), we obtain

$$I_{\varepsilon,l}^1+I_{\varepsilon,l}^2+I_{\varepsilon,l}^3+I_{\varepsilon,l}^4=I_{\varepsilon,l}^5 \tag{3.22}$$

where

$$\begin{aligned}I_{\varepsilon,l}^1&=\int_\Omega\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))h_l(u_\varepsilon)h(u)\phi\\ I_{\varepsilon,l}^2&=\varepsilon\int_\Omega\arctan(u_\varepsilon)h_l(u_\varepsilon)h(u)\phi\\ I_{\varepsilon,l}^3&=\int_\Omega a(x,Du_\varepsilon)\cdot D(h_l(u_\varepsilon)h(u)\phi)\\ I_{\varepsilon,l}^4&=\int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot D(h_l(u_\varepsilon)h(u)\phi)\\ I_{\varepsilon,l}^5&=\int_\Omega fh_l(u_\varepsilon)h(u)\phi.\end{aligned}$$

Step i: Passing to the limit with $\varepsilon\downarrow 0$ obviously,

$$\lim_{\varepsilon\to 0} I^2_{\varepsilon,l}=0. \tag{3.23}$$

Using the convergence results (3.11), (3.13) from Lemma 3.2, we can immediately calculate the following limits:

$$\lim_{\varepsilon\to 0} I^1_{\varepsilon,l}=\int_\Omega b h_l(u)h(u)\phi, \tag{3.24}$$

$$\lim_{\varepsilon\to 0} I^5_{\varepsilon,l}=\int_\Omega f h_l(u)h(u)\phi. \tag{3.25}$$

We write $I^3_{\varepsilon,l}=I^{3,1}_{\varepsilon,l}+I^{3,2}_{\varepsilon,l}$,
where $I^{3,1}_{\varepsilon,l}=\int_\Omega h'_l(u_\varepsilon)a(x,Du_\varepsilon)\cdot Du_\varepsilon h(u)\phi$ and $I^{3,2}_{\varepsilon,l}=\int_\Omega h_l(u_\varepsilon)a(x,Du_\varepsilon)\cdot D(h(u)\phi)$.
Using (3.8), we get the estimate

$$|\lim_{\varepsilon\to 0} I^{3,1}_{\varepsilon,l}|\le \|h\|_\infty\|\phi\|_\infty\cdot C_2(1)l^{-(p^*_s)^-}. \tag{3.26}$$

Since modular convergence is equivalent to norm convergence in $L^{p(\cdot)}(\Omega,\omega)$, by Lebesgue Dominated Convergence Theorem, it follows that for any $i\in\{1,...,N\}$, we have

$$h_l(u_\varepsilon)\frac{\partial}{\partial x_i}(h(u)\phi)\to h_l(u)\frac{\partial}{\partial x_i}(h(u)\phi)\,\text{in}\,L^{p(\cdot)}(\Omega,\omega)\,as\,\varepsilon\downarrow 0.$$

Keeping in mind that $I^{3,2}_{\varepsilon,l}=\int_\Omega h_l(u_\varepsilon)a(x,DT_{l+1}(u_\varepsilon))\cdot D(h(u)\phi)$.
By (3.15), we get

$$\lim_{\varepsilon\to 0} I^{3,2}_{\varepsilon,l}=\int_\Omega h_l(u)a(x,DT_{l+1}(u))\cdot D(h(u)\phi). \tag{3.27}$$

Let us write $I^4_{\varepsilon,l}=I^{4,1}_{\varepsilon,l}+I^{4,2}_{\varepsilon,l}$, where

$$I^{4,1}_{\varepsilon,l}=\int_\Omega h'_l(u_\varepsilon)F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot Du_\varepsilon h(u)\phi,$$
$$I^{4,2}_{\varepsilon,l}=\int_\Omega h_l(u_\varepsilon)F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot D(h(u)\phi).$$

For any $l\in\mathbb{N}$, there exists $\varepsilon_0(l)$; such that for all $\varepsilon<\varepsilon_0(l)$,

$$I^{4,1}_{\varepsilon,l}=\int_\Omega h'_l(T_{l+1}(u_\varepsilon))F(T_{l+1}(u_\varepsilon))\cdot DT_{l+1}(u_\varepsilon)h(u)\phi. \tag{3.28}$$

Using Gauss-Green Theorem for Sobolev functions in (3.28), we get

$$I^{4,1}_{\varepsilon,l}=-\int_\Omega\int_0^{T_{l+1}(u_\varepsilon)} h'_l(r)F(r)dr\cdot D(h(u)\phi). \tag{3.29}$$

