On Unitary Invariant Strongly Pseudoconvex Complex Landsberg Metrics

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Abstract. In this paper, we prove that a unitary invariant strongly pseudoconvex complex Finsler metric is a complex Landsberg metric if and if only if it comes from a unitary invariant Hermitian metric. This implies that there does not exist unitary invariant complex Landsberg metric unless it comes from a unitary invariant Hermitian metric.

AMS subject classifications: 53C60, 53C40.

Key words: Unitary invariant, complex Landsberg metric, complex Finsler metric.

1 Introduction

In real Finsler geometry, a Berwald metric is necessary a Landsberg metric. It is still an open problem that whether there exists a Landsberg metric which does not come from a Berwald metric [2]. This problem is also called the unicorn problem by D. Bao [3] and M. Matsumoto [2].

M. Matsumoto conjectured that there does not exist unicorn metric, which implies that every Landsberg metric comes from a Berwald metric. In 2008, Z. I. Szabó claimed that all regular Landsberg metrics are Berwald metrics [4]. A gap, however, was soon found in the proof by himself [5], thus leaving the problem still open.

On the other hand, in [6,7], G. S. Asnov constructed a family of almost regular unicorn metrics which come from (α,β)-metrics. In 2009, Z. Shen [8] characterized almost regular Landsberg (α,β)-metrics which generalized G. S. Asanovs results. For the spherically symmetric real Finsler metrics which are not necessary (α,β)-metrics, X.-H. Mo and L.-F. Zhou [9] proved that there does not exist any non-Berwaldian Landsberg metrics among the regular case.

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In complex Finsler geometry, there are also notions of complex Berwald metric, weakly complex Berwald metric, and complex Landsberg metric. It was known that every Kähler-Berwald metric is necessary a complex Landsberg metric [10]. One may wonder whether there exists a complex Landsberg metric which does not come from a Kähler-Berwald metric?

Unlike in real Finsler geometry, there are few explicit examples of strongly pseudo-convex complex Finsler metrics in literatures. This situation has already been changed because of the recent work of C.-P. Zhong [12], where unitary invariant strongly pse-duoconvex complex Finsler metrics were systematically studied and the explicit method of constructing strongly psuedoconvex or even strongly convex complex Finsler metrics were given. In [13], H.C. Xia and C.-P. Zhong gave a classification of unitary invariant weakly complex Berwald metrics which are of constant holomorphic curvatures. It was proved in [12] that there is neither complex Berwald metric nor Kähler Finsler metric which is unitary invariant and does not come from a Hermitian metric. There are, how-ever, lots of weakly complex Berwald metrics which are unitary invariant and they do not come from Hermitian metrics. One may wonder whether there exists unitary invariant complex Landsberg metric which does not come from a Kähler-Berwald metric.

In this paper, we prove that a unitary invariant strongly pseudoconvex complex Finsler metric is a complex Landsberg metric if and only if it comes from a unitary invariant Hermitian metric. This implies that there does not exist unitary invariant complex Landsberg metric unless it comes from a unitary invariant Hermitian metric.

2   Preliminary

Let $\mathbb{C}^n$ be a complex n dimensional linear space, denote by $\langle\cdot,\cdot\rangle$ the canonical complex Euclidean inner product and $\|\cdot\|$ the induced norm in $\mathbb{C}^n$. Let F be a strongly pseudoconvex complex Finsler metric on a unitary invariant domain $D\subset\mathbb{C}^n$. It was proved in [12] that F is unitary invariant if and only if there exists a smooth function $\phi(t,s):[0,+\infty)\times[0,+\infty)\to(0,+\infty)$ such that $F=\sqrt{r\phi(t,s)}$ with

$$r=\|v\|^2,\, t=\|z\|^2,\, s=\frac{|\langle z,v\rangle|^2}{\|v\|^2},$$

where $z=(z^1,\cdots,z^n)\in D$ and $v=(v^1,\cdots,v^n)\in T_z^{1,0}D$.

Lemma 2.1. *[12] Let $F=\sqrt{r\phi(t,s)}$ be a strongly pseudoconvex complex Finsler metrics defined on a domain $D\subset\mathbb{C}^n$. Then the fundamental tensor of F is*

$$G_{\gamma\bar{\tau}}=(\phi-s\phi_s)\delta_{\gamma\bar{\tau}}+r\phi_{ss}s_\gamma s_{\bar{\tau}}+\phi_s\overline{z^\gamma}z^\tau. \tag{2.1}$$

It is known that for a strongly pseudoconvex complex Finsler metric F, there are several complex Finsler connections associated to it. The most often used complex Finsler connections are the Chern-Finsler connection [1], the complex Rund connection and the

complex Berwald connection [11]. These connections are convenient respectively in considering different type of problems in complex Finsler geometry. In this paper, we need the complex Berwald connection associated to F. The connection 1-forms of complex Berwald connection are given by $\breve{\omega}^\alpha_\beta = \mathbb{G}^\alpha_{\beta\mu} dz^\mu$, where

$$\mathbb{G}^\alpha_{\beta\mu} = \dot\partial_\beta(\mathbb{G}^\alpha_\mu), \quad \mathbb{G}^\gamma_\mu = \dot\partial_\mu(\mathbb{G}^\gamma).$$

Here $\mathbb{G}^\gamma = \frac{1}{2}\Gamma^\gamma_{;\mu} v^\mu$ are the complex geodesic spray coefficients associated to F, and $\Gamma^\gamma_{;\mu} = G^{\bar\beta\gamma} G_{\bar\beta;\mu}$ are the Chern-Finsler nonlinear coefficients associated to F.

