J. Math. Study
doi: 10.4208/jms.v48n4.15.07

Vol. **48**, No. 4, pp. 406-418
December 2015

One-Side Derived Categories of Pre-Strict P-Semi-Abelian Categories

Liangyu Chen *and Lin Xin

Fuzhou College of Foreign Studies and Trade, Fuzhou 350202, Fujian Province, P. R. China.

Received 26 December, 2014; Accepted 23 October, 2015

Abstract. In this paper, we study a class of P-semi-abelian categories, as well as left and right cohomological functors. Then we establish the corresponding one-side derived categories.

AMS subject classifications: 18A05, 18A20, 18A22

Key words: P-semi-abelian category, cohomological functor, one-side derived category.

1 Introduction

The concept of derived categories seems to have first appeared in Verdier [9]. He introduced triangulated categories and developed localization theories to established derived categories. These theories have been studied by many mathematicians during the last four decades and applied in some branches of mathematics, such as representation theories of algebra and the algebraic geometry (see [1,2]).

The notion of semi-abelian categories was invented several times by different mathematicians under different names. At the end of the 1960's, Palamodov [8] introduced the same concept under the name of "semi-abelian categories". Therefore some authors use "P-semi-abelian category" to denote the categories above. The properties of P-semi-abelian categories was optimized by Kopylov [3,5] in recent years. In the sequel, we call the "P-semi-abelian category" as "semi-abelian category" for short.

Milicic [7] studied the derived category of abelian categories. The aim of this article is to generalize these properties to a class of semi-abelian categories, called pre-strict semi-abelian categories. In Section 2, we first give some necessary definitions or notations and recall basic facts. Then we construct the left and right cohomological functors in semi-abelian categoreis. In Section 3, we establish the pre-strict semi-abelian category and investigate some properties of it. Then we introduce the concept of left quasi-bimorphisms and right quasi-bimorphisms in pre-strict semi-abelian categories to form

*Corresponding author. *Email addresses:* `466984702@qq.com` (L. Y. Chen), `xinlin@fjnu.edu.cn` (L. Xin)

localizing classes compatible with triangulation respectively. Consequently, we obtain the corresponding one-side derived categories.

2 Right cohomological functor H_-^n and left cohomological functor H_+^n

The theory of P-semi-abelian categories is being optimized by mathematicians in recent years. Here the definition of P-semi-abelian category is in the sense of Palamodov [7]. In this paper, we call the P-semi-abelian category as semi-abelian category for short. Kopylov claimed in [5] that an additive category $\mathscr{E}$ is called pre-abelian category if every morphism in $\mathscr{E}$ has a kernel and a cokernel. Furthermore, in pre-abelian categories, each morphism α admits the canonical decomposition α=(imα)$\overline{\alpha}$(coimα), where imα=ker(cokerα), coimα=coker(kerα), and $\overline{\alpha}$ is unique. If the $\overline{\alpha}$ is an isomorphism, then α is called strict. A pre-abelian category is called semi-abelian category if for every morphism α, $\overline{\alpha}$ is bimorphism. The equivalent definition of semi-abelian categories can be seen in [3].

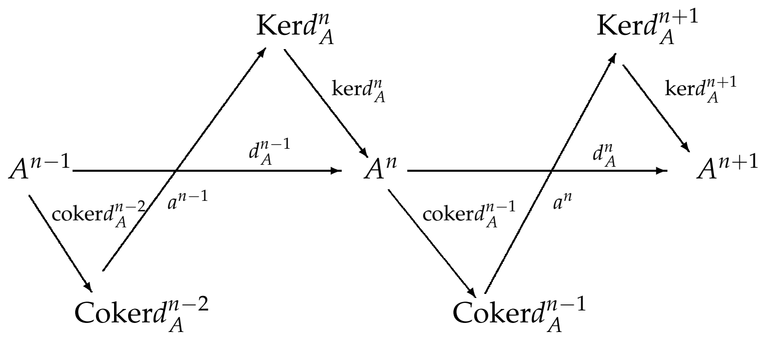
A preabelian category is semi-abelian if for every morphism α, $\overline{\alpha}$ is a bimorphism, i.e. $\overline{\alpha}$ is a monomorphism and an epimorphism simultaneously. We refer to [5] for equivalent definitions of semi-abelianity for a preabelian category.

For an additive category $\mathscr{C}$,$C(\mathscr{A})$ is the category of complexes of $\mathscr{C}$-objects with complexes of $\mathscr{C}$-objects as objects and morphisms of complexes as morphisms.

Let $\mathscr{C}$ be a semi-abelian category. Then so is $C(\mathscr{A})$. We denote a complex $A^\cdot = (A^n, d_A^n)_{n\in Z}$ in $C(\mathscr{A})$ to be as follows:

$$A^\cdot : \cdots \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \cdots$$

For each $n \in Z$, there is a commutative diagram



(2.1)

with determinate a^n and a^{n-1}. Evidently, Coker($a^{n-1}\cdot$cokerd^{n-2})=Cokera^{n-1} and Ker ((kerd^{n+1})$\cdot a^n$)=Kera^n. We denote $\mathrm{H}_-^n(A^\cdot) = \mathrm{Coker} a^{n-1}$ and $\mathrm{H}_+^n(A^\cdot) = \mathrm{Ker} a^n$. We call $\mathrm{H}_-^n(A^\cdot)$ the nth *right cohomology of complex* $A^\cdot$ and $\mathrm{H}_+^n(A^\cdot)$ the nth *left cohomology of complex* $A^\cdot$.

Remark 2.1. Kopylov [4] claimed that for any $n\in Z$, there exists an unique monomorphism m^n:$\mathrm{H}^n_-(A^\cdot)\to\mathrm{H}^n_+(A^\cdot)$ such that $(\ker a^n)m^n(\mathrm{coker} a^{n-1})=(\mathrm{coker} d_A^{n-1})(\ker d_A^n)$. Moreover, he proved in [5] that $\mathscr{A}$ is quasi-abelian if and only if m^n is an isomorphism for any $n\in Z$.

We need the following preliminaries for constructing the right and left cohomological functors. According to Kopylov's notations in [4], for a semi-abelian cagegory, M_c is the class of all strict monomorphisms; P_c is the class of all strict epimorphisms; O_c is the class of all strict morphisms.

Lemma 2.1. [4] *The following hold in a semi-abelian category.*

(1) If $gf\in M_c$, then $f\in M_c$; if $gf\in P_c$, then $g\in P_c$.

(2) If $f,g\in M_c$ and gf is defined, then $gf\in M_c$; if $f,g\in P_c$ and gf is defined, then $gf\in P_c$.

(3) If $fg\in O_c$ and $f\in M$, then $g\in O_c$; if $fg\in O_c$ and $g\in P$, then $f\in O_c$.

Remark 2.2. M_c is the class of all strict monomorphisms. P_c is the class of all strict epimorphisms. O_c is the class of all strict morphisms.

