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# On Strong Convergence Theorems for END Sequences

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**Abstract.** In this paper, we study the strong law of large numbers for a sequence of END random variables. Our results extend the corresponding ones for independent random variables and negatively orthant dependent (NOD, in short) random variables.

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**Key words**: Convergence theorem, three series theorems, END random variable.

## 1 Introduction

**Definition 1.1.** ([1]) Random variables $X_1,X_2,\cdots$ are said to be extended negatively dependent(END, in short), if there exists a constant $M>0$ such that both

$$P(X_1\le x_1,...,X_n\le x_n)\le M\prod_{i=1}^{n}P(X_i\le x_i), \tag{1.1}$$

$$P(X_1> x_1,...,X_n> x_n)\le M\prod_{i=1}^{n}P(X_i> x_i) \tag{1.2}$$

hold for each $n\ge 2$ and all real numbers $x_1,\cdots,x_n$.

The concept of END sequences was introduced by Liu [1]. When $M=1$, END random variables are negatively orthant dependent (NOD, in short) random variables, which was introduced by Joag-Dev and Proschan [2]. Some results for NOD sequences can be found in Ko and Kim [3], Fakoor and Azarnoosh [4], Ko *et al.* [5], Wu [6], Kim [7], Wu and Zhu [8]. As is mentioned in Liu [1], the END structure is substantially more comprehensive than the NOD structure in that it can reflect not only a negative dependence structure

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but also a positive one, to some extent. Liu [1] pointed out that the END random variables can be taken as negatively or positively dependent and provided some interesting examples to support this idea. Liu [1] obtained the precise large deviation results for END sequences. Liu [9] studied sufficient and necessary conditions for moderate deviations. Wu and Guan [10] discussed convergence properties of the partial sums for sequences of END random variables. Qiu *et al.* [11] obtained complete convergence for arrays of rowwise END random variables. For more details about strong convergence results for dependent sequence, one can refer to Sung [12], Wang *et al.* [13], Yang and Hu [14], Zhou *et al.* [15] and Zhou [16], and so forth.

The rest of the paper is organized as follows. In Section 2, some preliminary lemmas are presented. In Section 3, main results and their proofs are provided. Throughout the paper, let $I(A)$ be the indicator function of the set $A$. $C$ denotes a positive constant not depending on $n$.

## 2 Preliminaries

The following lemmas will be needed in this paper.

**Lemma 2.1** ([1]). *Let $\{X_n,n\geq 1\}$ be a sequence of END random variables, let $f_1,f_2,\ldots$be all nondecreasing (or all nonincreasing) functions, then $\{f_n(X_n),n\geq 1\}$ is still a sequence of END random variables.*

**Lemma 2.2** ([17]). *Let $\{X_n,n\geq 1\}$ be a sequence of END random variables. Assume that*

$$\sum_{n=1}^{\infty} log^2 n Var X_n<\infty, \tag{2.1}$$

*then $\sum_{n=1}^{\infty}(X_n-EX_n)$ converges almost surely.*

By Lemmas 2.1 and 2.2, we can get the following three series theorems for END sequences. The proof is standard, so we omit it.

**Lemma 2.3.** *Let $\{X_n,n\geq 1\}$ be a sequence of END random variables. For some $c>0$, denote $X_n^{(c)}=-cI(X_n<-c)+XI(|X_n|\leq c)+cI(X_n>c)$. If the following three conditions are satisfied:*

$$\sum_{n=1}^{\infty} P(|X_n|>c)<\infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} EX_n^{(c)} \text{ converges}, \tag{2.3}$$

$$\sum_{n=1}^{\infty} log^2 n Var X_n^{(c)}<\infty, \tag{2.4}$$

*then $\sum_{n=1}^{\infty}X_n$ converges almost surely.*

## 3 Main results

**Theorem 3.1.** *Let $\{X_n,n\geq 1\}$ be a sequence of END random variables. Assume that $\{g_n(x),n\geq 1\}$ is a sequence of even functions defined of $R$, positive and nondecreasing on the half-line $x>0$. Suppose that one or the other of the following conditions is satisfied for every $n\geq 1$,*

