Spectral Method for the Black-Scholes Model of American Options Valuation

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Abstract. In this paper, we devote ourselves to the research of numerical methods for American option pricing problems under the Black-Scholes model. The optimal exercise boundary which satisfies a nonlinear Volterra integral equation is resolved by a high-order collocation method based on graded meshes. For the other spatial domain boundary, an artificial boundary condition is applied to the pricing problem for the effective truncation of the semi-infinite domain. Then, the front-fixing and stretching transformations are employed to change the truncated problem in an irregular domain into a one-dimensional parabolic problem in $[-1,1]$. The Chebyshev spectral method coupled with fourth-order Runge-Kutta method is proposed for the resulting parabolic problem related to the options. The stability of the semi-discrete numerical method is established for the parabolic problem transformed from the original model. Numerical experiments are conducted to verify the performance of the proposed methods and compare them with some existing methods.

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1 Introduction

Options as a kind of important financial derivatives have a wide range of applications. The option pricing problem, especially the American option pricing problem, attracts the attention of more and more financial practitioners. The distinctive feature of an American option is its early exercise privilege, that is, the holder is endowed with the additional right to exercise the option at any time prior to the date of expiration, which makes an American option worth more than its European counterpart. Because of its early exercise

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privilege, American option pricing problem exists an optimal exercise boundary [3,26], which makes American pricing problem a nonlinear problem, and no analytical solution exists comparing to that of European options.

The researches of American option pricing problems have been extensively developed in recent decades. Traditionally, there are two ways—the analytical method and the numerical method. For analytical results, we refer to [1,9,10,13,25,33]. They managed to present the solutions in a closed form depending on the optimal exercise boundary, then the option value was determined as long as the optimal exercise boundary was given. However it was not known actually in practice. For numerical aspects, it can be divided into two categories in general, one is based on Monte Carlo approach [11,28,30], the other is based on partial differential equation (PDE) approach [2,3,19,20,22]. We would like to further adopt the PDE approach in this paper, since Monte Carlo method requires demanding computational resource due to its slow convergence.

The Black-Scholes equation is one of the most effective PDE models [5,21]. And numerical methods are of popular use and frequently resorted for the Black-Scholes model among financial practitioners. For example, lattice tree methods, finite difference methods and finite element methods have been developed and extensively studied in recent decades. Cox, Ross and Rubinstein firstly introduced the binomial model to price American options in their seminal paper [15]. Later, Amin and Khanna showed the convergence of the binomial method in [4]. Finite difference methods have been developed and discussed for a long time for American option pricing problem [8,19]. One may refer to [22] for the convergence analysis. Recently, finite element methods [3,20] have attracted more interest in this field for its solid theoretical framework, in particularly its efficiency and variability. Interested readers may refer to [26] and the references therein for a complete survey. In this paper, we discuss the Chebyshev spectral method to the same problem, which turns to be efficient and comparable to finite element method.

There exist four main challenges for the numerical treatment of the American option pricing problem:

- The optimal exercise boundary is unknown, which satisfies a highly nonlinear equation, so it is not easy to get a fine resolution of optimal exercise boundary.

- For the other boundary, since we cannot adopt numerical methods directly to unbounded domain, how to truncate the unbounded domain and control the truncated errors are the key issues.

- The solving domain is an irregular domain, and the partition has to be reconstructed for each time order in order to resolve the option values exactly.

- The last one is how to choose an efficient numerical method to solve the problem fast and accurately.

About the first challenge, Cox [15,23] has proved that the optimal exercise boundary satisfied a nonlinear Volterra integral equation, and Ma et al. [29] have solved the same

nonlinear Volterra integral equation by a high-order collocation method and presented numerical results. We shall follow the idea of [29] to deal with the first difficulty. For the second challenge, Holmes and Yang have introduced an artificial boundary condition to truncate the unbounded problem related to American call options in [20], we will use this method to deal with the put options. For the third challenge, front-fixing transformation technique [19, 20, 26, 27, 34], a very effective method for free boundary problems, is applied to convert the irregular domain into a regular one. Then the truncated problem can be transformed into a one-dimensional parabolic problem in $[-1,1]$ by stretching transformation.

The numerical method for solving the one-dimensional parabolic problem in $[-1,1]$ is our main task. To our knowledge, there are no pervious efforts about the spectral method applied to the American option pricing problems. Here, the Chebyshev spectral method [16, 32] with fourth-order Runge-Kutta method [31] is adopted to solve the parabolic problem related to option values. Numerical simulations verify the effectiveness of our algorithms and appear to be more accurate and efficient than the finite element method. Then the problem can be solved quickly by the spectral method achieving the same accuracy.