Lemma 2.2. *[12] Let $F = \sqrt{r\phi(t,s)}$ be a strongly pseudoconvex complex Finsler metrics defined on a domain $D \subset \mathbb{C}^n$. Then the complex Berwald nonlinear connection coefficients $\mathbb{G}^\gamma_\nu$ associated to F are given by*

$$2\mathbb{G}^\gamma_\nu = n_1\overline{z^\nu}v^\gamma - \frac{s}{r}\frac{\partial k_2}{\partial s}\overline{\langle z,v\rangle}\,\overline{v^\nu}v^\gamma + k_2\overline{\langle z,v\rangle}\delta^\gamma_\nu + (m_1+k_3)\overline{\langle z,v\rangle}\overline{z^\nu}z^\gamma - \frac{s}{r}\frac{\partial k_3}{\partial s}(\overline{\langle z,v\rangle})^2\overline{v^\nu}z^\gamma, \quad (2.2)$$

where

$$n_1 = s\frac{\partial k_2}{\partial s} + k_2, \quad k_1 = (\phi - \phi_s)[\phi + (t-s)\phi_s] + s(t-s)\phi\phi_{ss}, \quad (2.3)$$

$$k_2 = \frac{1}{k_1}\left\{[\phi + (t-s)\phi_s + s(t-s)\phi_{ss}](\phi_t + \phi_s) - s[\phi + (t-s)\phi_s](\phi_{st} + \phi_{ss})\right\}, \quad (2.4)$$

$$k_3 = \frac{1}{k_1}\left\{\phi(\phi_{st} + \phi_{ss}) - \phi_s(\phi_t + \phi_s)\right\}, \quad (2.5)$$

$$m_1 = s\frac{\partial k_3}{\partial s} + k_3. \quad (2.6)$$

Lemma 2.3. *Let $F = \sqrt{r\phi(t,s)}$ be a strongly pseudoconvex complex Finsler metric defined on a domain $D \subset \mathbb{C}^n$. Then the complex Berwald connection coefficients $\mathbb{G}^\gamma_{\nu\mu}$ are given by*

$$\begin{aligned}
2\mathbb{G}^\gamma_{\nu\mu} = {} & \frac{s}{r^2}m_2(\overline{\langle z,v\rangle})^2\overline{v^\mu v^\nu}z^\gamma - \frac{s}{r}m_3\overline{\langle z,v\rangle}\overline{z^\mu}\overline{v^\nu}z^\gamma - \frac{s}{r}m_3\overline{\langle z,v\rangle}\overline{z^\nu}\overline{v^\mu}z^\gamma \\
& + \frac{s}{r^2}n_2\overline{\langle z,v\rangle}\overline{v^\mu v^\nu}v^\gamma - \frac{s}{r}\frac{\partial k_2}{\partial s}\overline{\langle z,v\rangle}\overline{v^\nu}\delta^\gamma_\mu - \frac{s}{r}\frac{\partial k_2}{\partial s}\overline{\langle z,v\rangle}\overline{v^\mu}\delta^\gamma_\nu \\
& + \frac{1}{r}n_2\langle z,v\rangle\overline{z^\mu z^\nu}v^\gamma + (sm_3 + m_1 + k_3)\overline{z^\mu z^\nu}z^\gamma - \frac{s}{r}n_2\overline{z^\mu}\overline{v^\nu}v^\gamma \\
& - \frac{s}{r}n_2\overline{z^\nu}\overline{v^\mu}v^\gamma + n_1\delta^\gamma_\nu\overline{z^\mu} + n_1\delta^\gamma_\mu\overline{z^\nu}, \quad (2.7)
\end{aligned}$$

where

$$m_2 = s\frac{\partial^2 k_3}{\partial s^2} + 2\frac{\partial k_3}{\partial s}, \quad m_3 = s\frac{\partial^2 k_3}{\partial s^2} + 3\frac{\partial k_3}{\partial s}, \quad n_2 = s\frac{\partial^2 k_2}{\partial s^2} + 2\frac{\partial k_2}{\partial s}.$$

Proof. Note that

$$\frac{\partial n_1}{\partial v^\mu} = \left(s\frac{\partial^2 k_2}{\partial s^2} + 2\frac{\partial k_2}{\partial s}\right)s_\mu, \quad \frac{\partial m_1}{\partial v^\mu} = \left(s\frac{\partial^2 k_3}{\partial s^2} + 2\frac{\partial k_3}{\partial s}\right)s_\mu.$$

