Composite Implicit Iteration Process for Asymptotically Hemi-Pseudocontractive Mappings

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Abstract. In Banach space, the composite implicit iterative process for uniformly L-Lipschitzian asymptotically hemi-pseudocontractive mappings are studied, and the sufficient and necessary conditions of strong convergence for the composite implicit iterative process are obtained.

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1 Introduction and preliminaries

Throughout this work, we assume that E is a real Banach space. $E^*$ is the dual space of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^*: \langle x, f \rangle = \| x \| \| f \|, \| f \| = \| x \| \} \text{, } \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes duality pairing between E and $E^*$. A single-valued normalized duality mapping is denoted by $j$.

Let $C$ be a nonempty subset of $E$ and $T : C \to C$ a mapping, we denote the set of fixed points of $T$ by $F(T) = \{ x \in C; Tx = x \}$.

Definition 1.1. ([1]) $T$ is said to be asymptotically nonexpansive, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$\| T^n x - T^n y \| \leq k_n \| x - y \|, \text{ } \forall x, y \in C \text{ and } n \geq 1.$

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(2) ([2]) \( T \) is said to be uniformly \( L \)-Lipschitzian, if there exists \( L > 0 \) such that
\[
\|T^nx - T^ny\| \leq L\|x - y\|, \quad \forall x, y \in C \text{ and } n \geq 1.
\]

(3) ([3]) \( T \) is said to be asymptotically pseudocontractive, if there exists a sequence \( \{k_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} k_n = 1 \), for any \( x, y \in C \), there exists \( j(x - y) \in J(x - y) \) such that
\[
\langle T^nx - T^ny, j(x - y) \rangle \leq k_n\|x - y\|^2, \quad n \geq 1.
\]

(4) ([4]) \( T \) is said to be asymptotically hemi-pseudocontractive, if \( F(T) \neq \emptyset \) and there exists a sequence \( \{k_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} k_n = 1 \) such that, for any \( x \in C \) and \( p \in F(T) \), there exists \( j(x - p) \in J(x - p) \) such that
\[
\langle T^nx - T^np, j(x - p) \rangle \leq k_n\|x - p\|^2, \quad n \geq 1.
\]

**Remark 1.1.** It is easy to see that if \( T \) is an asymptotically nonexpansive mapping, then \( T \) is a uniformly \( L \)-Lipschitzian and asymptotically pseudocontractive mapping, where \( L = \sup_{n \geq 1} \{k_n\} \); if \( T \) is an asymptotically pseudocontractive mapping with \( F(T) \neq \emptyset \), then \( T \) is an asymptotically hemi-pseudocontractive mapping.

Let \( C \) be a nonempty closed convex subset of \( E \) and \( T : C \to C \) be a uniformly \( L \)-Lipschitzian asymptotically hemi-pseudocontractive mapping, for any given \( x_1 \in C \), we introduce a composite implicit iteration process \( \{x_n\} \) as follows:
\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nT^nx_n, \\
y_n &= (1 - \beta_n)x_n + \beta_nT^nx_{n+1},
\end{align*}
\]
where \( \{\alpha_n\}, \{\beta_n\} \) are two real sequences in \([0, 1]\).

As \( \beta_n = 0 \) for all \( n \geq 1 \), then (1.1) reduces to
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nT^nx_n.
\]

**Remark 1.2.** For any given \( x_n \in C \), define the mapping \( A_n : C \to C \), such as:
\[
A_nx = (1 - \alpha_n)x_n + \alpha_nT^n[(1 - \beta_n)x_n + \beta_nT^nx_n], \quad \forall x \in C,
\]
where \( C \) is a nonempty closed convex subset of \( E \) and \( T : C \to C \) is a uniformly \( L \)-Lipschitzian. Then
\[
\begin{align*}
\|A_nx - A_ny\| &= \|\alpha_n(T^n[(1 - \beta_n)x_n + \beta_nT^nx_n] - T^n[(1 - \beta_n)x_n + \beta_nT^ny_n])\| \\
&\leq \alpha_n\beta_nL\|T^nx - T^ny\| \\
&\leq \alpha_n\beta_nL^2\|x - y\|
\end{align*}
\]
for all \( x, y \in C \). Thus \( A_n \) is a contraction mapping if \( \alpha_n\beta_nL^2 < 1 \) for all \( n \geq 1 \), and so there exists a unique fixed point \( x_{n+1} \in C \) of \( A_n \), such that \( x_{n+1} = (1 - \alpha_n)x_n + \alpha_nT^n[(1 - \beta_n)x_n + \beta_nT^nx_{n+1}] \). This shows that the composite implicit iteration process (1.1) is well defined.
For any a point \( z \) and a set \( K \) in \( E \), we denote the distance between \( z \) and \( K \) by \( d(z,K) = \inf_{x \in K} \|z-x\| \).