Definition 2.1. [5] Let $\mathscr{E}$ be an additive category. An object $A=(A^n,d_A^n)_{n\in Z}$ in $C(\mathscr{E})$ is defined as *strict complex* if d_A^n is a strict morphism for each $n\in Z$.

Lemma 2.2. [5] *Let $\mathscr{A}$ be a semi-abeilan category and $A^\cdot=(A^n,d_A^n)_{n\in Z}$ be an object of $C(\mathscr{A})$. Then $A^\cdot$ is a strict complex if and only if for any $n\in Z$, a^n is a strict morphism (a^n is the same as figure* (2.1)).

Proposition 2.1. Let $\mathscr{C}$ be a semi-abelian category and $A^\cdot=(A^n,d_A^n)_{n\in Z}$ be a strict complex in $C(\mathscr{C})$. For each $n\in Z$

(1) there exists a strict monomorphism $g_A^n:\mathrm{Im} d_A^n\to\mathrm{Ker} d_A^{n+1}$ such that $\mathrm{H}_-^{n+1}(A^\cdot)=\mathrm{Coker} g_A^n$ and $g_A^n=\ker(\mathrm{coker} g_A^n)$;

(2) there exists a strict epimorphism t_A^n:$\mathrm{Coker} d_A^n\to\mathrm{Im} d_A^{n+1}$ such that $\mathrm{H}_+^{n+1}(A^\cdot)=\mathrm{Ker} t_A^n$ and $t_A^n=\mathrm{coker}(\ker t_A^n)$.

Proof. (1) First we show the existence of g_A^n. Let $b=\mathrm{coker} d_A^n, c=\mathrm{im} d_A^n$ and $e=\ker d_A^{n+1}$. Since $bd_A^n=0$, there exist an unique morphism $\lambda:A^n\to\mathrm{Im} d_A^n$ such that $d_A^n=c\lambda$. For $\mathscr{C}$ is a semi-abelian category, this implies that λ is an epimorphism. Since $d_A^{n+1}c\lambda=d_A^{n+1}d_A^n=0$, there exact an unique morphism $g_A^n:\mathrm{Im} d_A^n\to\mathrm{Ker} d_A^{n+1}$ such that $c=eg_A^n$. Moreover, c is a monomorphism, so g_A^n is a monomorphism.

Then we want to show that $\mathrm{H}_-^n(A^\cdot)=\mathrm{Coker} g_A^n$. It is enough to prove the existence of an epimorphism d:$\mathrm{Coker} d_A^{n-1}\to\mathrm{Im} d_A^n$ with the equality $a^n=g_A^n d$. Let $t=\mathrm{coker} d_A^{n-1}$. Since $bea^n t=bd_A^n=0$ and t is an epimorphism, $bea^n=0$. This implies that $d:\mathrm{Coker} d_A^{n-1}\to\mathrm{Im} d_A^n$ meet the equality $ea^n=cd=eg_A^n d$. Therefore, $a^n=g_A^n d$ as e is a monomorphism. Since

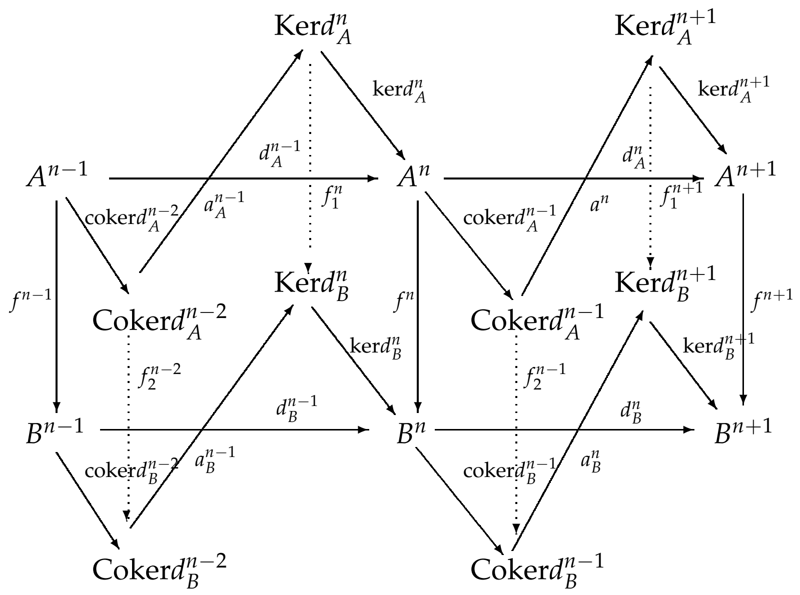
$cdt = ea^n t = d_A^n = c\lambda$ and c is a monomorphism, $dt = \lambda$. Moreover, λ is an epimorphism, so d is an epimorphism.

Finally we have to check that $g_A^n = \ker(\mathrm{coker} g_A^n)$. Since $eg_A^n = c$ is a strict morphism and e is a monomorphism, g_A^n is a strict morphism by Lemma 2.1. It follows that g_A^n is a strict monomorphism, i.e., a kernel.

(2) By the proof of (1), we have $a^n = g_A^n d$. Since g_A^n is a monomorphism, $\mathrm{H}_+^n(A^\cdot) = \mathrm{Ker} a^n = \mathrm{Ker} d$. $A^\cdot = (A^n, d_A^n)_{n\in Z}$ is a strict complex in $C(\mathscr{C})$. This implies that a^n is a strict morphism by Lemma 2.2. Moreover, g_A^n is a monomorphism, so d is a strict epimorphism, i.e., a cokernel. Put $t_A^n = d$, then the proposition follows. □

Now we can construct the corresponding right cohomological functor H_-^n and left cohomological functor H_+^n.

Let $A^\cdot = (A^n, d_A^n)_{n\in Z}$ and $B^\cdot = (B^n, d_B^n)_{n\in Z}$ be two complexes in $C(\mathscr{C})$. For any $n \in Z$ and an arbitrary morphism $f^\cdot : A^\cdot \to B^\cdot$, there is a commutative diagram



(2.2)

where $f_1^n, f_1^{n+1}, f_2^{n-1}$ and f_2^{n-2} are determined by $f^\cdot$. By the definition of kernels and cokernels, there exists an unique morphism $f_-^n : \mathrm{Coker} a_A^{n-1} \to \mathrm{Coker} a_B^{n-1}$ such that $f_-^n \cdot \mathrm{coker} a_A^{n-1} = \mathrm{coker} a_B^{n-1} \cdot f_1^n$, as well as an unique morphism $f_+^n : \mathrm{Ker} a_A^n \to \mathrm{Ker} a_B^n$ such that $\ker a_B^n \cdot f_+^n = f_2^{n-1} \cdot \ker a_A^n$. Put $H_-^n(f^\cdot) = f_-^n$ and $H_+^n(f^\cdot) = f_+^n$, then we have the following proposition.