Now, using (3.11) and the Gauss-Green Theorem, after letting $\varepsilon\downarrow 0$, we get

$$\lim_{\varepsilon\to 0} I^{4,1}_{\varepsilon,l}=\int_\Omega h'_l(u)F(u)\cdot Du\,h(u)\phi. \tag{3.30}$$

Choosing ε small enough, we can write

$$I^{4,2}_{\varepsilon,l}=\int_\Omega h_l(u_\varepsilon)F(T_{l+1}(u_\varepsilon))\cdot D(h(u)\phi) \tag{3.31}$$

and conclude

$$\lim_{\varepsilon\to 0} I^{4,2}_{\varepsilon,l}=\int_\Omega h_l(u)F(u)\cdot D(h(u)\phi). \tag{3.32}$$

Step ii: Passing to the limit with $l\to\infty$. Combining (3.22) and (3.23)- (3.32), we find

$$I^1_l+I^2_l+I^3_l+I^4_l+I^5_l=I^6_l \tag{3.33}$$

where

$$I^1_l=\int_\Omega bh_l(u)h(u)\phi, \qquad I^2_l=\int_\Omega h_l(u)a(x,DT_{l+1}(u))\cdot D(h(u)\phi),$$

$$|I^3_l|\le C_2(1)l^{-(p^*_s)^-}\|h\|_\infty\|\phi\|_\infty, \quad I^4_l=\int_\Omega h_l(u)F(u)\cdot D(h(u)\phi),$$

$$I^5_l=\int_\Omega h'_l(u)F(u)\cdot Du\,h(u)\phi, \quad I^6_l=\int_\Omega f\,h_l(u)h(u)\phi.$$

Obviously, we have

$$\lim_{l\to\infty} I^3_l=0. \tag{3.34}$$

Choosing $m>0$ such that $supp\,h\subset[-m,m]$, we can replace u by $T_m(u)$ in $I^1_l,I^2_l,\dots,I^6_l$, and $h'_l(u)=h'_l(T_m(u))=0$ if $l+1>m$, $h_l(u)=h_l(T_m(u))=1$, if $l>m$.
Therefore, letting $l\to\infty$ and combining (3.33) with (3.34), we obtain

$$\int_\Omega bh(u)\phi+\int_\Omega(a(x,Du)+F(u))\cdot D(h(u)\phi)=\int_\Omega fh(u)\phi, \tag{3.35}$$

for all $h\in C^1_c(\mathbb{R})$ and all $\phi\in W^{1,p(\cdot)}_0(\Omega,\omega)\cap L^\infty(\Omega)$.

Step iii: Subdifferential argument. It is left to prove that $u(x)\in D(\beta(x))$ and $b(x)\in\beta(u(x))$ for almost all $x\in\Omega$. Since β a is maximal monotone graph, exists a convex; l.s.c. and proper function $j:\mathbb{R}\to[0,\infty]$, such that $\beta(r)=\partial j(r)$ for all $r\in\mathbb{R}$. According to [6], for $0<\varepsilon\le 1$, $j_\varepsilon:\mathbb{R}\to\mathbb{R}$ defined by $j_\varepsilon(r)=\int_0^r\beta_\varepsilon(s)ds$ has the following properties: (see [19]). Using the same argument in [19], we can prove that for all $r\in\mathbb{R}$ and almost every $x\in\Omega$, $u\in D(\beta)$ and $b\in\beta(u)$ almost everywhere in Ω. With this last step the proof of Theorem 3.1 is completed.

4 Case where $f\in L^1(\Omega)$ -data

In this section we establish the existence and uniqueness of renormalized solution to the degenerated problem (E,f) with $f\in L^1(\Omega)$.

4.1 Results of existence and uniqueness

Theorem 4.1. *Under assumptions $(H_1)-(H_2)$, $(A_0)-(A_3)$ and $f\in L^1(\Omega)$, there exists at least one renormalized solution (u,b) to the degenerated problem (E,f).*

4.2 Proof of theorem 4.1

4.2.1 Approximate problem and a priori estimates

To prove Theorem 4.1, we will introduce and solve approximating problems. To this end, for $f\in L^1(\Omega)$, and $n,m\in\mathbb{N}$, we define $f_{m,n}:\Omega\to\mathbb{R}$ by

$$f_{m,n}=\max(\min(f(x),m),-n)$$

for almost every $x\in\Omega$, clearly $f_{m,n}\in L^\infty(\Omega)$ for each $m,n\in\mathbb{N}$, $|f_{m,n}(x)|\le|f(x)|$ a.e in Ω hence

$$\lim_{n\to\infty}\lim_{m\to\infty}f_{m,n}=f \text{ in } L^1(\Omega) \text{ for almost everywhere in } \Omega.$$