Thus differentiating (2.2) with respect to v^ν, we get

$$
\begin{aligned}
2\mathbb{G}^\gamma_{\nu\mu} = 2\dot{\partial}_\mu(\mathbb{G}^\gamma_\nu) \\
= \left(s\frac{\partial^2 k_2}{\partial s^2} + 2\frac{\partial k_2}{\partial s}\right)s_\mu \overline{z^\nu}v^\gamma + n_1\overline{z^\nu}\delta^\gamma_\mu - \frac{1}{r}\frac{\partial k_2}{\partial s}\overline{\langle z,v\rangle}s_\mu\overline{v^\nu}v^\gamma \\
+ \frac{s}{r^2}\frac{\partial k_2}{\partial s}\overline{\langle z,v\rangle}\overline{v^\mu v^\nu}v^\gamma - \frac{s}{r}\frac{\partial^2 k_2}{\partial s^2}\overline{\langle z,v\rangle}s_\mu\overline{v^\nu}v^\gamma - \frac{s}{r}\frac{\partial k_2}{\partial s}\overline{z^\mu v^\nu}v^\gamma \\
- \frac{s}{r}\frac{\partial k_2}{\partial s}\overline{\langle z,v\rangle}\overline{v^\nu}\delta^\gamma_\mu + \frac{\partial k_2}{\partial s}s_\mu\overline{\langle z,v\rangle}\delta^\gamma_\nu + k_2\overline{z^\mu}\delta^\gamma_\nu \\
+ \left(s\frac{\partial^2 k_3}{\partial s^2} + 3\frac{\partial k_3}{\partial s}\right)\overline{\langle z,v\rangle}s_\mu\overline{z^\nu}z^\gamma + (m_1+k_3)\overline{z^\mu z^\nu}z^\gamma \\
- \frac{1}{r}\frac{\partial k_3}{\partial s}(\overline{\langle z,v\rangle})^2 s_\mu\overline{v^\nu}z^\gamma + \frac{s}{r^2}\frac{\partial k_3}{\partial s}(\overline{\langle z,v\rangle})^2\overline{v^\mu v^\nu}z^\gamma \\
- \frac{s}{r}\frac{\partial^2 k_3}{\partial s^2}(\overline{\langle z,v\rangle})^2 s_\mu\overline{v^\nu}z^\gamma - \frac{2s}{r}\frac{\partial k_3}{\partial s}\overline{\langle z,v\rangle}\overline{z^\mu v^\nu}z^\gamma,
\end{aligned}
$$

from which we have

$$
\begin{aligned}
2\mathbb{G}^\gamma_{\nu\mu} = \frac{1}{r}\left(s\frac{\partial^2 k_2}{\partial s^2} + 2\frac{\partial k_2}{\partial s}\right)\langle z,v\rangle\overline{z^\mu z^\nu}v^\gamma - \frac{s}{r}\left(s\frac{\partial^2 k_2}{\partial s^2} + 2\frac{\partial k_2}{\partial s}\right)\overline{z^\nu v^\mu}v^\gamma \\
+ n_1\overline{z^\nu}\delta^\gamma_\mu - \frac{s}{r}\frac{\partial k_2}{\partial s}\overline{z^\mu v^\nu}v^\gamma + \frac{s}{r^2}\frac{\partial k_2}{\partial s}\overline{\langle z,v\rangle}\overline{v^\mu v^\nu}v^\gamma \\
+ \frac{s}{r^2}\frac{\partial k_2}{\partial s}\overline{\langle z,v\rangle}\overline{v^\mu v^\nu}v^\gamma - \frac{s^2}{r}\frac{\partial^2 k_2}{\partial s^2}\overline{z^\mu v^\nu}v^\gamma + \frac{s^2}{r^2}\frac{\partial^2 k_2}{\partial s^2}\overline{\langle z,v\rangle}\overline{v^\mu v^\nu}v^\gamma - \frac{s}{r}\frac{\partial k_2}{\partial s}\overline{z^\mu v^\nu}v^\gamma \\
- \frac{s}{r}\frac{\partial k_2}{\partial s}\overline{\langle z,v\rangle}\overline{v^\nu}\delta^\gamma_\mu + s\frac{\partial k_2}{\partial s}\overline{z^\mu}\delta^\gamma_\nu - \frac{s}{r}\frac{\partial k_2}{\partial s}\overline{\langle z,v\rangle}\overline{v^\mu}\delta^\gamma_\nu + k_2\overline{z^\mu}\delta^\gamma_\nu \\
+ s\left(s\frac{\partial^2 k_3}{\partial s^2} + 3\frac{\partial k_3}{\partial s}\right)\overline{z^\mu z^\nu}z^\gamma - \frac{s}{r}\left(s\frac{\partial^2 k_3}{\partial s^2} + 3\frac{\partial k_3}{\partial s}\right)\overline{\langle z,v\rangle}\overline{z^\nu v^\mu}z^\gamma + (m_1+k_3)\overline{z^\mu z^\nu}z^\gamma \\
- \frac{s}{r}\frac{\partial k_3}{\partial s}\overline{\langle z,v\rangle}\overline{z^\mu v^\nu}z^\gamma + \frac{s}{r^2}\frac{\partial k_3}{\partial s}(\overline{\langle z,v\rangle})^2\overline{v^\mu v^\nu}z^\gamma + \frac{s}{r^2}\frac{\partial k_3}{\partial s}(\overline{\langle z,v\rangle})^2\overline{v^\mu v^\nu}z^\gamma \\
- \frac{s^2}{r}\frac{\partial^2 k_3}{\partial s^2}\overline{\langle z,v\rangle}\overline{z^\mu v^\nu}z^\gamma + \frac{s^2}{r^2}\frac{\partial^2 k_3}{\partial s^2}(\overline{\langle z,v\rangle})^2\overline{v^\mu v^\nu}z^\gamma - \frac{2s}{r}\frac{\partial k_3}{\partial s}\overline{\langle z,v\rangle}\overline{z^\mu v^\nu}z^\gamma.
\end{aligned}
$$