The rest of the paper is organized as follows. In Section 2, we shall describe the Black-Scholes model for American put option, and present a high-order collocation method to solve the optimal exercise boundary. Artificial boundary condition, front-fixing and stretching transformations are employed to formulate a one-dimensional parabolic problem in $[-1,1]$ to replace the original problem in the unbounded domain In section 3. In section 4, the Chebyshev spectral method is applied to the one-dimensional parabolic problem in $[-1,1]$, and the stability of the method is also presented in this section. In Section 5, numerical simulations are implemented to test the performance of the proposed method and compare with some existing methods. Some concluded remark is collected in Section 6.

2   The optimal exercise boundary

In this section, we mainly introduce the optimal exercise boundary of American options. A derivation of this part can be found in [35]. For completeness, we outline the main process as follows. Assume that S and t are the underlying asset price and time respectively, then the put option value $P = P(S,t)$ satisfies the following free boundary problem

$$
\begin{cases}
P_t + \frac{\sigma^2}{2}S^2 P_{SS} + (r-d)S P_S - rP = 0, & B(t) < S < +\infty, \quad 0 \leq t < T, \\
P(S,T) = (K-S)^+, & B(T) \leq S < +\infty, \\
P(B(t),t) = K - B(t), & 0 \leq t \leq T, \\
P_S(B(t),t) = -1, & 0 \leq t \leq T, \\
\lim_{S \to +\infty} P(S,t) = 0, & 0 \leq t \leq T,
\end{cases}
\tag{2.1}
$$

where r, σ, d, K and T are the interest rate, volatility, dividend rate, exercise price and maturity date. $B(t)$ is the optimal exercise boundary and $\mathcal{F}^+ = \max(\mathcal{F},0)$. In [15], Cox, Ross, and Rubinstein proved that the optimal exercise boundary satisfied the following nonlinear Volterra integral equation [23,29]:

$$
\begin{aligned}
B(t) = K + B(t)e^{-d(T-t)}N\left(-\frac{\ln B(t)-\ln K+\beta_2(T-t)}{\sigma\sqrt{T-t}}\right) \\
- Ke^{-r(T-t)}N\left(-\frac{\ln B(t)-\ln K+\beta_1(T-t)}{\sigma\sqrt{T-t}}\right) \\
- Kr\int_t^T e^{-r(\eta-t)}\left[1-N\left(\frac{\ln B(t)-\ln B(\eta)+\beta_1(\eta-t)}{\sigma\sqrt{\eta-t}}\right)\right]d\eta \\
+ qB(t)\int_t^T e^{-d(\eta-t)}\left[1-N\left(\frac{\ln B(t)-\ln B(\eta)+\beta_2(\eta-t)}{\sigma\sqrt{\eta-t}}\right)\right]d\eta,
\end{aligned}
\tag{2.2}
$$

with

$$
B(T) = \min\left(K,\frac{rK}{d}\right), \quad N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{s^2}{2}}ds, \quad \beta_1 = r-d-\frac{\sigma^2}{2}, \quad \beta_2 = r-d+\frac{\sigma^2}{2}.
$$

In order to simplify the equation, let $t=T-\tau$, $y(\tau)=\ln\widetilde{B}(\tau)=\ln B(T-\tau)$, and combine with $1-N(x)=N(-x)$, we obtain

$$
\begin{aligned}
e^{y(\tau)} = K + e^{y(\tau)}e^{-d\tau}N\left(-\frac{y(\tau)-\ln K+\beta_2\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau}N\left(-\frac{y(\tau)-\ln K+\beta_1\tau}{\sigma\sqrt{\tau}}\right) \\
- Kr\int_0^\tau e^{-r(\tau-\eta)}N\left(-\frac{y(\tau)-y(\eta)+\beta_1(\tau-\eta)}{\sigma\sqrt{\tau-\eta}}\right)d\eta \\
+ de^{y(\tau)}\int_0^\tau e^{-d(\tau-\eta)}N\left(-\frac{y(\tau)-y(\eta)+\beta_2(\tau-\eta)}{\sigma\sqrt{\tau-\eta}}\right)d\eta.
\end{aligned}
\tag{2.3}
$$

Lemma 2.1. (*see* **[18,26]**) *The asymptotic expansion of optimal exercise boundary $\widetilde{B}(\tau)$ as $\tau\to 0^+$ has the following analytic representation*

$$
\frac{\frac{rK}{d}-\widetilde{B}(\tau)}{\frac{rK}{d}} \approx \frac{\sigma}{\sqrt{2}}\lambda\sqrt{\tau}, \qquad r<d;
$$

$$
\frac{K-\widetilde{B}(\tau)}{K} \approx \sqrt{2}\sigma\sqrt{\tau|\ln\tau|}, \quad r=d;
$$

$$
\frac{K-\widetilde{B}(\tau)}{K} \approx \sigma\sqrt{\tau|\ln\tau|}, \qquad r>d.
$$