Proposition 2.2. Let $\mathscr{C}$ be a semi-abelian category. Let $f^\cdot : A^\cdot \to B^\cdot$ be a morphism of complexes in $C(\mathscr{C})$. We define $H_-^n(A^\cdot) = \mathrm{H}_-^n(A^\cdot)$ and $H_+^n(A^\cdot) = \mathrm{H}_+^n(A^\cdot)$. Also, we define $H_-^n(f^\cdot)$ and $H_+^n(f^\cdot)$ as the above. Then $H_-^n : C(\mathscr{C}) \to \mathscr{C}$ and $H_+^n : C(\mathscr{C}) \to \mathscr{C}$ are additive functors.

Proof. According to the definition of additive functors in reference [11], we need to get the equality $H_+^n(f^\cdot+g^\cdot)=H_+^n(f^\cdot)+H_+^n(g^\cdot)$ and $H_-^n(f^\cdot+g^\cdot)=H_-^n(f^\cdot)+H_-^n(g^\cdot)$ for an arbitrary morphisms $f^\cdot:A^\cdot\to B^\cdot$ and $g^\cdot:A^\cdot\to B^\cdot$ in $C(\mathscr{C})$. Actually, following the notation of diagram (2.2), we get the equality $H_+^n(f^\cdot+g^\cdot)=f_+^n\oplus g_+^n=H_+^n(f^\cdot)\oplus H_+^n(g^\cdot)=H_+^n(f^\cdot)+H_+^n(g^\cdot)$, as well as $H_-^n(f^\cdot+g^\cdot)=f_-^n\oplus g_-^n=H_-^n(f^\cdot)\oplus H_-^n(g^\cdot)=H_-^n(f^\cdot)+H_-^n(g^\cdot)$. □

We call the functors $H_-^n:C(\mathscr{C})\to\mathscr{C}$ and $H_+^n:C(\mathscr{C})\to\mathscr{C}$ in Proposition 2.2 the right cohomological functor and the left cohomological functor respectively.

Let $\mathscr{C}$ be a semi-abelian category and $A^\cdot=(A^n,d_A^n)_{n\in Z}$ be a complex in $C(\mathscr{C})$. Assume that $T:C(\mathscr{C})\to C(\mathscr{C})$ is a reversible transformation such that $T(A^n)=A^{n+1}$ and $T(d_A^n)=-d_A^{n+1}$. On the other hand, for an arbitrary morphism of complexes $f^\cdot:A^\cdot\to B^\cdot$ in $C(\mathscr{C})$ and any $n\in\mathbb{Z}$, put $T(f^\cdot)^n=f^{n+1}$. We call such functor T the translation functor. If we put the notation $H_*^n:C(\mathscr{C})\to\mathscr{C}$ to represent $H_+^n:C(\mathscr{C})\to\mathscr{C}$ or $H_-^n:C(\mathscr{C})\to\mathscr{C}$ for any $n\in Z$, then it's easy to check the following corollary.

Corollary 2.1. $H_*^n\circ T=H_*^{n+1}$.

It's well-known that in abelian categories, homotopic morphisms applied by a cohomological functor are identical (see [11]). We have similar properties about the right and left cohomological functors in semi-abelian categories.

Proposition 2.3. If $f^\cdot,g^\cdot:A^\cdot\to B^\cdot$ are homotopic morphisms in $C(\mathscr{C})$, we have $H_*^n(f)=H_*^n(g)$ for any $n\in\mathbb{Z}$.

Proof. If $f^\cdot,g^\cdot:A^\cdot\to B^\cdot$ are homotopic morphisms, then there exists a morphism $s^\cdot:T(A^\cdot)\to B^\cdot$ such that $s^n:A^n\to B^{n-1}$ satisfies the equality $f^n-g^n=d_B^{n-1}s^n+s^{n+1}d_A^n$ for each $n\in Z$. We follow the notion of diagram (2.2), then $(\mathrm{ker}d_B^n)\cdot(f_1^n-g_1^n)=(f^n-g^n)\cdot\mathrm{ker}d_A^n=d_B^{n-1}\cdot s^n\cdot\mathrm{ker}d_A^n=\mathrm{ker}d_B^n\cdot a_B^{n-1}\cdot\mathrm{coker}d_B^{n-2}\cdot s^n\cdot\mathrm{ker}d_A^n$. Since $\mathrm{ker}d_B^n$ is a monomorphism, $f_1^n-g_1^n=a_B^{n-1}\cdot\mathrm{coker}d_B^{n-2}\cdot s^n\cdot\mathrm{ker}d_A^n$. This implies that $(f_-^n-g_-^n)\cdot\mathrm{coker}a_A^{n-1}=\mathrm{coker}a_B^{n-1}\cdot(f_1^n-g_1^n)=0$. Moreover, $\mathrm{coker}a_A^{n-1}$ is an epimorphism, so $f_-^n=g_-^n$. Therefore, $H_-^n(f)=H_-^n(g)$.

The other equality $H_+^n(f)=H_+^n(g)$ follows by duality. □

For an additive category $\mathscr{E}$, the homotopic category of complexes of $\mathscr{E}$ is the category $K(\mathscr{E})$ consisting of complexes of $\mathscr{E}$-objects as objects and classes of homotopic morphisms as morphisms. Let $\mathscr{C}$ be a semi-abelian category. By Proposition 2.3 we can generalize the right and left cohomological functors to the category $K(\mathscr{C})$. Consequently, we obtain two additive functors $H_-^n:K(\mathscr{C})\to\mathscr{C}$ and $H_+^n:K(\mathscr{C})\to\mathscr{C}$.

3 One-side derived categories of pre-strict P-semi-abelian categories

In this section, we will construct a class of semi-abelian categories called pre-strict semi-abelian categories. It is a generation of quasi-abelian categories, whose characterization can be seen in [11], and equipped with good properties. Furthermore, we will introduce left quasi-bimorphisms and right quasi-bimorphisms yielded by left cohomological functors and right cohomological functors to establish the corresponding one-side derived categories.

Definition 3.1. Let $\mathscr{C}$ be a semi-abelian category. We called $\mathscr{C}$ the *right−strict category* (resp. *left−strict category*) if it satisfies the following qualification: if $f:A\to B$ and $g:B\to C$ are two morphisms in $\mathscr{C}$ with gf be strict morphism, and g be strict epimorphism (resp. f be strict monomorphism), then f is a strict morphism (resp. g is a strict morphism).

We call $\mathscr{C}$ the pre-strict semi-abelian category if it is both left-strict and right-strict categories. We claim that pre-strict semi-abelian categories have some good properties.

Remark 3.1. Yaroslav Kopylov claimed in his article <Homology in P-semi-abelian categories> (Reference [5]) that the following holds in a P-semi-abelian categories: if the composite morphism fg is strict and f is a monomorphism, then g is a strict morphism. By duality, if the composite morphism fg is strict and g is a epimorphism, then f is a strict morphism. We got inspired to this property and study some similar issue, say, Definition 3. 1.

Proposition 3.1. Let $\mathscr{C}$ be a pre-strict semi-abelian category and $\alpha,\beta:A\to B$ be morphisms in $\mathscr{C}$. If α and β are both strict morphisms, then $\alpha+\beta$ is strict morphism.