The lemma is completed by rearranging terms in the above equality. $\qquad\square$

Lemma 2.4. *Let $F = \sqrt{r\phi(t,s)}$ be a strongly pseudoconvex complex Finsler metric defined on a*

domain $D \subset \mathbb{C}^n$. Then the complex Cartan tensor $G_{\mu\overline{\tau}\alpha}$ and $G_{\mu\overline{\tau};\nu}$ are given respectively by

$$
\begin{aligned}
G_{\mu\overline{\tau}\alpha} = \Bigg[& -s\phi_{ss}\delta_{\mu\overline{\tau}} - \frac{1}{r}(\phi_{ss} + s\phi_{sss})\overline{\langle z,v\rangle}\overline{v^\mu}z^\tau + \frac{s}{r}(2\phi_{ss} + s\phi_{sss})\overline{v^\mu}v^\tau \\
& -\frac{1}{r}(\phi_{ss} + s\phi_{sss})\langle z,v\rangle\overline{z^\mu}v^\tau + (\phi_{ss} + s\phi_{sss})\overline{z^\mu}z^\tau \Bigg]s_\alpha + \frac{1}{r}\phi_{ss}\langle z,v\rangle\overline{z^\mu z^\alpha}z^\tau \\
& -\frac{s}{r}\phi_{ss}\overline{z^\alpha v^\mu}z^\tau - \frac{s}{r}\phi_{ss}\overline{z^\mu v^\alpha}z^\tau + \frac{s}{r^2}\phi_{ss}\langle z,v\rangle\overline{z^\mu v^\alpha}v^\tau + \frac{s}{r^2}\phi_{ss}\overline{\langle z,v\rangle}\overline{v^\mu v^\alpha}z^\tau \\
& -\frac{s^2}{r^2}\phi_{ss}\overline{v^\mu v^\alpha}v^\tau - \frac{s}{r}\phi_{ss}\langle z,v\rangle\overline{z^\mu}\delta_{\tau\alpha} + \frac{s^2}{r}\phi_{ss}\overline{v^\mu}\delta_{\tau\alpha},
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
G_{\mu\overline{\tau};\nu} = {} & (\phi_s + s\phi_{ss})\delta_{\tau\nu}\overline{z^\mu} + (\phi_t - s\phi_{st})\delta_{\mu\overline{\tau}}\overline{z^\nu} - \frac{s}{r}\phi_{ss}\overline{\langle z,v\rangle}\delta_{\mu\overline{\tau}}\overline{v^\nu} - \frac{s}{r}\phi_{ss}\overline{\langle z,v\rangle}\delta_{\nu\tau}\overline{v^\mu} \\
& + (s\phi_{sst} + \phi_{st})\overline{z^\mu z^\nu}z^\tau + \frac{s^2}{r}\phi_{sst}\overline{v^\mu z^\nu}v^\tau - \frac{s}{r}(s\phi_{sss} + 2\phi_{ss})\overline{z^\mu v^\nu}v^\tau \\
& - \frac{s}{r}\phi_{sst}\langle z,v\rangle\overline{z^\mu z^\nu}v^\tau - \frac{s}{r}\phi_{sst}\overline{\langle z,v\rangle}\overline{v^\mu z^\nu}z^\tau + \frac{1}{r}(s\phi_{sss} + 2\phi_{ss})\overline{\langle z,v\rangle}\overline{z^\mu v^\nu}z^\tau \\
& + \frac{s}{r^2}(s\phi_{sss} + 2\phi_{ss})\overline{\langle z,v\rangle}\overline{v^\mu v^\nu}v^\tau - \frac{1}{r^2}(s\phi_{sss} + \phi_{ss})(\overline{\langle z,v\rangle})^2\overline{v^\mu v^\nu}z^\tau.
\end{aligned}
\tag{2.9}
$$