The comparison principle from Remark 3.1 will be the main tool in the second approximation procedure. For $f\in L^1(\Omega)$ and $m,n\in\mathbb{N}$, let $f_{m,n}\in L^\infty(\Omega)$ be defined as above. From Theorem 3.1, it follows that for any $m,n\in\mathbb{N}$, there exist $u_{m,n}\in W_0^{1,p(\cdot)}(\Omega,\omega)$, $b_{m,n}\in L^\infty(\Omega)$ such that $(u_{m,n},b_{m,n})$ is a renormalized solution of $(E,f_{m,n})$. Therefore

$$\int_\Omega b_{m,n}h(u_{m,n})\phi+\int_\Omega\big(a(x,Du_{m,n})+F(u_{m,n})\big)\cdot D(h(u_{m,n})\phi)=\int_\Omega f_{m,n}h(u_{m,n})\phi \tag{4.1}$$

holds for all $m,n\in\mathbb{N},h\in C_c^1(\mathbb{R})$, $\phi\in W_0^{1,p(\cdot)}(\Omega,\omega)\cap L^\infty(\Omega)$.

In the next lemma, we give a priori estimates that will be important in the following:

Lemma 4.1. *For $m,n\in\mathbb{N}$ let $(u_{m,n},b_{m,n})$ be a renormalized solution of $(E,f_{m,n})$. Then,*

i) For any $k>0$, we have

$$\int_\Omega|DT_k(u_{m,n})|^{p(x)}\omega(x)dx\le\frac{k}{\lambda}\|f\|_1. \tag{4.2}$$

ii) For $k>0$, there exists a constant $C_3(k)>0$, not depending on $m,n\in\mathbb{N}$, such that

$$\|DT_k(u_{m,n})\|_{W_0^{1,p(\cdot)}(\Omega,\omega)}\le C_3(k). \tag{4.3}$$

iii) For all $m,n \in \mathbb{N}$, we have:

$$\|b_{m,n}\|_1 \leq \|f\|_1. \tag{4.4}$$

Proof. For $l,k>0$, we plug $h_l(u_{m,n})T_k(u_{m,n})$ as a test function in (4.1). Then i) and ii) follow with similar arguments as used in the proof of Lemma 3.1.

To prove iii), we neglect the positive term $\int_\Omega a(x,DT_k(u_{m,n}))DT_k(u_{m,n})$ and keep

$$\int_\Omega b_{m,n}T_k(u_{m,n}) \leq \int_\Omega f_{m,n}u_{m,n}. \tag{4.5}$$

Since $b_{m,n} \in \beta(u_{m,n})$ a.e. in Ω, from (4.5) it follows that

$$\int_{\{|u_{m,n}|>k\}} |b_{m,n}| \leq \int_\Omega |f| \tag{4.6}$$

and we find iii) by passing to the limit with $k \downarrow 0$.

By definition we have

$$f_{m,n} \leq f_{m+1,n} \text{ and } f_{m,n+1} \leq f_{m,n}. \tag{4.7}$$

From Remark 3.1, it follows that

$$u^\varepsilon_{m,n} \leq u^\varepsilon_{m+1,n} \text{ and } u^\varepsilon_{m,n+1} \leq u^\varepsilon_{m,n} \tag{4.8}$$

almost everywhere in Ω for any $m,n \in \mathbb{N}$ and all $\varepsilon>0$, hence passing to the limit with $\varepsilon \downarrow 0$ in (4.8) yields

$$u_{m,n} \leq u_{m+1,n} \text{ and } u_{m,n+1} \leq u_{m,n} \tag{4.9}$$

almost everywhere in Ω for any $m,n \in \mathbb{N}$. Setting $b_\varepsilon = \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))$, using (4.8), Remark 3.2 and the fact that $b^\varepsilon_{m,n} \rightharpoonup b_{m,n}$ in $L^\infty(\Omega)$ and this convergence preserves order, we get