Here, λ is a constant that satisfies the following transcendental equation

$$
\lambda^3 e^{\frac{\lambda^2}{4}}\int_\lambda^\infty e^{-\frac{x^2}{4}}dx = 2(2-\lambda^2).
$$

Because of the singularity of the derivative of $\widetilde{B}(\tau)$ at $\tau=0$ by Lemma 2.1, standard numerical methods (such as collocation method) on uniform mesh for (2.3) can not guarantee the convergence. Fortunately, there exists a powerful and popular tool, named the graded meshes [6,7] which confirms sufficiently accurate approximation $\widetilde{B}(\tau_i)$ to $\widetilde{B}(\tau)$ on some small initial interval $[0,\tau^*]$.

Next, we present a high-order collocation method on graded meshes for solving equation (2.3). Let $\mathcal{I}_\tau = \{\tau_m = T(\frac{m}{M})^2, \ m=0,1,\dots,M\}$ be the graded meshes on $[0,T]$ and set $k_m := (\tau_m,\tau_{m+1}], \ m=0,\dots,M-1$. We will be concerned with the collocation solution $\mathcal{Y}_\tau$ lying in the piecewise polynomial space

$$S_J^{(-1)}(\mathcal{I}_\tau) = \{v, v|_{k_m} \in \mathcal{P}_J(k_m), \ 0 \le m \le M-1\},$$

where the superscript (-1) in $S_J^{(-1)}(\mathcal{I}_\tau)$ means discontinuous and $\mathcal{P}_J$ denotes the set of polynomials of degree not exceeding J. The dimension of the space $S_J^{(-1)}(\mathcal{I}_\tau)$ equals to $M(J+1)$, and it is natural to choose the set of collocation points as

$$\mathcal{T}_\tau := \{\tau_{m,j} = \tau_m + c_j(\tau_{m+1}-\tau_m): \ 0 < c_1 < \cdots < c_{J+1} \le 1, m=0,\cdots,M-1\},$$

since its cardinality is also $M(J+1)$. Here, $\{c_j = \frac{j}{J+1}\}_{j=1}^{J+1}$ is a given set of collocation parameters in $(0,1]$.

The collocation method is given as: Find $\mathcal{Y}_\tau \in S_J^{(-1)}(\mathcal{I}_\tau)$ satisfying Eq. (2.3) exactly at the collocation points $\tau_{m,j} \in \mathcal{T}_\tau$, that is

$$
\begin{aligned}
e^{\mathcal{Y}_\tau(\tau_{m,j})} =& K + e^{\mathcal{Y}_\tau(\tau_{m,j})} e^{-d\tau_{m,j}} N\left(-\frac{\mathcal{Y}_\tau(\tau_{m,j})-\ln K+\beta_2\tau_{m,j}}{\sigma\sqrt{\tau_{m,j}}}\right) \\
&- Ke^{-r\tau_{m,j}} N\left(-\frac{\mathcal{Y}_\tau(\tau_{m,j})-\ln K+\beta_1\tau_{m,j}}{\sigma\sqrt{\tau_{m,j}}}\right) \\
&- Kr\int_0^{\tau_{m,j}} e^{-r(\tau_{m,j}-\eta)} N\left(-\frac{\mathcal{Y}_\tau(\tau_{m,j})-\mathcal{Y}_\tau(\eta)+\beta_1(\tau_{m,j}-\eta)}{\sigma\sqrt{\tau_{m,j}-\eta}}\right) d\eta \\
&+ de^{\mathcal{Y}_\tau(\tau_{m,j})}\int_0^{\tau_{m,j}} e^{-d(\tau_{m,j}-\eta)} N\left(-\frac{\mathcal{Y}_\tau(\tau_{m,j})-\mathcal{Y}_\tau(\eta)+\beta_2(\tau_{m,j}-\eta)}{\sigma\sqrt{\tau_{m,j}-\eta}}\right) d\eta.
\end{aligned}
\tag{2.4}
$$