Recently, Kan Xuzhou and Guo Weiping [5] proved the sufficient and necessary condition for the strong convergence of the composite implicit iterative process for a Lipschitzian pseudocontractive mapping in Banach space.

In this work, we obtained sufficient and necessary conditions of the strong convergence of the iterations sequences (1.1) and (1.2) for uniformly L-Lipschitzian asymptotically hemi-pseudocontractive mappings in Banach spaces.

**Lemma 1.1.** [6] Let \( \{a_n\}, \{b_n\}, \{c_n\} \) be sequences of nonnegative real numbers satisfying the inequality
\[
a_{n+1} \leq (1+c_n)a_n + b_n, \quad \forall n \geq n_0,
\]
where \( n_0 \) is some positive integer, \( \sum_{n=1}^{\infty} c_n < \infty \) and \( \sum_{n=1}^{\infty} b_n < \infty \). Then \( \lim_{n \to \infty} a_n \) exists.

**Lemma 1.2.** [7] Let \( C \) be a nonempty subset of a Banach space \( E \) and \( T: C \to C \) be an asymptotically hemi-pseudocontractive mapping with the sequence \( \{k_n\} \subset [1,\infty) \), \( \lim_{n \to \infty} k_n = 1 \), then
\[
\|x - p\| \leq \|x - p + r[(k_n I - T^n)x - (k_n I - T^n)p]\|
\]
for all \( x \in C, p \in F(T), r > 0 \) and \( n \geq 1 \), where \( I \) is a identity mapping.

## 2 Main results

**Lemma 2.1.** Let \( E \) be a real Banach space and \( C \) be a nonempty closed convex subset of \( E \). Let \( T: C \to C \) be a uniformly L-Lipschitzian asymptotically hemi-pseudocontractive mapping with the sequence \( \{k_n\} \subset [1,\infty) \), \( \lim_{n \to \infty} k_n = 1 \) and Lipschitz constant \( L > 1 \). Suppose that the sequence \( \{x_n\} \) is defined by (1.1) satisfying the following conditions:

(i) \( \sum_{n=1}^{\infty} a_n \beta_n < \infty \) and \( \sum_{n=1}^{\infty} a_n^2 < \infty \);
(ii) \( \sum_{n=1}^{\infty} a_n (k_n - 1) < \infty \);
(iii) \( a_n \beta_n L^2 < 1 \) for all \( n \geq 1 \).

Then

(1) there exists a sequence \( \{\gamma_n\} \subseteq [0,\infty) \) and some positive integer \( n_0 \), such that \( \sum_{n=1}^{\infty} \gamma_n < \infty \) and
\[
\|x_{n+1} - p\| \leq (1 + \gamma_n)\|x_n - p\|
\]
for all \( p \in F(T) \) and \( n \geq n_0 \).

(2) The limit \( \lim_{n \to \infty} d(x_n, F(T)) \) exists.
Proof. It follows from the condition (iii) and Remark 1.2 that the sequence (1.1) is well defined. By (1.1), we have

\[
x_n = x_{n+1} + \alpha_n(x_n - \alpha_n T^n y_n)
\]

\[
= (1 + \alpha_n) x_{n+1} + \alpha_n((k_n I - T^n)x_{n+1} - (1 + k_n)\alpha_n x_{n+1} + \alpha_n(T^n x_{n+1} - T^n y_n))
\]

\[
= (1 + \alpha_n) x_{n+1} + \alpha_n((k_n I - T^n)x_{n+1} - (1 + k_n)\alpha_n[x_n + \alpha_n(T^n y_n - x_n)])
\]

\[
+ \alpha_n(x_{n+1} + \alpha_n(T^n x_{n+1} - T^n y_n))
\]

\[
= (1 + \alpha_n) x_{n+1} + \alpha_n((k_n I - T^n)x_{n+1} - k_n\alpha_n x_n + (1 + k_n)\alpha_n^2(x_n - T^n y_n))
\]

\[
+ \alpha_n(T^n x_{n+1} - T^n y_n)
\]