$$b_{m,n} \leq b_{m+1,n} \text{ and } b_{m,n+1} \leq b_{m,n} \tag{4.10}$$

almost everywhere in Ω for any $m,n \in \mathbb{N}$. By (4.10) and (4.4), for any $n \in \mathbb{N}$, there exists $b^n \in L^1(\Omega)$ such that $b_{m,n} \to b^n$ as $m \to \infty$ in $L^1(\Omega)$ and almost everywhere in Ω and $b \in L^1(\Omega)$, such that $b^n \to b$ as $n \to \infty$ in $L^1(\Omega)$ and almost everywhere in Ω. By (4.9), the sequence $(u_{m,n})_m$ is monotone increasing, hence, for any $n \in \mathbb{N}, u_{m,n} \to u^n$ almost everywhere in Ω, where $u^n : \Omega \to \overline{\mathbb{R}}$ is a measurable function. Using (4.9) again, we conclude that the sequence $(u^n)_n$ is monotone decreasing, hence $u^n \to u$ almost everywhere in Ω, where $u : \Omega \to \overline{\mathbb{R}}$ is a measurable function. In order to show that u is finite almost everywhere, we will give an estimate on the level sets of $u_{m,n}$ in the next lemma. $\square$

Lemma 4.2. *For $m,n \in \mathbb{N}$ let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(E, f_{m,n})$. Then, there exists a constant $C_4 > 0$, not depending on $m,n \in \mathbb{N}$, such that*

$$|\{|u_{m,n}| \geq l\}| \leq C_4 l^{\frac{1}{\kappa}-1} \tag{4.11}$$

for all $l \geq 1$.

Proof. With the same arguments as in Remark 3.3, we obtain

$$\begin{aligned} l|\{|u_{m,n}| \geq l\}| &= \int_{\{u_{m,n} \geq l\}} |T_l(u_{m,n})| dx \\ &\leq C \|DT_l(u_{m,n})\|_{L^{p(x)}(\Omega,\omega)} \\ &\leq C \Big(\int_\Omega |DT_l(u_{m,n})|^{p(x)} \omega(x) dx \Big)^\kappa \\ &\leq C l^{\frac{1}{\kappa}}. \end{aligned}$$

Where

$$\kappa = \begin{cases} p^- & \text{if} \quad \|DT_l(u_{m,n})\|_{L^{p(x)}(\Omega,\omega)} \leq 1 \\ p^+ & \text{if} \quad \|DT_l(u_{m,n})\|_{L^{p(x)}(\Omega,\omega)} > 1, \end{cases}$$

which implies that

$$|\{|u_{m,n}| \geq l\}| \leq \frac{C_4}{l^{1-\frac{1}{\kappa}}}, \qquad \forall l > 1.$$

Note that, as $(u_{m,n})_m$ is pointwise increasing with respect to m,

$$\lim_{m \to \infty} |\{|u_{m,n}| \geq l\}| = |\{|u^n| \geq l\}| \tag{4.12}$$

and

$$\lim_{m \to \infty} |\{|u_{m,n}| \geq -l\}| = |\{|u^n| \geq -l\}|. \tag{4.13}$$

combining (4.11) with (4.12) and (4.13), we get

$$|\{|u^n| \geq l\}| + |\{|u^n| \geq -l\}| \leq C_4 l^{\frac{1}{\kappa}-1}, \tag{4.14}$$

for any $l \geq 1$, hence u^n is finite almost everywhere for any $n \in \mathbb{N}$. By the same arguments we get

$$|\{|u| \geq l\}| + |\{|u| \geq -l\}| \leq C_4 l^{\frac{1}{\kappa}-1} \tag{4.15}$$

from (4.14), hence u is finite almost everywhere. Now, since $b_{m,n} \in \beta(u_{m,n})$ almost everywhere in Ω it follows by a subdifferential argument that $b^n \in \beta(u^n)$ and $b \in \beta(u)$ almost everywhere in Ω. $\square$

Remark 4.1.

$$\int_{\{l<|u_{m,n}|<l+k\}} a(x,Du_{m,n})\cdot Du_{m,n} \leq k\left(\int_{\{|u_{m,n}|>l\}\cap\{|f|<\delta\}}|f|+\int_{\{|f|>\delta\}}|f|\right) \tag{4.16}$$

for any $k,l,\delta>0$. Now, applying (4.11) to (4.16), we find that

$$\int_{\{l<|u_{m,n}|<l+k\}} a(x,Du_{m,n})\cdot Du_{m,n} \leq k\delta C_4 l^{\frac{1}{\kappa}-1}+k\int_{\{|f|>\delta\}}|f| \tag{4.17}$$

holds for any $k,\delta>0$, $l\geq 1$ uniformly in $m,n\in\mathbb{N}$.