Proof. Assume that $\alpha=\alpha_1\alpha_0$ and $\beta=\beta_1\beta_0$, where α_1 and β_1 are strict monomorphisms, α_0 and β_0 are strict epimorphisms. Then $\alpha+\beta=\alpha_1\alpha_0+\beta_1\beta_0=(\alpha_1,\beta_1)(\alpha_0,\beta_0)^T$. Since $\alpha_0=(1,0)(\alpha_0,\beta_0)^T$ and α_0 is a strict morphism while (1,0) is a strict epimorphism, $(\alpha_0,\beta_0)^T$ is a strict epimorphism.

On the other hand, it is easy to check that (α_1,β_1) is a strict monomorphism by duality. Hence, $\alpha+\beta$ is a strict morphism. □

Let $\mathscr{C}$ be a pre-strict semi-abelian category. We denote $S(\mathscr{C})$ to be a full subcategory of $K(\mathscr{C})$, consisting of strict complexes in $K(\mathscr{C})$. Let $f^{\cdot}:X^{\cdot}\to Y^{\cdot}$ be a morphism of complexes. As in abelian categories, we define $C_f^n=X^{n+1}\oplus Y^n$ for every $n\in Z$, and $d_{C_f}^n:C_f^n\to C_f^{n+1}$ such that $d_{C_f}^n=\begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}$. Obviously, $C_f^{\cdot}$ is a complex, which we call the corn of morphism $f^{\cdot}$.

Lemma 3.1. [3] *Let $\mathscr{C}$ be a semi-abelian category. If $f:X\to Y$ is a strict morphism, $g:W\to X$ is a strict epimorphism and $h:Y\to Z$ is a strict monomorphism, then fg and hf are strict morphisms.*

Proposition 3.2. Let $\mathscr{C}$ be a pre-strict semi-abelian category. If $f^{\cdot}:X^{\cdot}\to Y^{\cdot}$ is a morphism in $S(\mathscr{C})$, then $C_f^{\cdot}=(C_f^n,d_{C_f}^n)_{n\in Z}$ is an object of $S(\mathscr{C})$.

Proof. Since $-d_X^n=(1,0)(-d_X^n,f^n)^T$ is a strict morphism and (1,0) is a strict epimorphism, $(-d_X^n,f^n)^T$ is a strict morphism. For any $n\in Z$, compute

$$d_{C_f}^n=\begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}=\begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & d_Y^n \end{pmatrix}=\begin{pmatrix} -d_X^{n+1} \\ f^{n+1} \end{pmatrix}(1,0)+\begin{pmatrix} 0 \\ 1 \end{pmatrix}d_Y^n(0,1).$$

We have known that (1,0) is a strict epimorphism, $(0,1)^T$ is a strict monomorphism, d_Y^n is a strict morphism and (0,1) is a strict epimorphism. Therefore, $(-d_X^{n+1},f^{n+1})^T(1,0)$ and $(1,0)^T d_Y^n(0,1)$ are strict morphisms by Lemma 3.1. This implies that $d_{C_f}^n$ is a strict morphism by proposition 3.1, i.e. $C_f^\cdot=(C_f^n,d_{C_f}^n)_{n\in Z}$ is an object of $S(\mathscr{C})$. $\square$

Let $\mathscr{C}$ be a semi-abelian category and $f^\cdot:X^\cdot\to Y^\cdot$ be a morphism in $C(\mathscr{C})$. Given the morphism $i_f^\cdot:Y^\cdot\to C_f^\cdot$ with $i_f^n:Y^n\to C_f^n$ a canonical injection and the morphism $p_f^\cdot:C_f^\cdot\to T(X^\cdot)$ with $p_f^n:C_f^n\to X^{n+1}$ a canonical projection, where the translation functor T was announced in Section 2. It's easy to check that $i_f^\cdot$ and $p_f^\cdot$ are morphisms of complexes and consequently $0\to Y^\cdot\xrightarrow{i_f^\cdot}C_f^\cdot\xrightarrow{p_f^\cdot}T(X^\cdot)\to 0$ is an exact sequence of complexes.

Lemma 3.2. [4] *Let $\mathscr{C}$ be a semi-abelian category and $0\to A^\cdot\to B^\cdot\to C^\cdot\to 0$ be an exact sequence of complexes in $C(\mathscr{C})$.*

(1) If $A^\cdot$ is a strict complex, then $\mathrm{H}_+^n(A^\cdot)\to\mathrm{H}_+^n(B^\cdot)\to\mathrm{H}_+^n(C^\cdot)$ are exact sequences for all $n\in Z$.

(2) If $C^\cdot$ is a strict complex, then $\mathrm{H}_-^n(A^\cdot)\to\mathrm{H}_-^n(B^\cdot)\to\mathrm{H}_-^n(C^\cdot)$ are exact sequences for all $n\in Z$.

Corollary 3.1. If $0\to Y^\cdot\xrightarrow{i_f^\cdot}C_f^\cdot\xrightarrow{p_f^\cdot}T(X^\cdot)\to 0$ is a sequence of complexes in $S(\mathscr{C})$, then the sequences $\mathrm{H}_-^n(Y^\cdot)\xrightarrow{H_-^n(i_f^\cdot)}\mathrm{H}_-^n(C_f^\cdot)\xrightarrow{H_-^n(p_f^\cdot)}\mathrm{H}_-^n(T(X^\cdot))$ and $\mathrm{H}_+^n(Y^\cdot)\xrightarrow{H_+^n(i_f^\cdot)}\mathrm{H}_+^n(C_f^\cdot)\xrightarrow{H_+^n(p_f^\cdot)}\mathrm{H}_+^n(T(X^\cdot))$ are exact for all $n\in Z$.

In order to introduce the concept of left and right quasi-bimorphisms we need some preliminary results. We refer to [7] for the notion of localizing classes and triangulated categories.

Definition 3.2. [7] Let $\mathscr{C}$ be an additive category and $f^\cdot:X^\cdot\to Y^\cdot$ be a morphism in $C(\mathscr{C})$. Let T be the corresponding translation functor on $C(\mathscr{C})$. We call $X^\cdot\to Y^\cdot\to Z^\cdot\to T(X^\cdot)$ a distinguished triangle in $K(\mathscr{C})$ if it is isomorphic to the image of a standard triangle $X^\cdot\xrightarrow{f^\cdot}Y^\cdot\xrightarrow{i_f^\cdot}C_f^\cdot\xrightarrow{p_f^\cdot}T(X^\cdot)$ in $K(\mathscr{C})$. In [6], Milicic asserted that the addictive category $K(\mathscr{C})$ equipped with the translation functor T and the class of distinguished triangles in $K(\mathscr{C})$ is a triangulated category.