(2.1)

and

\[
p = (1 + \alpha_n)p + \alpha_n(k_n I - T^n)p - k_n \alpha_n p
\]

(2.2)

for all \( p \in F(T) \). Together with (2.1) and (2.2), we can obtain

\[
x_n - p = (1 + \alpha_n)(x_{n+1} - p) + \alpha_n[(k_n I - T^n)x_{n+1} - (k_n I - T^n)p] - k_n \alpha_n(x_n - p)
\]

\[
+ (1 + k_n)\alpha_n^2(x_n - T^n y_n) + \alpha_n(T^n x_{n+1} - T^n y_n).
\]

(2.3)

Notice that

\[
(1 + \alpha_n)(x_{n+1} - p) + \alpha_n[(k_n I - T^n)x_{n+1} - (k_n I - T^n)p]
\]

\[
= (1 + \alpha_n)[(x_{n+1} - p) + \frac{\alpha_n}{1 + \alpha_n}((k_n I - T^n)x_{n+1} - (k_n I - T^n)p)].
\]

Using Lemma 1.2, we obtain that

\[
\|(1 + \alpha_n)(x_{n+1} - p) + \alpha_n(k_n I - T^n)(x_{n+1} - p)\| \geq (1 + \alpha_n)\|x_{n+1} - p\|.
\]

(2.4)

It follows from (2.3) and (2.4) that

\[
\|x_n - p\| \geq (1 + \alpha_n)\|x_{n+1} - p\| - k_n \alpha_n\|x_n - p\| - (1 + k_n)\alpha_n^2\|x_n - T^n y_n\| - \alpha_n\|T^n x_{n+1} - T^n y_n\|.
\]

This implies that

\[
(1 + \alpha_n)\|x_{n+1} - p\| \leq (1 + k_n \alpha_n)\|x_n - p\| + (1 + k_n)\alpha_n^2\|x_n - T^n y_n\|
\]

\[
+ \alpha_n\|T^n x_{n+1} - T^n y_n\|.
\]

(2.5)

Next, we make the following estimations:

\[
\|y_n - p\| = \|(1 - \beta_n)(x_n - p) + \beta_n(T^n x_{n+1} - p)\|
\]

\[
\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|x_{n+1} - p\|
\]
and
\[
\|x_n - T^u y_n\| \leq \|x_n - p\| + \|T^u y_n - p\| \\
\leq \|x_n - p\| + L\|y_n - p\| \\
\leq [1 + L(1 - \beta_n)]\|x_n - p\| + \beta_n L^2\|x_{n+1} - p\|.
\] (2.6)

Furthermore,
\[
\|x_n - y_n\| = \beta_n\|x_n - T^u x_{n+1}\| \\
\leq \beta_n(\|x_n - p\| + \|T^u x_{n+1} - p\|) \\
\leq \beta_n\|x_n - p\| + L\beta_n\|x_{n+1} - p\| \\
\] (2.7)

and
\[
\|T^u x_{n+1} - T^u y_n\| \leq L\|x_{n+1} - y_n\| \\
= L\|x_n - y_n + \alpha_n(T^u y_n - x_n)\| \\
\leq L\|x_n - y_n\| + \alpha_n L\|T^u y_n - x_n\|.
\] (2.8)

Substituting (2.6) and (2.7) into (2.8), we have
\[
\|T^u x_{n+1} - T^u y_n\| \leq \left[\alpha_n L + \beta_n L + \alpha_n L^2(1 - \beta_n)\right]\|x_n - p\| + \beta_n L^2(1 + \alpha_n L)\|x_{n+1} - p\|. \\
\] (2.9)