*(i) for some $0<r\leq 1$, $x^r/g_n(x)$ is a nondecreasing function of $x$ on the half-line $x>0$;*

*(ii) for some $1<r\leq 2$, $x/g_n(x)$ and $g_n(x)/x^r$ are nonincreasing functions of $x$ on the half-line $x>0$, $EX_n=0$.*

*For any positive number sequence $\{a_n,n\geq 1\}$ with $a_n\uparrow\infty$, if we assume that*

$$\sum_{n=1}^{\infty}\log^2 nEg_n(X_n)/g_n(a_n)<\infty, \tag{3.1}$$

*then $\sum_{n=1}^{\infty}\frac{X_n}{a_n}$ converges almost surely and therefore*

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{i=1}^{n}X_i=0, \quad a.s.$$

*Proof.* Let

$$Z_n=\frac{X_n}{a_n},\ Z_n^{(1)}=-I(Z_n<-1)+Z_nI(|Z_n|\leq 1)+I(Z_n>1).$$

By Lemma 2.1, we can get that $\{Z_n,n\geq 1\}$ is a sequence of END random variables. So by Lemma 2.3 in order to prove $\sum_{n=1}^{\infty}\frac{X_n}{a_n}$ converges almost surely, we need only to prove (2.2)-(2.4), where $c=1$.

Firstly, observe that $\{g_n(x),n\geq 1\}$ is a sequence of even functions defined of $R$, positive and nondecreasing on the half-line $x>0$, $g_n(X_n)\geq g_n(a_n)$ as $|Z_n|\geq 1$. Then

$$P(|Z_n|\geq 1)\leq P(g_n(X_n)\geq g_n(a_n))\leq\frac{Eg_n(X_n)}{g_n(a_n)},$$

so by (3.1) we can get that

$$\sum_{n=1}^{\infty}P(|Z_n|\geq 1)<\infty. \tag{3.2}$$

Secondly, if the function $g_n(x)$ satisfies condition (i), when $|x|\leq a_n$, we have

$$\frac{|x|^r}{g_n(x)}\leq\frac{a_n^r}{g_n(a_n)}.$$

Then

$$\frac{|x|^r}{a_n^r}\leq\frac{g_n(x)}{g_n(a_n)}, \quad \frac{x^2}{a_n^2}\leq\frac{(g_n(x))^{2/r}}{(g_n(a_n))^{2/r}}.$$

Observe that $\{g_n(x), n\geq 1\}$ is a sequence of even functions, positive and nondecreasing on the half-line $x>0$, so

$$0\leq \frac{g_n(x)}{g_n(a_n)}\leq 1, \text{ for } |x|\leq a_n.$$

Consequently,

$$\frac{x^2}{a_n^2}\leq \frac{(g_n(x))^{2/r}}{(g_n(a_n))^{2/r}}\leq \frac{g_n(x)}{g_n(a_n)}, \text{ for } 0<r\leq 1.$$

If the function $g_n(x)$ satisfies condition (ii), when $|x|\leq a_n$, we have $\frac{g_n(x)}{|x|^r}\geq \frac{g_n(a_n)}{a_n^r}$. Then

$$\frac{|x|^r}{a_n^r}\leq \frac{g_n(x)}{g_n(a_n)}, \quad \frac{x^2}{a_n^2}\leq \frac{(g_n(x))^{2/r}}{(g_n(a_n))^{2/r}}.$$

Notice that when $|x|\leq a_n$, $0<\frac{g_n(x)}{g_n(a_n)}\leq 1$. So

$$\frac{x^2}{a_n^2}\leq \frac{(g_n(x))^{2/r}}{(g_n(a_n))^{2/r}}\leq \frac{g_n(x)}{g_n(a_n)}, \text{ for } 1<r\leq 2.$$

Therefore, whether even function $g_n(x)$ satisfies condition (i) or condition (ii), we can obtain

$$\begin{aligned}
\mathrm{Var}Z_n^{(1)} &\leq E(Z_n^{(1)})^2 \\
&= E\left(\frac{X_n^2}{a_n^2} I(|X_n|\leq a_n)\right) + EI(|X_n|>a_n) \\
&\leq E\left(\frac{g_n(X_n)}{g_n(a_n)} I(|X_n|\leq a_n)\right) + E\left(\frac{g_n(X_n)}{g_n(a_n)} I(|X_n|>a_n)\right) \\
&= \frac{Eg_n(X_n)}{g_n(a_n)}.
\end{aligned}$$

