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On $\mathfrak{F}_\tau$-s-supplemented Subgroups of Finite Groups

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Abstract. Let $\mathfrak{F}$ be a non-empty formation of groups, τ a subgroup functor and H a p-subgroup of a finite group G. Let $\bar{G}=G/H_G$ and $\bar{H}=H/H_G$. We say that H is $\mathfrak{F}_\tau$-s-supplemented in G if for some subgroup $\bar{T}$ and some τ-subgroup $\bar{S}$ of $\bar{G}$ contained in $\bar{H}$, $\bar{H}\bar{T}$ is subnormal in $\bar{G}$ and $\bar{H}\cap\bar{T}\leq\bar{S}Z_{\mathfrak{F}}(\bar{G})$. In this paper, we investigate the influence of $\mathfrak{F}_\tau$-*s*-supplemented subgroups on the structure of finite groups. Some new characterizations about solubility of finite groups are obtained.

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1 Introduction

Throughout this paper, all groups considered are finite and G always denotes a group, π denotes a set of primes and p denotes a prime. Let $|G|_p$ denote the order of Sylow p-subgroups of G. All unexplained notation and terminology are standard, as in [1] and [2].

For a class of groups $\mathfrak{F}$, a chief factor L/K of G is said to be $\mathfrak{F}$-central in G if $L/K\rtimes G/C_G(L/K)\in\mathfrak{F}$. A normal subgroup N of G is called $\mathfrak{F}$-hypercentral in G if either $N=1$ or every chief factor of G below N is $\mathfrak{F}$-central in G. Let $Z_{\mathfrak{F}}(G)$ denote the $\mathfrak{F}$-hypercentre of G, that is, the product of all $\mathfrak{F}$-hypercentral normal subgroups of G. We use $\mathfrak{N}_p$ and $\mathfrak{S}$ to denote the classes of all p-nilpotent groups and soluble groups, respectively. It is well known that $\mathfrak{N}_p$ and $\mathfrak{S}$ are all S-closed saturated formations. Following Guo [3], a subgroup functor is a function τ which assigns to each group G a set of subgroups $\tau(G)$ of G satisfying that $1\in\tau(G)$ and $\theta(\tau(G))=\tau(\theta(G))$ for any isomorphism θ: $G\to G^*$. If $H\in\tau(G)$, then H is called a τ-subgroup of G. If τ is a subgroup functor, then τ is said to be

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(1) inductive if for any group G, whenever $H\in\tau(G)$ is a p-group and $N\trianglelefteq G$, then $HN/N\in\tau(G/N)$.

(2) hereditary if for group G, whenever $H\in\tau(G)$ is a p-group and $H\leq E\leq G$, then $H\in\tau(E)$.

(3) Φ-regular if any primitive group G, whenever $H\in\tau(G)$ is a p-group and N is a minimal normal subgroup of G, then $|G:N_G(H\cap N)|$ is a power of p.

Recall that a subgroup H of G is said to complemented in G if G has a subgroup K such that $G=HK$ and $H\cap K=1$. A subgroup H of G is said to be supplement in G if there exists a subgroup K such that $G=HK$. A subgroup H of G is said to be c-supplemented in G [4] if there exists a normal subgroup N of G such that $G=HN$ and $H\cap N\leq H_G$, where H_G is the largest normal subgroup of G contained in H. For a formation $\mathfrak{F}$, a subgroup H of G is said to be $\mathfrak{F}$-supplement in G [5] if there exists a subgroup K of G such that $G=HK$ and $(H\cap K)H_G/H_G\leq Z_{\mathfrak{F}}(G/H_G)$, where $Z_{\mathfrak{F}}(G/H_G)$ is the $\mathfrak{F}$-hypercenter of G/H_G. By using the above supplement subgroups, people have obtain many interesting results (see, for example, [4], [5] and [6]). As a continuation of the above researches, by using Guo-Skiba's method (see [7]), we now introduce the following notion:

Definition 1.1. Let $\mathfrak{F}$ be a non-empty formation of groups, τ a subgroup functor and H a p-subgroup of a finite group G. Let $\bar{G}=G/H_G$ and $\bar{H}=H/H_G$. We say that H is $\mathfrak{F}_\tau$-s-supplemented in G if for some subgroup $\bar{T}$ and some τ-subgroup $\bar{S}$ of $\bar{G}$ contained in $\bar{H}$, $\bar{H}\bar{T}$ is subnormal in $\bar{G}$ and $\bar{H}\cap\bar{T}\leq\bar{S}Z_{\mathfrak{F}}(\bar{G})$.