4.2.2 Basic convergence

Lemma 4.3. *For $m,n\in\mathbb{N}$, let $(u_{m,n},b_{m,n})$ be a renormalized solution of $(E,f_{m,n})$. There exists a subsequence $(m(n))_n$ such that setting $f_n:=f_{m(n),n}, b_n:=b_{m(n),n}$, $u_n:=u_{m(n),n}$, we have*

$$u_n\to u \quad \textit{almost everywhere in } \Omega. \tag{4.18}$$

Moreover, for any $k>0$,

$$T_k(u_n)\to T_k(u) \quad in \quad W_0^{1,p(\cdot)}(\Omega,\omega) \tag{4.19}$$

$$DT_k(u_n)\rightharpoonup DT_k(u) \quad in \quad (L^{p(\cdot)}(\Omega,\omega))^N \tag{4.20}$$

$$a(x,DT_k(u_n))\rightharpoonup a(x,DT_k(u)) \quad in \quad (L^{p'(\cdot)}(\Omega,\omega^*))^N. \tag{4.21}$$

as $n\to\infty$.

Proof. We construct a subsequence $(m(n))_n$, such that

$$\arctan(u_{m(n),n})\to\arctan(u), b_n:=b_{m(n),n}\to b, f_n:=f_{m(n),n}\to f$$

as $n\to\infty$ in $L^1(\Omega)$ and almost everywhere in Ω. It follows that (4.18) and (4.19) hold. Combining (4.19) with (4.3), we get $T_k(u)\in W_0^{1,p(\cdot)}(\Omega,\omega)$, $T_k(u_n)\to T_k(u)$ in $W_0^{1,p(\cdot)}(\Omega,\omega)$ and (4.20) holds for any $k>0$. From (4.2) and (A2), it follows that for fixed $k>0$, given any subsequence of $(a(x,DT_k(u_n)))_n$ there exists a subsequence, still denoted by $a(x,DT_k(u_n))_n$, such that

$$a(x,DT_k(u_n)))_n\rightharpoonup\Phi_k \, in \quad (L^{p'(\cdot)}(\Omega,\omega^*))^N.$$

as $n\to\infty$. Since $h_l(u_n)(T_k(u_n)-T_k(u))$ is an admissible test function in (4.1), we obtain

$$\limsup_{n\to\infty}\int_\Omega a(x,DT_k(u_n))\cdot D(T_k(u_n)-T_k(u))\leq 0 \tag{4.22}$$

holds. Then, (4.21) follows with the same arguments as in the proof of Lemma 3.2 . □

Remark 4.2. With the same arguments as in Remark 3.4 and Lemma 3.3, we have

$$\lim_{n\to\infty}\int_\Omega\Big(a(x,DT_k(u_n))-a(x,DT_k(u))\Big)\cdot D(T_k(u_n)-T_k(u))=0. \tag{4.23}$$

$$\lim_{l\to\infty}\int_{\{l<|u|<l+1\}}a(x,Du)\cdot Du=0. \tag{4.24}$$

4.2.3 Conclusion of the proof of Theorem 4.1

It is left to prove that (u,b) satisfies

$$\int_\Omega bh(u)\phi+\int_\Omega \big(a(x,Du)+F(u)\big)\cdot D(h(u)\phi)=\int_\Omega fh(u)\phi, \tag{4.25}$$

for all $h\in C_c^1(\mathbb{R})$, $\phi\in W_0^{1,p(\cdot)}(\Omega,\omega)\cap L^\infty(\Omega)$. To this end, we take $h\in C_c^1(\mathbb{R})$, $\phi\in W_0^{1,p(\cdot)}(\Omega,\omega)\cap L^\infty(\Omega)$ arbitrary and plug $h_l(u_n)h(u)\phi$ into (4.1) to obtain

$$I_{n,l}^1+I_{n,l}^2+I_{n,l}^3=I_{n,l}^4, \tag{4.26}$$

where

$$\begin{aligned}&I_{n,l}^1=\int_\Omega b_n h_l(u_n)h(u)\phi,\ \ I_{n,l}^2=\int_\Omega a(x,Du_n)\cdot D(h_l(u_n)h(u)\phi)\\ &I_{n,l}^3=\int_\Omega F(u_n)\cdot D(h_l(u_n)h(u)\phi),\ \ I_{n,l}^4=\int_\Omega fh_l(u_n)h(u)\phi.\end{aligned}$$