Substituting (2.6) and (2.9) into (2.5), we have
\[
(1 + \alpha_n)\|x_{n+1} - p\| \leq (1 + k_n \alpha_n)\|x_n - p\| + (1 + k_n)\alpha_n^2[1 + L(1 - \beta_n)]\|x_n - p\| \\
+ (1 + k_n)\alpha_n^2\beta_n L^2\|x_{n+1} - p\| \\
+ \alpha_n \alpha_n L + \beta_n L + \alpha_n L^2(1 - \beta_n)\|x_n - p\| \\
+ \alpha_n \beta_n L^2(1 + \alpha_n L)\|x_{n+1} - p\|.
\]

Since \(1 + \alpha_n \geq 1\), this implies that
\[
\|x_{n+1} - p\| \leq (1 + (k_n - 1) \alpha_n)\|x_n - p\| + \alpha_n \alpha_n L + \beta_n L + \alpha_n L^2(1 - \beta_n)\|x_n - p\| \\
+ \alpha_n \alpha_n L + \beta_n L + \alpha_n L^2(1 - \beta_n)\|x_n - p\| \\
+ \alpha_n \alpha_n L + \beta_n L + \alpha_n L^2(1 - \beta_n)\|x_{n+1} - p\| \\
\leq (1 + (k_n - 1) \alpha_n)\|x_n - p\| + (1 + k_n)\alpha_n^2[1 + L(1 - \beta_n)]\|x_n - p\| \\
+ \alpha_n \alpha_n L + \beta_n L + \alpha_n L^2(1 - \beta_n)\|x_n - p\| \\
+ \alpha_n \alpha_n L + \beta_n L + \alpha_n L^2(1 - \beta_n)\|x_{n+1} - p\| \\
\leq (1 + (k_n - 1) \alpha_n)\|x_n - p\| + (1 + k_n)\alpha_n^2[1 + L(1 + L)\alpha_n L^2\|x_n - p\| \\
+ \alpha_n \alpha_n L + \beta_n L + \alpha_n L^2(1 + \alpha_n L)\|x_{n+1} - p\| \\
+ \alpha_n \alpha_n L + \beta_n L + \alpha_n L^2(1 + \alpha_n L)\|x_{n+1} - p\|. \\
\]
and so
\[
\left[1 - (1+k_n)\alpha_n^2 \beta_n L^2 - \alpha_n \beta_n L^2 - \alpha_n^2 \beta_n L^3\right] \|x_{n+1} - p\| \\
\leq \left[1 + (k_n - 1)\alpha_n + (1+k_n + L)(1+L)\alpha_n^2 + (L - L\alpha_n - Lk_n\alpha_n - L^2 \alpha_n)\alpha_n \beta_n\right] \|x_n - p\|.
\]
Since \(\lim_{n \to \infty} \alpha_n \beta_n = 0\) and \(\lim_{n \to \infty} k_n = 1\), there exists some positive integer \(n_0\), such that 

\[
\alpha_n \beta_n \leq \frac{1}{8L^3} \quad \text{and} \quad k_n < L \quad \text{for all} \quad n \geq n_0. 
\]
Thus

\[
1 - (1+k_n)\alpha_n^2 \beta_n L^2 - \alpha_n \beta_n L^2 - \alpha_n^2 \beta_n L^3 \geq 1 - \frac{1+k_n}{8L^3} L^2 - \frac{1}{8L^3} L^3 = \frac{8L - 2L - k_n}{8L} \geq \frac{8L - 4L}{8L} = \frac{1}{2}.
\]

After finishing deformation, we have

\[
\|x_{n+1} - p\| \leq \{1 + 2[(k_n - 1)\alpha_n + (1+k_n + L)(1+L)\alpha_n^2 + (L + (2+k_n)L^2 + L^3)\alpha_n \beta_n]\} \|x_n - p\| \\
= (1 + \gamma_n)\|x_n - p\|, \quad n \geq n_0, 
\]
where \(\gamma_n = 2[(k_n - 1)\alpha_n + (1+k_n + L)(1+L)\alpha_n^2 + (L + (2+k_n)L^2 + L^3)\alpha_n \beta_n]\), and \(\sum_{n=1}^{\infty} \gamma_n < \infty\) by conditions (i) and (ii), Thus (1) is proved.