It is clear that c-supplemented subgroups and $\mathfrak{F}$-supplement subgroups are all $\mathfrak{F}_\tau$-s-supplemented subgroups. But the following example shows that the converse is not true.

Example 1.1. Let $G=A\rtimes B$, where A is a cyclic group of order 5 and $B=\langle\alpha\rangle\in Aut(A)$ with $|\alpha|=4$. Put $H=\langle\alpha^2\rangle$. Since $|G:HA|=2$, HA is normal in G. It is easy to see that $H_G=Z_\infty(G)=1$. If $H_{sG}\neq 1$, then by [8, Lemma A], $O^2(G)\leq N_G(H_{sG})$ and so $H_{sG}\trianglelefteq G$, which is impossible. Hence $H_{sG}=1$. Let $\tau(G)$ be the set of all S-qusinormal subgroups of G. If $S\leq H$ and $S\in\tau(G)$, then $S\leq H_{sG}=1$. Hence H is $\mathfrak{F}_\tau$-s-supplemented in G. But H is not $\mathfrak{F}$-supplement in G. Assume that H is $\mathfrak{F}$-supplement in G. Then G has a subgroup K such that $G=HK$ and $H\cap K=1$. It implies that H is complemented in G, and so H is complemented in B. This contradicts that B is cyclic. Therefore, H is not $\mathfrak{F}$-supplement in G. Clearly, $O_2(G)=1$, so H is not c-supplement in G.

In this paper, we investigate the influence of the $\mathfrak{F}_\tau$-s-supplemented subgroups on the structure of finite groups. Some new results of soluble groups are obtained.

2 Preliminaries

Lemma 2.1. [9, Lemma 2.5] *Let U be a subnormal subgroup of G.*

(1) If $V \leq G$, then $U \cap V$ is subnormal in V.

(2) If $N \trianglelefteq G$, then UN/N is subnormal in G/N.

(3) If U is a π-subgroup, then $U \leq O_\pi(G)$.

(4) If U is soluble, then U is contained in some normal soluble subgroup of G.

Lemma 2.2. [5, Lemma 2.1] *Let $\mathfrak{F}$ be a non-empty saturated formation, $H \leq G$ and $N \trianglelefteq G$. Then:*

(1) $Z_{\mathfrak{F}}(G)N/N \leq Z_{\mathfrak{F}}(G/N)$.

(2) If $\mathfrak{F}$ is S-closed, then $Z_{\mathfrak{F}}(G) \cap H \leq Z_{\mathfrak{F}}(H)$.

Lemma 2.3. *Let $\mathfrak{F}$ be a non-empty formation of groups and τ an inductive subgroup functor. Suppose that H is a p-subgroup of G and H is $\mathfrak{F}_\tau$-s-supplemented in G.*

(1) If $N \trianglelefteq G$ and either $N \leq H$ or $(|H|,|N|)=1$, then HN/N is $\mathfrak{F}_\tau$-s-supplemented in G/N.

(2) If $\mathfrak{F}$ is an s-closed saturated formation, τ is hereditary and $H \leq K \leq G$, then H is $\mathfrak{F}_\tau$-s-supplemented in K.

Proof. Let $\bar{G} = G/H_G$ and $\bar{H} = H/H_G$. Since H is $\mathfrak{F}_\tau$-s-supplemented in G, $\bar{G}$ has a subgroup $\bar{T}$ and a τ-subgroup $\bar{S}$ contained in $\bar{H}$ such that $\bar{H}\bar{T}$ is subnormal in $\bar{G}$ and $\bar{H} \cap \bar{T} \leq \bar{S} Z_{\mathfrak{F}}(\bar{G})$, where $\bar{S} = S/H_G$ and $\bar{T} = T/H_G$.