According to the definition of triangulated categories, $X^\cdot\xrightarrow{f^\cdot}Y^\cdot\xrightarrow{g^\cdot}Z^\cdot\xrightarrow{h^\cdot}T(X^\cdot)$ is a distinguished triangle if and only if $Y^\cdot\xrightarrow{g^\cdot}Z^\cdot\xrightarrow{h^\cdot}T(X^\cdot)\xrightarrow{-T(f)^\cdot}T(Y^\cdot)$ is a distinguished triangle, where T is the lifting functor. On the other hand, we call $X^\cdot\to Y^\cdot\to Z^\cdot\to T(X^\cdot)$ a distinguished triangle in $K(\mathscr{C})$ if it is isomorphic to the image of a standard triangle $X^\cdot\xrightarrow{f^\cdot}Y^\cdot\xrightarrow{i_f^\cdot}C_f^\cdot\xrightarrow{p_f^\cdot}T(X^\cdot)$ in $K(\mathscr{C})$, where $C_f^\cdot=(C_f^n,d_{C_f}^n)_{n\in Z}$ is the corn of morphism

$f^\cdot$. Therefore, we can apply Corollary 3.6 to the distinguished triangle $X^\cdot \xrightarrow{f^\cdot} Y^\cdot \xrightarrow{g^\cdot} Z^\cdot \xrightarrow{h^\cdot} T(X^\cdot)$ and get an exact sequence $\mathrm{H}^n_-(Y^\cdot) \xrightarrow{H^n_-(g^\cdot)} \mathrm{H}^n_-(Z^\cdot) \xrightarrow{H^n_-(h^\cdot)} \mathrm{H}^n_-(T(X^\cdot))$. Once more, we apply Corollary 3.6 to the distinguished triangle $Y^\cdot \xrightarrow{g^\cdot} Z^\cdot \xrightarrow{h^\cdot} T(X^\cdot) \xrightarrow{-T(f)^\cdot} T(X^\cdot)$ and get an exact sequence $\mathrm{H}^n_-(Z^\cdot) \xrightarrow{H^n_-(h^\cdot)} \mathrm{H}^n_-(T(X^\cdot)) \xrightarrow{H^n_-(-T(f^\cdot))} \mathrm{H}^n_-(T(Y^\cdot))$. Combine the above two exact sequences we get a longer exact sequence $\mathrm{H}^n_-(Y^\cdot) \xrightarrow{H^n_-(g^\cdot)} \mathrm{H}^n_-(Z^\cdot) \xrightarrow{H^n_-(h^\cdot)} \mathrm{H}^n_-(T(X^\cdot)) \xrightarrow{H^n_-(-T(f^\cdot))} \mathrm{H}^n_-(T(Y^\cdot))$. By Corollary 2.7, $\mathrm{H}^n_-(Y^\cdot) \xrightarrow{H^n_-(g^\cdot)} \mathrm{H}^n_-(Z^\cdot) \xrightarrow{H^n_-(h^\cdot)} \mathrm{H}^{n+1}_-(X^\cdot) \xrightarrow{H^{n+1}_-(-f^\cdot)} \mathrm{H}^{n+1}_-(Y^\cdot)$ is an exact sequence.

To use the above analogy we can get the following corollary.

Corollary 3.2. If $X^\cdot \xrightarrow{f^\cdot} Y^\cdot \xrightarrow{g^\cdot} Z^\cdot \xrightarrow{h^\cdot} T(X^\cdot)$ is a distinguished triangle in $K(\mathscr{C})$, we have the following two long exact sequences in $\mathscr{C}$.

$$\cdots \to \mathrm{H}^n_-(X^\cdot) \xrightarrow{H^n_-(f)} \mathrm{H}^n_-(Y^\cdot) \xrightarrow{H^n_-(g)} \mathrm{H}^n_-(Z^\cdot) \xrightarrow{H^n_-(h)} \mathrm{H}^{n+1}_-(X^\cdot) \to \cdots$$
$$\cdots \to \mathrm{H}^n_+(X^\cdot) \xrightarrow{H^n_+(f)} \mathrm{H}^n_+(Y^\cdot) \xrightarrow{H^n_+(g)} \mathrm{H}^n_+(Z^\cdot) \xrightarrow{H^n_+(h)} \mathrm{H}^{n+1}_+(X^\cdot) \to \cdots.$$

We call them the right cohomology long exact sequence and the left cohomology long exact sequence of the distinguished triangle $X^\cdot \xrightarrow{f^\cdot} Y^\cdot \xrightarrow{g^\cdot} Z^\cdot \xrightarrow{h^\cdot} T(X^\cdot)$ respectively.

Now we introduce the concept of left and right quasi-bimorphisms in pre-strict semi-abelian categories.

Definition 3.3. Let $\mathscr{C}$ be a pre-strict semi-abelian category and $f^\cdot : X^\cdot \to Y^\cdot$ be a morphism in $K(\mathscr{C})$.

(1) $f^\cdot$ is called a $right\ quasi-bimorphisms$ if $H^n_-(f^\cdot)$ are bimorphisms for all $n \in Z$.

(2) $f^\cdot$ is called a $left\ quasi-bimorphisms$ if $H^n_+(f^\cdot)$ are bimorphisms for all $n \in Z$.

Definition 3.4. Let $\mathscr{C}$ be a pre-strict semi-abelian category and $A^\cdot = (A^n, d^n_A)_{n\in Z}$ be an object of $K(\mathscr{C})$.

(1) $A^\cdot$ is called a *right acyclic* if $\mathrm{H}^n_-(A^\cdot) = 0$ for all $n \in Z$;

(2) $A^\cdot$ is called a $left\ acyclic$ if $\mathrm{H}^n_+(A^\cdot) = 0$ for all $n \in Z$.

Lemma 3.3. *Let $f^\cdot : X^\cdot \to Y^\cdot$ is a morphism in $S(\mathscr{C})$, which is consist of strict complexes in $K(\mathscr{C})$.*

(1) $f^\cdot$ is a right quasi-bimorphism if and only if $C^\cdot_f = (C^n_f, d^\cdot_{C_f})_{n\in Z}$ is a right acyclic.

(2) $f^\cdot$ is a left quasi-bimorphism if and only if $C^\cdot_f = (C^n_f, d^\cdot_{C_f})_{n\in Z}$ is a left acyclic.

Proof. The corn of a morphism $f^{\cdot}$ is unique up to isomorphic, seen in [6]. Namely, in $K(\mathscr{C})$, if $X^{\cdot} \xrightarrow{f^{\cdot}} Y^{\cdot} \xrightarrow{g^{\cdot}} Z^{\cdot} \xrightarrow{h^{\cdot}} T(X^{\cdot})$ is a distinguished triangle then $Z^{\cdot}$ is isomorphic to $C_f^{\cdot}$. Applying Corollary 3.8 to the distinguished triangle $X^{\cdot} \xrightarrow{f^{\cdot}} Y^{\cdot} \xrightarrow{g^{\cdot}} C_f \xrightarrow{h^{\cdot}} T(X^{\cdot})$, we get the long exact sequence $\cdots \to \mathrm{H}_{-}^{n}(X^{\cdot}) \xrightarrow{H_{-}^{n}(f)} \mathrm{H}_{-}^{n}(Y^{\cdot}) \xrightarrow{H_{-}^{n}(g)} \mathrm{H}_{-}^{n}(C_f^{\cdot}) \xrightarrow{H_{-}^{n}(h)} \mathrm{H}_{-}^{n+1}(X^{\cdot}) \to \cdots$.