Proof. By a simple calculation, we have

$$
s_\mu = \frac{1}{r}[\langle z,v\rangle\overline{z^\mu} - s\overline{v^\mu}], \quad s_{\mu\alpha} = -\frac{1}{r}(s_\mu\overline{v^\alpha} + s_\alpha\overline{v^\mu}),
\tag{2.10}
$$

$$
s_{\overline{\tau}} = \frac{1}{r}[\overline{\langle z,v\rangle}z^\tau - sv^\tau], \quad s_{\overline{\tau}\alpha} = \frac{1}{r}(\overline{z^\alpha}z^\tau - s_\alpha v^\tau - s_{\overline{\tau}}\overline{v^\alpha} - s\delta_{\tau\alpha}),
\tag{2.11}
$$

$$
s_{;\nu} = \frac{1}{r}\langle z,v\rangle\overline{v^\nu}, \quad s_{\mu;\nu} = \frac{1}{r}[\overline{z^\mu v^\nu} - \frac{1}{r}\langle z,v\rangle\overline{v^\mu v^\nu}],
\tag{2.12}
$$

$$
s_{\overline{\tau};\nu} = \frac{1}{r}[\overline{\langle z,v\rangle}\delta_{\tau\nu} - \frac{1}{r}\overline{\langle z,v\rangle}\overline{v^\nu}v^\tau].
\tag{2.13}
$$

Differentiating (2.1) with respect to v^α, we get

$$
\begin{aligned}
G_{\mu\overline{\tau}\alpha} = {} & -s\phi_{ss}\delta_{\mu\overline{\tau}}s_\alpha - \phi_{ss}s_{\overline{\tau}}\overline{v^\mu}s_\alpha + \phi_{ss}\overline{z^\mu}z^\tau s_\alpha + \phi_{ss}\overline{z^\alpha}z^\tau s_\mu \\
& - \phi_{ss}s_\mu s_\alpha v^\tau + r\phi_{sss}s_\mu s_{\overline{\tau}}s_\alpha - \phi_{ss}s_\mu s_{\overline{\tau}}\overline{v^\alpha} - s\phi_{ss}\delta_{\tau\alpha}s_\mu.
\end{aligned}
\tag{2.14}
$$

Substituting (2.10)-(2.13) into (2.14), we obtain (2.8).

Next differentiating (2.1) with respect to z^ν, we have

$$
\begin{aligned}
G_{\mu\overline{\tau};\nu} = {} & (\phi_t\overline{z^\nu} - s\phi_{st}\overline{z^\nu} - s\phi_{ss}s_{;\nu})\delta_{\mu\overline{\tau}} + r\phi_{sst}s_\mu s_{\overline{\tau}}\overline{z^\nu} + r\phi_{sss}s_\mu s_{\overline{\tau}}s_{;\nu} \\
& + r\phi_{ss}s_{\mu;\nu}s_{\overline{\tau}} + r\phi_{ss}s_{\overline{\tau};\nu}s_\mu + \phi_{st}\overline{z^\mu z^\nu}z^\tau + \phi_{ss}\overline{z^\mu}z^\tau s_{;\nu} + \phi_s\delta_{\tau\nu}\overline{z^\mu}.
\end{aligned}
\tag{2.15}
$$

Substituting (2.10)-(2.13) into (2.15), we obtain (2.9). □

Theorem 2.1. ([12]) *A strongly pseudoconvex complex Finsler metrics $F = \sqrt{r\phi(t,s)}$ defined on a domain $D \subset \mathbb{C}^n$ is a complex Berwald metric if and only if*

$$\phi_{ss} = 0, \tag{2.16}$$

if and only if $\phi(t,s) = a_0(t) + a_1(t)s$ for smooth real-valued functions $a_0(t)$ and $a_1(t)$ satisfying $a_0(t) > 0$ and $a_0(t) + ta_1(t) > 0$.

3   Complex Landersberg metrics

Complex Landsberg metric is an analogue notion in complex Finsler geometry, which was introduced by M. Aldea and M. Munteanu in [10]. Denote by $\check{\delta}_\nu = \partial_\nu - \mathbb{G}_\nu^\alpha \dot{\partial}_\alpha$ for $\nu = 1, \cdots, n$ and define

$$\mathbb{L}_{\nu\mu}^\gamma = \frac{1}{2} G^{\overline{\tau}\gamma} (\check{\delta}_\nu G_{\mu\overline{\tau}} + \check{\delta}_\mu G_{\nu\overline{\tau}}).$$

A strongly pseudoconvex complex Finsler metric F is called a complex Landsberg metric if $\mathbb{G}_{\nu\mu}^\gamma = \mathbb{L}_{\nu\mu}^\gamma$, where $\mathbb{G}_{\nu\mu}^\gamma$ are the complex Berwald connection coefficients. $\mathbb{L}_{\nu\mu}^\gamma$ are the horizontal connection coefficients of the Rund type complex linear connection in the sense of M. Munteanu [11].