(2) Taking the infimum over all \(p \in F(T)\) on both sides in (2.10), we get

\[
d(x_{n+1}, F(T)) \leq (1 + \gamma_n) d(x_n, F(T)), \quad n \geq n_0.
\]
It follows from Lemma 1.1 that the limit \(\lim_{n \to \infty} d(x_n, F(T))\) exists. This completes the proof.

**Theorem 2.1.** Let \(E\) be a real Banach space and \(C\) be a nonempty closed convex subset of \(E\). Let \(T: C \to C\) be a uniformly \(L\)-Lipschitzian asymptotically hemi-pseudocontractive mapping with the sequence \(\{k_n\} \subset [1, \infty)\), \(\lim_{n \to \infty} k_n = 1\) and Lipschitz constant \(L > 1\). Suppose that the sequence \(\{x_n\}\) is defined by (1.1) satisfying the following conditions:

(i) \(\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty\) and \(\sum_{n=1}^{\infty} \alpha_n^2 < \infty\);

(ii) \(\sum_{n=1}^{\infty} \alpha_n (k_n - 1) < \infty\);

(iii) \(\alpha_n \beta_n L^2 < 1\) for all \(n \geq 1\).

Then \(\{x_n\}\) converges strongly to some fixed point of \(T\) if and only if \(\liminf_{n \to \infty} d(x_n, F(T)) = 0\).
Proof. The necessary of Theorem 2.1 is obvious. we just need to prove the sufficiency.
From Lemma 2.1 (2) and the condition \( \liminf_{n \to \infty} d(x_n, F(T)) = 0 \), we have
\[
\lim_{n \to \infty} d(x_n, F(T)) = 0.
\]
Next, we show that \( \{ x_n \} \) is a Cauchy sequence. In fact, using Lemma 2.1(1), for any \( p \in F(T) \) and any positive integers \( m,n,m > n \geq n_0 \), we have
\[
\| x_m - p \| \leq (1 + \gamma_m)\| x_{m-1} - p \|
\]
\[
\leq e^{\gamma_{m-1}}\| x_{m-1} - p \|
\]
\[
\leq e^{\sum_{j=0}^{m-1} \gamma_j} \| x_n - p \|
\]
\[
\leq M \| x_n - p \|,
\]
where \( M = e^{\sum_{j=0}^{\infty} \gamma_j} \). Thus, we have
\[
\| x_n - x_m \| \leq \| x_n - p \| + \| x_m - p \| \leq (1 + M)\| x_n - p \|.
\]
Taking the infimum over all \( p \in F(T) \), we have
\[
\| x_n - x_m \| \leq (1 + M) d(x_n, F(T)).
\]
It follows from \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \) that \( \{ x_n \} \) is a Cauchy sequence. Since \( C \) is closed subset of \( E \), so there exists a \( p_0 \in C \) such that \( x_n \to p_0 \) as \( n \to \infty \). Further, since \( T \) is uniformly L-Lipschitzian, it is easy to prove that \( F(T) \) is closed. Again since \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \) and \( p_0 \in F(T) \), this shows that \( \{ x_n \} \) converges strongly to a fixed point of \( T \), this completes the proof.

Using Theorem 2.1, we have the following:

**Theorem 2.2.** Let \( E \) be a real Banach space and \( C \) be a nonempty closed convex subset of \( E \). Let \( T:C \to C \) be a uniformly L-Lipschitzian asymptotically hemi-pseudocontractive mapping with the sequence \( \{ k_n \} \subset [1,\infty) \), \( \lim_{n \to \infty} k_n = 1 \) and Lipschitz constant \( L > 1 \). Suppose that the sequence \( \{ x_n \} \) is defined by (1.2) satisfying the following conditions:

(i) \( \sum_{n=1}^{\infty} a_n^2 < \infty \);

(ii) \( \sum_{n=1}^{\infty} a_n (k_n - 1) < \infty \).

Then \( \{ x_n \} \) converges strongly to some fixed point of \( T \) if and only if \( \liminf_{n \to \infty} d(x_n, F(T)) = 0 \).

**Remark 2.1.** By Remark 1.1, clearly, Theorem 2.1 and Theorem 2.2 hold for uniformly L-Lipschitzian and asymptotically pseudocontractive mappings with \( F(T) \neq \emptyset \) and for asymptotically nonexpansive mappings with \( F(T) \neq \emptyset \).
References