Consequently,

$$\sum_{n=1}^{\infty} \log^2 n \mathrm{Var}Z_n^{(1)} \leq \sum_{n=1}^{\infty} \log^2 n \frac{Eg_n(X_n)}{g_n(a_n)} < \infty. \tag{3.3}$$

Finally, if condition (i) is satisfied, when $|x|\leq a_n$, we have $\frac{|x|}{a_n}\leq \frac{|x|^r}{a_n^r}$, for $0<r\leq 1$. By the fact that $\frac{x^r}{g_n(x)}$ $(0<r\leq 1)$ is a nondecreasing function of $x$ on the half-line $x>0$, then

$$\begin{aligned}
|EZ_n^{(1)}| &\leq E(|Z_n| I(|Z_n|\leq 1)) + EI(|Z_n|>1) \\
&\leq E\left(\frac{|X_n|^r}{a_n^r} I(|X_n|\leq a_n)\right) + E\left(\frac{g_n(X_n)}{g_n(a_n)} I(|X_n|>a_n)\right) \\
&\leq E\left(\frac{g_n(X_n)}{g_n(a_n)} I(|X_n|\leq a_n)\right) + E\left(\frac{g_n(X_n)}{g_n(a_n)} I(|X_n|>a_n)\right) \\
&= \frac{Eg_n(X_n)}{g_n(a_n)},
\end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty} |EZ_n^{(1)}| \leq \sum_{n=1}^{\infty} \frac{Eg_n(X_n)}{g_n(a_n)} < \infty.$$

If condition (ii) is satisfied, then by the facts that $EX_n = 0$ and $x/g_n(x)$ is a nonincreasing function of $x$ on the half-line $x>0$, we have

$$\begin{aligned} |EZ_n^{(1)}| &\leq E(|Z_n| I(|Z_n|>1)) + EI(|Z_n|>1) \\ &\leq E\left(\frac{g_n(X_n)}{g_n(a_n)} I(|X_n|>a_n)\right) + E\left(\frac{g_n(X_n)}{g_n(a_n)} I(|X_n|>a_n)\right) \\ &\leq 2\,\frac{Eg_n(X_n)}{g_n(a_n)}, \end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty} |EZ_n^{(1)}| \leq 2 \sum_{n=1}^{\infty} \frac{Eg_n(X_n)}{g_n(a_n)} < \infty.$$

Therefore, whether $g_n(x)$ satisfies condition (i) or condition (ii), we have

$$\sum_{n=1}^{\infty} |EZ_n^{(1)}| < \infty.$$

Consequently,

$$\sum_{n=1}^{\infty} EZ_n^{(1)} \text{ converges.} \tag{3.4}$$

By (3.2)-(3.4), we can get that $\sum_{n=1}^{\infty} \frac{X_n}{a_n}$ converges almost surely. □

If we take $g_n(x) = |x|^p$ in Theorem 3.1, we can get Corollary 3.1.

**Corollary 3.1.** *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables and $\{a_n, n \geq 1\}$ be a sequence of positive numbers with $a_n \uparrow \infty$. There exists some $0 < p \leq 2$ such that*

$$\sum_{n=1}^{\infty} \log^2 n \frac{E|X_n|^p}{a_n^p} < \infty.$$

*If $1 < p \leq 2$, we further assume that $EX_n = 0$. Then*

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} X_i = 0, \quad a.s. \tag{3.5}$$

If taking $a_n(x) = n^{1/p}$ in Corollary 3.1, then we can obtain Corollary 3.2.

**Corollary 3.2.** *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables. There exist some $0 < p \leq 2$ and $\delta > 0$ such that*

$$E|X_n|^p \leq C(logn)^{-3-\delta}.$$

*If $1 < p \leq 2$, we further assume that $EX_n = 0$. Then*

$$\lim_{n\to\infty} n^{-1/p} \sum_{i=1}^{n} X_i = 0, \quad a.s. \tag{3.6}$$

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