   Among unitary invariant strongly pseudoconvex complex Finsler metrics there exists no complex Berwald metric which does not come from a unitary invariant Hermitian metric. One may wonder that among unitary invariant strongly pseudoconvex complex Finsler metrics, whether there exists a complex Landsberg metric which does not come from a unitary invariant Hermitian metric? If there exists such a metric, then it implies that the unicorn metric problem in complex Finsler geometry does not hold.

Theorem 3.1. *A strongly pseudoconvex complex Finsler metrics $F = \sqrt{r\phi(t,s)}$ defined on a domain $D \subset \mathbb{C}^n$ is a complex Landsberg metric if and only if*

$$\phi_{ss} = 0, \tag{3.1}$$

if and only if $\phi(t,s) = a_0(t) + a_1(t)s$ for smooth real-valued functions $a_0(t)$ and $a_1(t)$ satisfying $a_0(t) > 0$ and $a_0(t) + ta_1(t) > 0$.

Proof. If $\phi_{ss} = 0$, then (2.1) gives

$$G_{\gamma\overline{\tau}} = (\phi - s\phi_s)\delta_{\gamma\overline{\tau}} + \phi_s \overline{z^\gamma} z^\tau.$$

since

$$\frac{\partial(\phi - s\phi_s)}{\partial s} = -s\phi_{ss} = 0,$$

thus, $G_{\alpha\overline{\beta}}$ actually depends only on $z = (z^1 \cdots z^n)$, i.e., F comes from a Hermitian metric.

Next we shall prove the necessity part of the theorem. Let $F = \sqrt{r\phi(t,s)}$ be a complex Landsberg metric, then we have

$$2G_{\gamma\bar{\tau}}\mathbb{G}^{\gamma}_{\nu\mu} = (G_{\mu\bar{\tau};\nu} + G_{\nu\bar{\tau};\mu}) - (\mathbb{G}^{\alpha}_{\nu}G_{\mu\bar{\tau}\alpha} + \mathbb{G}^{\beta}_{\mu}G_{\nu\bar{\tau}\beta}) \tag{3.2}$$

Using (2.1) and (2.7), after a long calculation, we have

$$
\begin{aligned}
4G_{\gamma\bar{\tau}}\mathbb{G}^{\gamma}_{\nu\mu} =\ & 2(\phi - s\phi_s)\left[n_1\delta_{\nu\tau}\overline{z^{\mu}} + n_1\delta_{\mu\tau}\overline{z^{\nu}} - \frac{1}{r}(n_1 - k_2)\overline{\langle z,v \rangle}(\overline{v^{\mu}}\delta_{\nu\tau} - \overline{v^{\nu}}\delta_{\mu\tau})\right] \\
& + \frac{2s}{r}[s^2(t-s)\phi_{ss}m_3 - k_4n_2 + (2n_1 - k_2)s\phi_{ss}]\overline{z^{\mu}v^{\nu}}v^{\tau} \\
& + \frac{2s}{r}[s^2(t-s)\phi_{ss}m_3 - k_4n_2 + (2n_1 - k_2)s\phi_{ss}]\overline{z^{\nu}v^{\mu}}v^{\tau} \\
& + [2k_5(sm_3 + m_1 + k_3) + 2s\phi_s n_2 + 4k_6 n_1]\overline{z^{\mu}z^{\nu}}z^{\tau} \\
& + \frac{2}{r}[-s(t-s)\phi_{ss}(sm_3 + m_1 + k_3) + k_4n_2 - 2s\phi_{ss}n_1]\langle z,v \rangle\overline{z^{\mu}z^{\nu}}v^{\tau} \\
& + \frac{2s}{r^2}[-s^2(t-s)\phi_{ss}m_2 + k_4n_2 - 2s\phi_{ss}(n_1 - k_2)]\overline{\langle z,v \rangle}\,\overline{v^{\mu}v^{\nu}}v^{\tau} \\
& - \frac{2s}{r}[k_5m_3 + \phi_s n_3 + \phi_{ss}(2n_1 - k_2)]\overline{\langle z,v \rangle}\,\overline{z^{\nu}v^{\mu}}z^{\tau} \\
& - \frac{2s}{r}[k_5m_3 + \phi_s n_3 + \phi_{ss}(2n_1 - k_2)]\overline{\langle z,v \rangle}\,\overline{z^{\mu}v^{\nu}}z^{\tau} \\
& + \frac{2s}{r^2}[k_5m_2 + \phi_s n_2 + 2\phi_{ss}(n_1 - k_2)](\overline{\langle z,v \rangle})^2\overline{v^{\mu}v^{\nu}}z^{\tau},
\end{aligned}
\tag{3.3}
$$

where we denote

$$n_3 = s\frac{\partial^2 k_2}{\partial s^2} + 3\frac{\partial k_2}{\partial s}, \quad k_4 = \phi - s\phi_s, \quad k_5 = \phi + (t-s)\phi_s + s(t-s)\phi_{ss}, \quad k_6 = \phi_s + s\phi_{ss}.$$