(1) Let $\widehat{G} = G/(HN)_G$, $\widehat{HN} = HN/(HN)_G$, $\widehat{T} = T(HN)_G/(HN)_G$ and $\widehat{S} = S(HN)_G/(HN)_G$. Clearly, $H_G \leq (HN)_G$. Then $\widehat{S} \in \tau(\widehat{G})$ for τ is inductive. By Lemma 2.1 (2), $\widehat{HN}\widehat{T}$ is subnormal in $\widehat{G}$. Since $(|N|,|H|)=1$, $(|HN \cap T : T \cap N|, |HN \cap T : T \cap H|)=1$. Hence $(HN \cap T) = (H \cap T)(N \cap T)$. By Lemma 2.2 (1), it is easy to see that $(Z_{\mathfrak{F}}(G/H_G))((HN)_G/H_G)/((HN)_G/H_G) \leq Z_{\mathfrak{F}}(G/(HN)_G)$. It follows that

$$\begin{aligned}\widehat{HN} \cap \widehat{T} &= HN/(HN)_G \cap T(HN)_G/(HN)_G = (H \cap T)(HN)_G/(HN)_G \\ &\leq (S(HN)_G/(HN)_G)(Z_{\mathfrak{F}}(G/(HN)_G)) = \widehat{S} Z_{\mathfrak{F}}(\widehat{G}).\end{aligned}$$

Therefore, HN/N is $\mathfrak{F}_\tau$-s-supplemented in G/N.

(2) It is easy to see that $H_G \leq H_K$. Let $\widetilde{K} = K/H_K$, $\widetilde{H} = H/H_K$, $\widetilde{T} = (TH_K/H_K) \cap (K/H_K)$ and $\widetilde{S} = SH_K/H_K$. Since τ is hereditary and inductive, $\widetilde{S} \in \tau(\widetilde{K})$. By Lemma 2.1 (1) (2), $\widetilde{H}\widetilde{T} = (H/H_K)(TH_K/\ H_K \cap K/H_K) = H(T \cap K)/H_K = (HT \cap K)/H_K$ is subnormal in $\widetilde{K}$. Since $\mathfrak{F}$ is an s-closed saturated formation, $Z_{\mathfrak{F}}(G/H_K) \cap K/H_K \leq Z_{\mathfrak{F}}(K/H_K)$ by Lemma 2.2 (2). It implies that $\widetilde{H} \cap \widetilde{T} = H/H_K \cap TH_K/H_K = (H \cap T)H_K/H_K \leq (SH_K/H_K)(Z_{\mathfrak{F}}(G/H_K) \cap K/H_K) \leq (SH_K/H_K)(Z_{\mathfrak{F}}(K/H_K)) = \widetilde{S} Z_{\mathfrak{F}}(\widetilde{K})$. Hence H is $\mathfrak{F}_\tau$-s-supplemented in K. □

Lemma 2.4. [1, Chapter I, 3.5] *Suppose that* $|G|=p_1p_2\cdots p_s$. *Then* G *is soluble if and only if* G *has* p_i'*-Hall subgroup for every* $i=1,2,\cdots,s$.

Lemma 2.5. [1, Chapter A, 14.3] *Let* A *be a subnormal subgroup of* G *and* N *a minimal normal subgroup of* G, *then* $N\leq N_G(A)$.

3 Main Results

Theorem 3.1. *Suppose that* τ *is a* Φ*-regular inductive subgroup functor. A group* G *is soluble if and only if every Sylow subgroup of* G *is* $\mathfrak{S}_\tau$*-s-supplemented in* G.

Proof. The necessity is obvious. We need only prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order.

First we show that G has a unique minimal normal subgroup N, G/N is soluble and G is a primitive group. Let N be a minimal normal subgroup of G, and let H/N be a Sylow p-subgroup G/N, where p is a prime divisor of $|G|$. Then there exists a Sylow p-subgroup G_p of G such that $H=G_pN$. Let $\bar{G}=G/(G_p)_G$ and $\bar{G_p}=G_p/(G_p)_G$. By the hypothesis, $\bar{G}$ has a subgroup $\bar{T}$ and a τ-subgroup $\bar{S}$ contained in $\bar{G_p}$ such that $\bar{G_p}\bar{T}$ is subnormal in $\bar{G}$ and $\bar{G_p}\cap\bar{T}\leq\bar{S}Z_{\mathfrak{S}}(\bar{G})$, where $\bar{S}=S/(G_p)_G$ and $\bar{T}=T/(G_p)_G$. Let $\widehat{G}=G/(G_pN)_G$, $\widehat{G_pN}=G_pN/(G_pN)_G$, $\widehat{T}=T(G_pN)_G/(G_pN)_G$ and $\widehat{S}=S(G_pN)_G/(G_pN)_G$. Obviously, $(G_p)_G\leq(G_pN)_G$. Since τ is inductive, $\widehat{S}\in\tau(\widehat{G})$. By Lemma 2.1 (2), $\widehat{G_pN}\widehat{T}=(G_pN/(G_pN)_G)(T(G_pN)_G/(G_pN)_G)$ is subnormal in $\widehat{G}$. Since $(|G_pN\cap T:G_p\cap T|,|G_pN\cap T:N\cap T|)=1$, $(G_pN\cap T)=(G_p\cap T)(N\cap T)$. By Lemma 2.2 (1), it is clear to see that