If $f^{\cdot}$ is a right quasi-bimorphism, by Definition 3.3 $H_{-}^{n}(f)$ is a bimorphism, and then $\mathrm{H}_{-}^{p}(C_f^{\cdot})=0$, which means that $C_f^{\cdot}$ is a right acyclic. Conversely, if $C_f^{\cdot}$ is a right acyclic, by Definition 3.4 $\mathrm{H}_{-}^{p}(C_f^{\cdot})=0$, then $H_{-}^{n}(f)$ is a monomorphism and an epimorphism, namely, a bimorphism, which means $f^{\cdot}$ is a right quasi-bimorphism.

The second result can be proved by dual. $\square$

Remark 3.2. If $f^{\cdot}: A^{\cdot} \to B^{\cdot}$ is a bimorphism, it is naturally a right (left) quasi-bimorphism. In fact, since $0 \to A^{\cdot} \xrightarrow{f^{\cdot}} B^{\cdot} \to 0$ is an exact sequence, by Lemma 3.2 we get exact sequences $0 \to \mathrm{H}_{+}^{n}(A^{\cdot}) \xrightarrow{H_{+}^{n}(f^{\cdot})} \mathrm{H}_{+}^{n}(B^{\cdot}) \to 0$ and $0 \to \mathrm{H}_{-}^{n}(A^{\cdot}) \xrightarrow{H_{-}^{n}(f^{\cdot})} \mathrm{H}_{-}^{n}(B^{\cdot}) \to 0$ for any $n \in Z$. This implies that $H_{+}^{n}(f)$ and $H_{-}^{n}(f)$ are bimorphisms for any $n \in Z$.

In the sequel, we introduce the concept of one-side derived categories based on the above preparations.

The following theorem is a main result of this paper. Let's mention at this point that the notations "LC" and "LT" in Theorem 3.1 are the characterizations of localizing classes and localizing classes compatible with triangulation respectively (see [7, p. 4 and 66]).

Theorem 3.1. *Let $\mathscr{C}$ be a pre-strict semi-abelian category. Put $S_{-}^{*} = \{f^{\cdot}: X^{\cdot} \to Y^{\cdot} \mid f^{\cdot}$ is a right quasi-bimorphism in $S(\mathscr{C})\}$. Then S_{-}^{*} is a localizing class compatible with triangulation in $S(\mathscr{C})$.*

Proof. First we show that S_{-}^{*} is a localizing class.

(LC1) Let $X^{\cdot}$ be an arbitrary object in $S(\mathscr{C})$. Then $C_{1_X}^{\cdot} = 0$. By Lemma 3.3, $1_X^{\cdot}$ is in S_{-}^{*}.

(LC2) If $t^{\cdot}: X^{\cdot} \to Y^{\cdot}$ and $s^{\cdot}: Y^{\cdot} \to Z^{\cdot}$ are two morphisms in S_{-}^{*}, then $H_{-}^{n}(s)$ and $H_{-}^{n}(t)$ are bimorphisms for any $n \in Z$. Since H_{-}^{n} is a functor, $H_{-}^{n}(st)$ is a bimorphism. This implies that $s^{\cdot} \cdot t^{\cdot}$ is in S_{-}^{*}.

(LC3a) Let $f^{\cdot}: Z^{\cdot} \to Y^{\cdot}$ be a morphism in $S(\mathscr{C})$ and $s^{\cdot}: X^{\cdot} \to Y^{\cdot}$ be in S_{-}^{*}. Consider the distinguished triangle $X^{\cdot} \xrightarrow{s^{\cdot}} Y^{\cdot} \xrightarrow{i_s} C_s^{\cdot} \xrightarrow{p_s} T(X^{\cdot})$ based on $s^{\cdot}$. By Lemma 3.3, $C_s^{\cdot}$ is a right acyclic. Since $Y^{\cdot} \xrightarrow{i_s} C_s^{\cdot} \xrightarrow{p_s} T(X^{\cdot}) \xrightarrow{-T(s)} T(Y^{\cdot})$ is again a distinguished triangle, we have a commutative diagram

$$\begin{array}{ccccccc}
Z^{\cdot} & \xrightarrow{i_s f} & C_s^{\cdot} & \longrightarrow & C_{i_s f}^{\cdot} & \xrightarrow{u^{\cdot}} & T(Z^{\cdot}) \\
\downarrow{\scriptstyle f^{\cdot}} & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle T(f^{\cdot})} \\
Y^{\cdot} & \xrightarrow{i_s} & C_s^{\cdot} & \xrightarrow{p} & T(X^{\cdot}) & \xrightarrow{-T(s)} & T(Y^{\cdot})
\end{array}$$

where the rows are distinguished triangles. By the characterization of triangulated categories, we can complete this diagram by a morphism $v^{\cdot}:C^{\cdot}_{i_sf}\to T(X^{\cdot})$ to the morphism of distinguished triangles. Since $C^{\cdot}_s$ is a right acyclic, $T^{-1}(u^{\cdot})$ is a right quasi-bimorphism. Put $W=T^{-1}(C^{\cdot}_{i_sf})$,$t^{\cdot}=T^{-1}(u^{\cdot})$ and $g^{\cdot}=-T^{-1}(v^{\cdot})$. Consequently, we get the commutative diagram

$$\begin{array}{ccc} W^{\cdot} & \xrightarrow{t^{\cdot}} & Z^{\cdot} \\ {\scriptstyle g^{\cdot}}\downarrow & & \downarrow{\scriptstyle f^{\cdot}} \\ X^{\cdot} & \xrightarrow{s^{\cdot}} & Y^{\cdot} \end{array}$$

where $s^{\cdot}$ and $t^{\cdot}$ are in S^*_-.

(LC3b) Let $f^{\cdot}:X^{\cdot}\to Z^{\cdot}$ be a morphism in $S(\mathscr{C})$ and $s^{\cdot}:X^{\cdot}\to Y^{\cdot}$ be in S^*_-. Consider a distinguished triangle $X^{\cdot}\xrightarrow{s^{\cdot}}Y^{\cdot}\xrightarrow{i_s}C^{\cdot}_s\xrightarrow{p_s}T(X^{\cdot})$ based on $s^{\cdot}$. By Lemma 3.3, $C^{\cdot}_s$ is a right acyclic. Thus $T^{-1}(C^{\cdot}_s)\xrightarrow{-T^{-1}(p_s)}X^{\cdot}\xrightarrow{s^{\cdot}}Y^{\cdot}\xrightarrow{i_s}C^{\cdot}_s$ is a distinguished triangle. Then we have a commutative diagram

$$\begin{array}{ccccccc} T^{-1}(C^{\cdot}_s) & \xrightarrow{-T^{-1}(p_s)} & X^{\cdot} & \xrightarrow{s^{\cdot}} & Y^{\cdot} & \xrightarrow{i^{\cdot}_s} & C^{\cdot}_s \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle f^{\cdot}} & & & & \downarrow{\scriptstyle 1} \\ T^{-1}(C^{\cdot}_s) & \xrightarrow[-fT^{-1}(p_s)^{\cdot}]{} & Z^{\cdot} & \xrightarrow[t]{} & C^{\cdot}_{-fT^{-1}(p_s)} & \longrightarrow & C^{\cdot}_s \end{array}$$