Interchanging the indices μ and ν in (2.9), and then adding the obtained equality to (2.9), we get

$$
\begin{aligned}
G_{\mu\bar{\tau};\nu} + G_{\nu\bar{\tau};\mu} =\ & (\phi_s + \phi_t + s\phi_{ss} - s\phi_{st})(\delta_{\tau\nu}\overline{z^{\mu}} + \delta_{\mu\bar{\tau}}\overline{z^{\nu}}) - \frac{2s}{r}\phi_{ss}\overline{\langle z,v \rangle}(\delta_{\mu\bar{\tau}}\overline{v^{\nu}} + \delta_{\nu\tau}\overline{v^{\mu}}) \\
& + 2(s\phi_{sst} + \phi_{st})\overline{z^{\mu}z^{\nu}}z^{\tau} + \frac{1}{r}sk_7(\overline{z^{\mu}v^{\nu}}v^{\tau} + \overline{z^{\nu}v^{\mu}}v^{\tau}) \\
& - \frac{2s}{r}\phi_{sst}\langle z,v \rangle\overline{z^{\mu}z^{\nu}}v^{\tau} - \frac{1}{r}k_7\overline{\langle z,v \rangle}(\overline{z^{\mu}v^{\nu}}z^{\tau} + \overline{z^{\nu}v^{\mu}}z^{\tau}) \\
& + \frac{2}{r^2}sk_8\overline{\langle z,v \rangle}\,\overline{v^{\mu}v^{\nu}}v^{\tau} - \frac{2}{r^2}(k_8 - \phi_{ss})(\overline{\langle z,v \rangle})^2\overline{v^{\mu}v^{\nu}}z^{\tau},
\end{aligned}
\tag{3.4}
$$

where we denote

$$k_7 = s\phi_{sst} - s\phi_{sss} - 2\phi_{ss}, \quad k_8 = s\phi_{sss} + 2\phi_{ss}.$$

Using (2.7), (2.8), and notice that $G_{\mu\bar{\tau}\alpha}v^{\alpha}=0$, $G_{\mu\bar{\tau}\alpha}v^{\bar{\tau}}=G_{\mu\alpha}=r\phi_{ss}s_{\alpha}s_{\mu}$, we obtain

$$
\begin{aligned}
2G_{\nu}^{\alpha}G_{\mu\bar{\tau}\alpha}+2G_{\mu}^{\beta}G_{\nu\bar{\tau}\beta}=&-s^2[(t-s)(m_1+k_3)+2k_2](\phi_{ss}\overline{z^{\mu}}\delta_{\nu\tau}+\phi_{ss}\overline{z^{\nu}}\delta_{\mu\bar{\tau}})\\
&+\frac{s^2}{r}[(t-s)(m_1-k_3)+2k_2](\phi_{ss}\overline{\langle z,v\rangle}\overline{v^{\mu}}\delta_{\nu\tau}+\phi_{ss}\overline{\langle z,v\rangle}\overline{v^{\nu}}\delta_{\mu\bar{\tau}})\\
&+\frac{s^2}{r}[2k_8k_2+2k_9k_3+(k_9+k_{10})(m_1-k_3)](\overline{z^{\mu}v^{\nu}}v^{\tau}+\overline{z^{\nu}v^{\mu}}v^{\tau})\\
&+2[sk_8k_2+sk_9(m_1+k_3)]\overline{z^{\mu}z^{\nu}}z^{\tau}\\
&-\frac{2s}{r}[(k_8-\phi_{ss})k_2+k_{10}(m_1+k_3)]\langle z,v\rangle\overline{z^{\mu}z^{\nu}}v^{\tau}\\
&-\frac{2s^2}{r^2}[(k_8+\phi_{ss})k_2+k_9(m_1-k_3)]\overline{\langle z,v\rangle}\overline{v^{\mu}v^{\nu}}v^{\tau}\\
&-\frac{2s}{r}(k_8k_2+k_9m_1)\overline{\langle z,v\rangle}\overline{z^{\nu}v^{\mu}}z^{\tau}-\frac{2s}{r}(k_8k_2+k_9m_1)\overline{\langle z,v\rangle}\overline{z^{\mu}v^{\nu}}z^{\tau}\\
&+\frac{2s}{r^2}[k_8k_2+k_9(m_1-k_3)](\overline{\langle z,v\rangle})^2\overline{v^{\mu}v^{\nu}}z^{\tau},
\end{aligned}
\tag{3.5}
$$

where we denote

$$
k_9=(2t-3s)\phi_{ss}+s(t-s)\phi_{sss}, \quad k_{10}=(t-2s)\phi_{ss}+s(t-s)\phi_{sss}.
\tag{3.6}
$$