Step 1: Passing to the limit with $n\to\infty$, applying the convergence results from Lemma 4.3, we get

$$\lim_{n\to\infty} I_{n,l}^1=\int_\Omega bh_l(u)h(u)\phi \tag{4.27}$$

$$\lim_{n\to\infty} I_{n,l}^4=\int_\Omega fh_l(u)h(u)\phi. \tag{4.28}$$

Let us write

$$I_{n,l}^2=I_{n,l}^{2,1}+I_{n,l}^{2,2},$$

where

$$\begin{aligned}I_{n,l}^{2,1}&=\int_\Omega h_l(u_n)a(x,Du_n)\cdot D(h(u)\phi),\\ I_{n,l}^{2,2}&=\int_\Omega h_l'(u_n)a(x,Du_n)\cdot Du_n h(u)\phi.\end{aligned}$$

With similar arguments as in the proof of (3.27), it follows that

$$\lim_{n\to\infty} I_{n,l}^{2,1}=\int_\Omega h_l(u)a(x,Du)\cdot D(h(u)\phi). \tag{4.29}$$

By (4.17), we get the estimate

$$|\lim_{n\to\infty} I_{n,l}^{2,2}|\le \|h\|_\infty\|\phi\|_\infty\Big(\delta C_4 l^{\frac{1}{\kappa}-1}+\int_{\{|f|>\delta\}}|f|\Big), \tag{4.30}$$

for all $n\in\mathbb{N}$ and all $l\ge 1$, $\delta>0$.

Next, we write

$$I_{n,l}^{3}=I_{n,l}^{3,1}+I_{n,l}^{3,2},$$

where

$$\lim_{n\to\infty} I_{n,l}^{3,1}=\int_\Omega h_l(u)F(u)\cdot D(h(u)\phi), \tag{4.31}$$

$$\lim_{n\to\infty} I_{n,l}^{3,2}=\int_\Omega h_l'(u)F(u)\cdot Du\,h(u)\phi \tag{4.32}$$

follows with the same arguments as in (3.28)-(3.32).

Step 2: Passing to the limit with $l\to\infty$, combining (4.26) with (4.27)-(4.32), we get for all $\delta>0$ and all $l\geq 1$,

$$I_l^1+I_l^2+I_l^3+I_l^4+I_l^5=I_l^6 \tag{4.33}$$

where

$$\begin{aligned}
&I_l^1=\int_\Omega bh_l(u)h(u)\phi,\\
&I_l^2=\int_\Omega h_l(u)a(x,DT_{l+1}(u))\cdot D(h(u)\phi),\\
&|I_l^3|\leq \|h\|_\infty\|\phi\|_\infty\Big(\delta C_4 l^{\frac{1}{\kappa}-1}+\int_{\{|f|>\delta\}}|f|\Big),
\end{aligned}$$

for any $\delta>0$ and

$$\begin{aligned}
&I_l^4=\int_\Omega h_l'(u)F(u)h(u)\phi Du,\\
&I_l^5=\int_\Omega h_l(u)F(u)\cdot D(h(u)\phi),\\
&|I_l^6|=\int_\Omega fh_l(u)h(u)\phi.
\end{aligned}$$

Choosing $m>0$, such that $supp\,h\subset[-m,m]$, we can replace u by $T_m(u)$ in $I_l^1,\dots,I_l^6$, hence

$$\lim_{l\to\infty} I_l^1=\int_\Omega bh(u)\phi, \tag{4.34}$$

$$\lim_{l\to\infty} I_l^2=\int_\Omega a(x,Du)\cdot D(h(u)\phi), \tag{4.35}$$

$$\lim_{l\to\infty} |I_l^3|\leq \|h\|_\infty\|\phi\|_\infty\int_{\{|f|>\sigma\}}|f|, \tag{4.36}$$

$$\lim_{l\to\infty} I_l^4=0, \tag{4.37}$$

$$\lim_{l\to\infty} I_l^5=\int_\Omega F(u)\cdot D(h(u)\phi), \tag{4.38}$$

$$\lim_{l\to\infty} |I_l^6|=\int_\Omega fh(u)\phi, \tag{4.39}$$

for all $\delta>0$. Combining (4.33) with (4.34) - (4.39), we finally obtain that (4.25) holds for all $h\in C_c^1(\mathbb{R})$ and all $\phi\in W_0^{1,p(\cdot)}(\Omega,\omega)\cap L^\infty(\Omega)$.

Hence (u,b) satisfies $(R1)$,$(R2)$ and $(R3)$ and proof of theorem is completed.

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