where the rows are distinguished triangles. By the characterization of triangulated categories, we can complete this diagram by a morphism $g^{\cdot}:Y^{\cdot}\to C^{\cdot}_{-fT^{-1}(p_s)}$ to the morphism of distinguished triangles. Since $C^{\cdot}_s$ is a right acyclic, $t^{\cdot}$ is a right quasi-bimorphism. This result yields a commutative diagram

$$\begin{array}{ccc} X^{\cdot} & \xrightarrow{s^{\cdot}} & Y^{\cdot} \\ {\scriptstyle f^{\cdot}}\downarrow & & \downarrow{\scriptstyle g^{\cdot}} \\ Z^{\cdot} & \xrightarrow{t^{\cdot}} & C^{\cdot}_{-fT^{-1}(p_s)} \end{array}$$

where $s^{\cdot}$ and $t^{\cdot}$ are in S^*_-.

(LC4a) Let $f^{\cdot}:X^{\cdot}\to Y^{\cdot}$ be a morphism in $S(\mathscr{C})$ and $s^{\cdot}:Y^{\cdot}\to Z^{\cdot}$ be in S^*_- with $s^{\cdot}\cdot f^{\cdot}=0^{\cdot}$. By [7, Chapter 2, Lemma 2.1.2], $X^{\cdot}\xrightarrow{1_X}X^{\cdot}\to 0^{\cdot}\to T(X^{\cdot})$ is a distinguished triangle. This implies that $X^{\cdot}\to 0^{\cdot}\to T(X^{\cdot})\xrightarrow{-1_{T(X)}}T(X^{\cdot})$ is a distinguished triangle. Hence, we can construct a commutative diagram

$$\begin{array}{ccccccc}
X^{\cdot} & \longrightarrow & 0^{\cdot} & \longrightarrow & T(X^{\cdot}) & \xrightarrow{-1} & T(X^{\cdot}) \\
\Big\downarrow{\scriptstyle f^{\cdot}} & & \Big\downarrow & & & & \Big\downarrow{\scriptstyle T(f^{\cdot})} \\
Y^{\cdot} & \xrightarrow{s} & Z^{\cdot} & \xrightarrow{i_s} & C_s^{\cdot} & \xrightarrow{p_s} & T(Y^{\cdot})
\end{array}$$

where the rows are distinguished triangles. By the characterization of triangulated categories, we can complete this diagram by a morphism $-v^{\cdot}: T(X^{\cdot}) \to C_s^{\cdot}$ to the morphism of distinguished triangles. This in turn implies that $f^{\cdot} = T^{-1}(p_s)T^{-1}(v^{\cdot})$. Consider the distinguished triangle $X^{\cdot} \xrightarrow{T^{-1}(v^{\cdot})} T^{-1}(C_s^{\cdot}) \to C^{\cdot}_{T^{-1}(v^{\cdot})} \xrightarrow{T(t)} T(X^{\cdot})$ based on $T^{-1}(v^{\cdot})$. Since $s^{\cdot}$ is a right quasi-bimorphism, $C_s^{\cdot}$ is a right acyclic. Therefore $t^{\cdot}$ is a right quasi-bimorphism. By [7, Chapter 1, Lemma 1.3.1], $T^{-1}(v^{\cdot})t^{\cdot} = 0$. Thus $f^{\cdot}t^{\cdot} = T^{-1}(p_s)T^{-1}(v^{\cdot})t^{\cdot} = 0$.

(LC4b) Let $f^{\cdot}: Y^{\cdot} \to Z^{\cdot}$ be a morphism in $S(\mathscr{C})$ and $t^{\cdot}: X^{\cdot} \to Y^{\cdot}$ be in S_-^* with $f^{\cdot} \cdot t^{\cdot} = 0^{\cdot}$. We have a commutative diagram

$$\begin{array}{ccccccc}
X^{\cdot} & \xrightarrow{t^{\cdot}} & Y^{\cdot} & \xrightarrow{u^{\cdot}} & C_t^{\cdot} & \longrightarrow & T(X^{\cdot}) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle f^{\cdot}} & & & & \Big\downarrow \\
0^{\cdot} & \longrightarrow & Z^{\cdot} & \xrightarrow{1} & Z^{\cdot} & \longrightarrow & 0^{\cdot}
\end{array}$$

where the rows are distinguished triangles. By the characterization of triangulated categories, we can complete this diagram by a morphism $v^{\cdot}: C_t^{\cdot} \to Z^{\cdot}$ to the morphism of distinguished triangles. This implies that $f^{\cdot} = v^{\cdot}u^{\cdot}$. Consider the distinguished triangle $C_t^{\cdot} \xrightarrow{v^{\cdot}} Z^{\cdot} \xrightarrow{s^{\cdot}} C_{v^{\cdot}}^{\cdot} \to T(C_t^{\cdot})$ based on $v^{\cdot}$. Since $t^{\cdot}$ is a right quasi-bimorphism, $C_t^{\cdot}$ is a right acyclic. Therefore $s^{\cdot}$ is a right quasi-bimorphism and $s^{\cdot}f^{\cdot} = s^{\cdot}v^{\cdot}u^{\cdot} = 0$.

From what has been discussed above, we proved that S_-^* is a localizing class. Then we have to check that S_-^* is compatible with triangulation.

(LT1) Obviously, S_-^* is invariant under the translation functor T. This implies that for an arbitrary morphism $s^{\cdot}$ in S_-^*, $T(s^{\cdot})$ is in S_-^*.

(LT2) Consider the morphism of distinguished triangles

$$\begin{array}{ccccccc}
X^{\cdot} & \longrightarrow & Y^{\cdot} & \longrightarrow & Z^{\cdot} & \longrightarrow & T(X^{\cdot}) \\
\Big\downarrow{\scriptstyle s^{\cdot}} & & \Big\downarrow{\scriptstyle t^{\cdot}} & & \Big\downarrow{\scriptstyle u^{\cdot}} & & \Big\downarrow{\scriptstyle T(s^{\cdot})} \\
X_1^{\cdot} & \longrightarrow & Y_1^{\cdot} & \longrightarrow & Z_1^{\cdot} & \longrightarrow & T(X_1^{\cdot})
\end{array}$$

where $s^{\cdot}$ and $t^{\cdot}$ are in S_-^*. By [10, Remark 2.2], there exists a commutative diagram