Substituting (3.3), (3.4) and (3.5) into (3.2), and comparing coefficients of the same types on both sides of the resulted equality, we obtain the following system of equations

$$
\begin{cases}
2k_4n_1=2(\phi_s+s\phi_{ss}+\phi_t-s\phi_{st})+s^2(t-s)\phi_{ss}(m_1+k_3)+2s^2\phi_{ss}k_2,\\
2k_4(n_1-k_2)=4s\phi_{ss}+s^2(t-s)\phi_{ss}(m_1-k_3)+2s^2\phi_{ss}k_2,\\
2s^2(t-s)\phi_{ss}m_3-2k_4n_2+2(2n_1-k_2)s\phi_{ss}=2k_7-2sk_8k_2-2sk_9k_3-s(k_9+k_{10})(m_1-k_3),\\
k_5(sm_3+m_1+k_3)+s\phi_sn_2+2k_6n_1=2(s\phi_{sst}+\phi_{st})-sk_8k_2-sk_9(m_1+k_3),\\
-s(t-s)\phi_{ss}(sm_3+m_1+k_3)+k_4n_2-2s\phi_{ss}n_1=-2s\phi_{sst}+s(k_8-\phi_{ss})k_2+sk_{10}(m_1+k_3),\\
-s^2(t-s)\phi_{ss}m_2+k_4n_2-2s\phi_{ss}(n_1-k_2)=2k_8+s(k_8+\phi_{ss})k_2+sk_9(m_1-k_3),\\
k_5sm_3+s\phi_sn_3+s\phi_{ss}(2n_1-k_2)=k_7-sk_8k_2-sk_9m_1,\\
k_5sm_2+s\phi_sn_2+2s\phi_{ss}(n_1-k_2)=-2(k_8-\phi_{ss})-sk_8k_2-sk_9(m_1-k_3).
\end{cases}
\tag{3.7}
$$

The first equation of (3.7) minus the second equation of (3.7) yields

$$
s^2(t-s)\phi_{ss}k_3=k_4k_2-(\phi_s-s\phi_{ss}+\phi_t-s\phi_{st}).
\tag{3.8}
$$

Taking the derivative with respect to the variable s, we obtain

$$
s^2(t-s)\phi_{ss}\frac{\partial k_3}{\partial s}=k_4\frac{\partial k_2}{\partial s}-s\phi_{ss}k_2+s\phi_{sss}+s\phi_{sst}-sk_9k_3.
\tag{3.9}
$$

The third equation of (3.7) plus two times of the sixth equation of (3.7) yields

$$
s^2(t-s)\phi_{ss}\frac{\partial k_3}{\partial s}=2(k_7+2k_8)-2sk_9k_3.
\tag{3.10}
$$

Substituting (3.10) into the second equation of (3.7), one gets

$$2k_4\frac{\partial k_2}{\partial s}=4\phi_{ss}+2(k_7+2k_8)-2sk_9k_3-s(t-s)\phi_{ss}k_3+2s\phi_{ss}k_2. \tag{3.11}$$

By (3.9)-(3.11), it follows that

$$s(t-s)\phi_{ss}k_3=0. \tag{3.12}$$

Now it is easy to see that the Eq. (3.12) holds if and only if either

$$\phi_{ss}=0, \tag{3.13}$$

or

$$k_3=0 \tag{3.14}$$

holds.

If (3.13) holds, we immediately have

$$\phi(t,s)=a_0(t)+a_1(t)s,$$

where $a_0(t)$ and $a_1(t)$ are two smooth real-valued functions satisfying $a_0(t)>0$ and $a_0(t)+ta_1(t)>0$.

If (3.14) holds, according to (2.5), we have

$$\phi_{ss}+\phi_{st}=\frac{\phi_s(\phi_t+\phi_s)}{\phi}. \tag{3.15}$$

Substituting (3.15) into (2.4), we get

$$k_2=\frac{\phi_t+\phi_s}{\phi k_1}\left\{(\phi-s\phi_s)[\phi+(t-s)\phi_s]+s(t-s)\phi\phi_{ss}\right\}=\frac{\phi_t+\phi_s}{\phi k_1}k_1=\frac{\phi_t+\phi_s}{\phi}. \tag{3.16}$$

Thus

$$\frac{\partial k_2}{\partial s}=\frac{1}{\phi^2}[\phi(\phi_{st}+\phi_{ss})-\phi_s(\phi_t+\phi_s)]=\frac{1}{\phi}k_1k_3=0. \tag{3.17}$$

Substituting (3.14) and (3.17) into the second and eighth equations of (3.7), respectively, we obtain

$$\begin{cases} sk_2=-2, \\ 2\phi_{ss}=(sk_2+2)k_8, \end{cases} \tag{3.18}$$

from which we get (3.13). This completes the proof of the Theorem. $\qquad\square$

Remark 3.1. Theorem 3.1 implies as far as that unitary invariant strongly pseudoconvex complex metrics are concerned, $F=\sqrt{r\phi(t,s)}$ is a complex Landsberg metric if and only if it comes from a unitary invariant Hermitian metric. Thus there is no unicorn metric among unitary invariant strongly pseudoconvex complex Finsler metrics.

Acknowledgments

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