$$(Z_{\mathfrak{S}}(G/(G_p)_G)((G_pN)_G/(G_p)_G)/((G_pN)_G/(G_p)_G)\leq Z_{\mathfrak{S}}(G/(G_pN)_G).$$

It follows that $\widehat{G_pN}\cap\widehat{T}=(G_pN)_G(G_pN\cap T)/(G_pN)_G\leq(S(G_pN)_G/(G_pN)_G)Z_{\mathfrak{S}}(G/(G_pN)_G)=\widehat{S}Z_{\mathfrak{S}}(\widehat{G})$. Hence H/N is $\mathfrak{S}_\tau$-s-supplemented in G/N. The choice of G implies that G/N is soluble. If N is soluble, then G is soluble, a contradiction. Therefore, N is not soluble. Since the class of all soluble groups is closed under subdirect product, N is the unique minimal normal subgroup of G. Clearly, $N\nleq\Phi(G)$. There exists a maximal subgroup M of G such that $N\nleq M$ and so $M_G=1$. Hence G is a primitive group.

Let N_p be Sylow p-subgroup N, where p is any prime divisor of $|N|$. Then there exists a Sylow p-subgroup P of G such that $N_p=N\cap P$. Since N is not soluble and the unique minimal normal subgroup of G, $P_G=Z_{\mathfrak{S}}(G)=1$ and $N=N_1\times N_2\times\cdots\times N_t$, where N_i $(i=1,2,\cdots,t)$ are isomorphic non-abelian simple groups. By the hypothesis, G has a subgroup T and a τ-subgroup S contained in P such that PT is subnormal in G and $P\cap T\leq S$. By Lemma 2.1(1), $PT\cap N_i$ is subnormal in N_i for every i, and so either $PT\cap N_i=1$ or $N_i\leq PT$. If $PT\cap N_i=1$, then $P\cap N=1$, which is impossible. Assume that $N_i\leq PT$ for every i, then $N\leq PT$. It is easy to see that $(|N\cap PT:N\cap P|:|N\cap PT:N\cap T|)=1$, so $N=N\cap PT=(N\cap P)(N\cap T)$. Since τ is a Φ-regular subgroup functor, $|G:N_G(S\cap N)|$ is a power of p. If $N\cap S>1$, then $N=(S\cap N)^G=(S\cap N)^P\leq S^P\leq P$, a contradiction. Hence

$N\cap S=1$. It implies that $N\cap P\cap T\leq N\cap S=1$. This shows that every Sylow subgroup N is complemented in N. Hence, by Lemma 2.4, N is soluble. The final contradiction completed the proof of the theorem. □

Corollary 3.1. [4, Theorem 2.4] A group G is soluble if and only if every Sylow subgroup of G is c-supplement in G.

Corollary 3.2. [5, Theorem 4.2] A group G is soluble if and only if every Sylow subgroup of G is $\mathfrak{S}$-supplement in G.

Theorem 3.2. *Suppose that τ is a Φ-regular inductive and hereditary subgroup functor. Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|,p-1)=1$. If every maximal subgroup of P is $\mathfrak{S}_\tau$-s-supplemented in G, then G is soluble.*

Proof. Suppose that the theorem is false and let G is a counterexample with minimal order. Then $p=2$ by Feit-Thompson's Theorem. We prove theorem via the following steps.