$$\begin{array}{ccccccc}
X^{\cdot} & \longrightarrow & Y^{\cdot} & \longrightarrow & Z^{\cdot} & \longrightarrow & T(X^{\cdot}) \\
\downarrow{\scriptstyle s^{\cdot}} & & \downarrow{\scriptstyle t^{\cdot}} & & \downarrow{\scriptstyle u^{\cdot}} & & \downarrow{\scriptstyle T(s^{\cdot})} \\
X_1^{\cdot} & \longrightarrow & Y_1^{\cdot} & \longrightarrow & Z_1^{\cdot} & \longrightarrow & T(X_1^{\cdot}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_s^{\cdot} & \longrightarrow & C_t^{\cdot} & \longrightarrow & C_u^{\cdot} & \longrightarrow & T(C_s^{\cdot})
\end{array}$$

such that the rows and columns are distinguished triangles. For any $p \in Z$, this leads to a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}_-^p(X^{\cdot}) & \longrightarrow & \mathrm{H}_-^p(Y^{\cdot}) & \longrightarrow & \mathrm{H}_-^p(Z^{\cdot}) & \longrightarrow & \mathrm{H}_-^p(T(X^{\cdot})) \\
\downarrow{\scriptstyle H_-^p(s^{\cdot})} & & \downarrow{\scriptstyle H_-^p(t^{\cdot})} & & \downarrow{\scriptstyle H_-^p(u^{\cdot})} & & \downarrow{\scriptstyle H_-^p(T(s^{\cdot}))} \\
\mathrm{H}_-^p(X_1^{\cdot}) & \longrightarrow & \mathrm{H}_-^p(Y_1^{\cdot}) & \longrightarrow & \mathrm{H}_-^p(Z_1^{\cdot}) & \longrightarrow & \mathrm{H}_-^p(T(X_1^{\cdot})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}_-^p(C_s^{\cdot}) & \longrightarrow & \mathrm{H}_-^p(C_t^{\cdot}) & \longrightarrow & \mathrm{H}_-^p(C_u^{\cdot}) & \longrightarrow & \mathrm{H}_-^{p+1}(C_s^{\cdot})
\end{array}$$

where $\mathrm{H}_-^p(C_t^{\cdot}) = \mathrm{H}_-^{p+1}(C_s^{\cdot}) = 0$. Since the bottom row is exact, $\mathrm{H}_-^p(C_u^{\cdot}) = 0$ hold. This in turn implies that $u^{\cdot}$ is in S_-^*. $\square$

By Theorem 3.1, we obtain the localization of $S(\mathscr{C})$ with respect to S_-^* and denote it by $\mathrm{DS}_-(\mathscr{C})$. We call $\mathrm{DS}_-(\mathscr{C})$ the *right derived category* of $S(\mathscr{C})$, which is again a triangulated category (see [7, Chapter 2, Theorem 1.6.1]).

Dually, we obtain the *left derived category* of $S(\mathscr{C})$ and denote it by $\mathrm{DS}_+(\mathscr{C})$, which is also a triangulated category. The corresponding localizing class is formulated below.

Theorem 3.2. *Let $\mathscr{C}$ be a pre-strict semi-abelian category. Put $S_+^* = \{f^{\cdot} : X^{\cdot} \to Y^{\cdot} \mid f^{\cdot}$ is a left quasi-bimorphism in $S(\mathscr{C})\}$. Then S_+^* is a localizing class compatible with triangulation in $S(\mathscr{C})$*
.

Milicic [7] proved that the derived category of an abelian category is unique up to isomorphism. For a pre-strict semi-abelian category $\mathscr{C}$, we claim that $\mathrm{DS}_+(\mathscr{C})$ can be regarded as a subcategory of $\mathrm{DS}_-(\mathscr{C})$. The following theorem is another main result of this paper.

Theorem 3.3. *Let $\mathscr{C}$ be a pre-strict semi-abelian category. There exists a dense and faithful functor $F : \mathrm{DS}_+(\mathscr{C}) \to \mathrm{DS}_-(\mathscr{C})$.*

Proof. The objects of $\mathrm{DS}_+(\mathscr{C})$ and $\mathrm{DS}_-(\mathscr{C})$ are the same, consisting of strict complexes in $K(\mathscr{C})$. Assume that a morphism in $\mathrm{DS}_+(\mathscr{C})$ is represented by the left roof

$$M \xleftarrow[\sim]{s} L \xrightarrow{f} N$$

where s is a left quasi-bimorhism. For any $n \in Z$, we have $\mathrm{H}^n_+(C^{\cdot}_s)=0$. Since $m^n:\mathrm{H}^n_-(C^{\cdot}_s)\to \mathrm{H}^n_+(C^{\cdot}_s)$ is a bimorphism (see Remark 2.1), $\mathrm{H}^n_-(C^{\cdot}_s)=0$ hold. This implies that s is a right quasi-bimorphism. Thus the above-mentioned left roof is a morphism in $\mathrm{DS}_-(\mathscr{C})$. Hence, $\mathrm{MorDS}_+(\mathscr{C})\subseteq \mathrm{MorDS}_-(\mathscr{C})$. Put F being the embedding morphism and consequently the theorem followed. $\square$

Remark 3.3. If $\mathscr{C}$ is a quasi-abelian category, then $m^n:\mathrm{H}^n_-(C^{\cdot}_s)\to \mathrm{H}^n_+(C^{\cdot}_s)$ are isomorphisms for all $n \in Z$. Therefore, the above-mentioned functor F is an equivalent functor. This implies that $\mathrm{DS}_+(\mathscr{C})$ is equivalent to $\mathrm{DS}_-(\mathscr{C})$ as categories.

Acknowledgments

The authors are grateful to the referee for valuable comments and suggestions, which improved the manuscript, and for pointing out some inaccuracies.

References

[1] H. Krause and S. Iyengar. Acyclicity versus total acyclicity for complexes over noetherian rings(J). Doc. Math., 11: 207–240, 2006.
[2] M. Kashiwara and P. Schapira. Categories and Sheaves(M). Berlin: Springer-Verlag, 2006.
[3] Y. A. Kopylov and V. I. Kuz0minov. On the Ker-Coker sequence in a semiabelian category(J). Sib. Math. Z. 41(3): 615–624, 2000; English translation: Sib. Math. Z., 41(3): 509–517, 2000.
[4] Y. A. Kopylov. Homology in P-semi-abelian categoreis(J). Math. Sci., 17: 105–114, 2009.
[5] Y. A. Kopylov. On the homology sequence in a P-semi-abelian category(J). Sib. Elec. Math. Izv., 9: 190–200, 2012.
[6] v Y. A. Kopylo and S. A. Wegner. On the notion of a semi-abelian cagegory in the sense of Palamodov(J). Applied Categorical Structures, 20(5): 531–541, 2012.
[7] D. Milicic. Lectures on derived categories(M). New York: Cambridge University Press, 1982.
[8] V. P. Palamodov. The projective limit functor in the category of linear topological spaces(J). Math. Sb.(N.S.) 117(4): 567–603, 1968.
[9] J. Verdier. Cat*é*gories D*é*rivees *é*tat 0(J). Lecture Notes in Math., 569: 262–311, 1977.
[10] L. Xin and Y. Lin. On one-sided torsion pair(J). Science in China Series A: Mathematics, 50(1): 13–26, 2007.
[11] W. Tong. An introduction to homology algebra(M). Beijing: Higher Education Press, 1998.