(1) $O_{2'}(G)=1$

Suppose that $O_{2'}(G)\neq 1$. Let $M/O_{2'}(G)$ be a maximal subgroup of $PO_{2'}(G)/O_{2'}(G)$. Then $M=P_1O_{2'}(G)$ for some maximal subgroup P_1 of P. By the Lemma 2.3 (1) and the hypothesis, $P_1O_{2'}(G)/O_{2'}(G)$ is $\mathfrak{S}_\tau$-s-supplemented in $G/O_{2'}(G)$. This shows that $G/O_{2'}(G)$ satisfies the hypothesis of the theorem. The choice of G implies that $G/O_{2'}(G)$ is soluble, and so G is soluble, a contradiction. Hence $O_{2'}(G)=1$.

(2) $O_2(G)=1$

Assume that $O_2(G)\neq 1$. Obviously, $P\neq O_2(G)$ and $|P/O_2(G)|\geq 2$. Let $P_1/O_2(G)$ be a maximal subgroup of $P/O_2(G)$. Then P_1 is a maximal subgroup of P. By the hypothesis and Lemma 2.3 (1), $P_1/O_2(G)$ is $\mathfrak{S}_\tau$-s-supplemented in $G/O_2(G)$. The choice of G implies that $G/O_2(G)$ is soluble, and so G is soluble, a contradiction. Therefore, $O_2(G)=1$.

(3) If $1\neq H\trianglelefteq G$, then H is not soluble and $G=PH$.

Suppose that H is soluble. Then $O_2(H)\neq 1$ or $O_{2'}(H)\neq 1$. Without loss of generality, assume that $O_2(H)\neq 1$. Since $O_2(H)$ char $H\trianglelefteq G$, we get $O_2(H)\leq O_2(G)$, which contradicts (2). Thus H is not soluble. Assume that $PH<G$. Then by Lemma 2.3 (2), every maximal subgroup of P is $\mathfrak{S}_\tau$-s-supplemented in PH. Therefore PH satisfies the hypothesis. By the choice of G, we have that PH is soluble, and so H is soluble. This contradiction implies that $G=PH$.

(4) G has a unique minimal normal subgroup, denote by N and $N=N_1\times N_2\times\cdots\times N_t$, where N_i $(i=1,2,\cdots,t)$ are isomorphic non-abelian simple groups.

Let N be a minimal normal subgroup of G. Then by (3), $G=PN$. It is clear that $G/N\cong P/P\cap N$ is soluble. Since the class of all soluble groups is closed under subdirect product, G has the unique minimal normal subgroup. Clearly, N is non-abelian, therefore $N=N_1\times N_2\times\cdots\times N_t$, where N_i $(i=1,2,\cdots,t)$ are isomorphic non-abelian simple groups.

(5) Final contradiction

Let P_1 be a maximal subgroup of P. By (4), $(P_1)_G=Z_{\mathfrak{S}}(G)=1$. By the hypothesis, G has a subgroup T_1 and a τ-subgroup S_1 contained in P_1 such that P_1T_1 is subnormal in G and $P_1\cap T_1\le S_1$. If $T_1=1$, then P_1 is subnormal in G, by Lemma 2.1 (3), $P_1\le O_2(G)=1$. It follows that P is cyclic, and thereby G is 2-nilpotent by [10, 10.1.9]. Then G is soluble, a contradiction. Hence $T_1\ne 1$. By Lemma 2.1 (1), $P_1T_1\cap N_i$ is subnormal in N_i for every i, and so either $P_1T_1\cap N_i=1$ or $N_i\le P_1T_1$. If $P_1T_1\cap N_i=1$, then $P_1\cap N_i=1$, which implies that $|N_i|_2\le 2$. Then by [10, 10.1.9] again, N_i is 2-nilpotent, and so N_i is soluble, a contradiction. Hence $N_i\le P_1T_1$ for every i, then $N\le P_1T_1$, and thereby $G=PT_1$ by (3). Since $(|T_1:T_1\cap P|,|T_1:T_1\cap N|)=(|PT_1:P|,|NT_1:N|)=1$, we have $T_1=(T_1\cap P)(T_1\cap N)$. Obviously, G is a primitive group. Assume that $N\cap S_1>1$. Since τ is a Φ-regular subgroup functor, then by (4), $N=(S_1\cap N)^G=(S_1\cap N)^P\le(P_1)^P=P_1$, a contradiction. Hence $N\cap S_1=1$. It implies that $N\cap P_1\cap T_1\le N\cap S_1=1$. Clearly, $P\cap T_1$ is a Sylow p-subgroup of T_1, and thereby $N\cap P\cap T_1$ is a Sylow p-subgroup of $N\cap T_1$. Since $|N\cap P\cap T_1|\le 2$, $N\cap T_1$ is 2-nilpotent. Let V_1 be a normal Hall $2'$-subgroup of $N\cap T_1$. If $V_1=1$, then T_1 is a 2-subgroup for $T_1=(T_1\cap P)(T_1\cap N)$, which is impossible. Hence $V_1\ne 1$. Since $|T_1:V_1|=|(T_1\cap P)(T_1\cap N):V_1|$ is a 2-subgroup and $V_1\trianglelefteq T_1$ for V_1 char $N\cap T_1\trianglelefteq T_1$, V_1 is a normal Hall $2'$-subgroup. By (3), $G=PT_1$. It follows that $N=N\cap PT_1=(N\cap P)(N\cap T_1)$, and thereby V_1 is a Hall $2'$-subgroup of N. Put $H=N_G(V_1)$. Then by Frattini argument, $G=NH$. It is easy to see that $(|N:N\cap P|,|N:V_1|)=1$, so $N=(N\cap P)V_1$. It follows that $G=H(N\cap P)$, and thereby $P=P\cap G=P\cap H(N\cap P)=(P\cap H)(N\cap P)$. Since $(|G:P|,|G:V_1|)=(|G:P|,|PN:V_1|)=1$, $G=PV_1=PH$. It follows that $(|H:P\cap H|,|H:V_1|)=1$, and so $H=(P\cap H)V_1$. If $P\cap H=P$, then $P\le H$. It implies that $G=H$, then V_1 be a normal Hall $2'$-subgroup of G. Therefore, $V_1\le O_{2'}(G)=1$, a contradiction. Hence $P\cap H<P$. Then there exists a maximal subgroup P_2 of P such that $P\cap H\le P_2$. Clearly, $(P_2)_G=1$. By the hypothesis, G has a subgroup T_2 and a τ-subgroup S_2 contained in P_2 such that P_2T_2 is subnormal in G and $P_2\cap T_2\le S_2$. A similar discussion as above, we have $N\le P_2T_2$ and $N\cap T_2$ is 2-nilpotent. Let V_2 be a normal Hall $2'$-subgroup $N\cap T_2$. Obviously, $V_2\ne 1$. The same argument as above, V_2 is a normal Hall $2'$-subgroup of T_2. Since $P=(P\cap H)(N\cap P)=(N\cap P)P_2$, we have $P\le P_2T_2$. By Lemma 2.3(2), P_2T_2 satisfies the hypothesis of the theorem. If $P_2T_2<G$, then the choice of the G implies that P_2T_2 is soluble and so N is soluble, which contradicts (3). Therefore, $G=P_2T_2$. Since $G=PN=PT_2$, it is obvious that V_2 is a Hall $2'$-subgroup of G. Therefore there an element $x\in P$ such that $V_1=(V_2)^x$. Hence

$$G=(P_2T_2)^x=P_2N_G(V_2^x)=P_2N_G(V_1)=P_2H=P_2(P\cap H)V_1=P_2V_1.$$

Then $|G|=|P_2||V_1|<|P||V_1|=|G|$, a contradiction. This completes the proof of the theorem. □

Corollary 3.3. Let M be a maximal subgroup of G and P a Sylow p-subgroup of M, where p is the smallest prime dividing $|M|$. If every maximal subgroup of P is $\mathfrak{F}_\tau$-s-supplemented in G, then G is soluble.

Proof. Suppose that the result is false and let G be a counterexample of minimal order. By Feit-Theopson's theorem, we know $2\in\pi(G)$. By Lemma 2.3(2), every maximal subgroup of P is $\mathfrak{S}_\tau$-semiembedded in M. The choice of G implies that M is soluble. If $|G:M|=2$, then $M\trianglelefteq G$, and so G is soluble, a contradiction. Hence $|G:M|>2$, then P is a Sylow p-subgroup of G, by Theorem 3.1, G is soluble, a contradiction. Hence the theorem holds. □

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