Setting $\mathcal{Y}_{m,j} = \mathcal{Y}_\tau(\tau_m + c_j(\tau_{m+1}-\tau_m)) \ (j=1,\dots,J+1)$, we can express $\mathcal{Y}_\tau^m$ (the restriction of $\mathcal{Y}_\tau$ on interval k_m) by interpolation

$$\mathcal{Y}_\tau|_{k_m} = \mathcal{Y}_\tau^m(\tau) = \mathcal{Y}_\tau(\tau_m+s(\tau_{m+1}-\tau_m)) = \sum_{j=1}^{l+1} \mathcal{Y}_{m,j} L_j(s), \quad 0 < s \le 1, \tag{2.5}$$

with Lagrange interpolation polynomials

$$L_j(s) = \prod_{k=1, k\neq j}^{l+1} \frac{s-c_k}{c_j-c_k}, \quad 0 < s \le 1, \quad j=1,\cdots,J+1. \tag{2.6}$$

Therefore, the global collocation solution $\mathcal{Y}_\tau$ on $[0,T]$ is given by

$$\mathcal{Y}_\tau = \sum_{m=0}^{M-1} \Phi_m(\tau) \mathcal{Y}_\tau^m(\tau),$$

where $\Phi_m(\tau)$ is the characteristic function on k_m.

Substituting $\mathcal{Y}_\tau$ into (2.3), we can derive the following block nonlinear systems

$$\mathcal{F}_{m,j}(\mathcal{Y}_{m,1}, \mathcal{Y}_{m,2}, \cdots, \mathcal{Y}_{m,J+1}) = 0, \quad j = 1,2,\cdots,J+1, \tag{2.7}$$

for each $m = 0,1,\ldots,M-1$, where $\mathcal{F}_{m,j}$ are given by

$$\mathcal{F}_{m,j} = e^{\mathcal{Y}_{m,j}} - K - e^{\mathcal{Y}_{m,j} - d\tau_{m,j}} N(\varepsilon_2) + K e^{-r\tau_{m,j}} N(\varepsilon_1)$$
$$- d e^{\mathcal{Y}_{m,j}} \int_0^{\tau_{m,j}} e^{-d(\tau_{m,j}-\eta)} N(\eta_2) \mathrm{d}\eta + rK \int_0^{\tau_{m,j}} e^{-r(\tau_{m,j}-\eta)} N(\eta_1) \mathrm{d}\eta$$

with

$$\varepsilon_2 = -\frac{\mathcal{Y}_{m,j} - \ln K + \beta_2 \tau_{m,j}}{\sigma\sqrt{\tau_{m,j}}}, \qquad \varepsilon_1 = -\frac{\mathcal{Y}_{m,j} - \ln K + \beta_1 \tau_{m,j}}{\sigma\sqrt{\tau_{m,j}}},$$

$$\eta_2 = -\frac{\mathcal{Y}_{m,j} - \mathcal{Y}(\eta) + \beta_2(\tau_{m,j}-\eta)}{\sigma\sqrt{\tau_{m,j}-\eta}}, \qquad \eta_1 = -\frac{\mathcal{Y}_{m,j} - \mathcal{Y}(\eta) + \beta_1(\tau_{m,j}-\eta)}{\sigma\sqrt{\tau_{m,j}-\eta}}.$$

Here, we adopt simplified Newton's method to solve the nonlinear systems (2.7), where the Jacobian is calculated as follows:

$$\frac{\partial \mathcal{F}_{m,j}}{\partial \mathcal{Y}_{m,j}} \approx e^{\mathcal{Y}_{m,j}} - e^{\mathcal{Y}_{m,j} - d\tau_{m,j}} N(\varepsilon_2) + \frac{e^{\mathcal{Y}_{m,j} - d\tau_{m,j} - \frac{\varepsilon_2^2}{2}} - K e^{-r\tau_{m,j} - \frac{\varepsilon_1^2}{2}}}{\sigma\sqrt{2\pi\tau_{m,j}}}$$

$$- d e^{\mathcal{Y}_{m,j}} \int_0^{\tau_{m,j}} e^{-d(\tau_{m,j}-\eta)} N(\eta_2) \mathrm{d}\eta + d e^{\mathcal{Y}_{m,j}} \int_0^{\tau_m} \frac{e^{-d(\tau_{m,j}-\eta) - \frac{\eta_2^2}{2}}}{\sigma\sqrt{2\pi(\tau_{m,j}-\eta)}} \mathrm{d}\eta$$

$$- rK \int_0^{\tau_m} \frac{e^{-r(\tau_{m,j}-\eta) - \frac{\eta_1^2}{2}}}{\sigma\sqrt{2\pi(\tau_{m,j}-\eta)}} \mathrm{d}\eta,$$

$$\frac{\partial \mathcal{F}_{m,j}}{\partial \mathcal{Y}_{m,k}} \approx 0, \quad \text{for} \quad k \neq j.$$

Set $\mathcal{Y}^m = (\mathcal{Y}_{m,1}, \mathcal{Y}_{m,2}, \ldots, \mathcal{Y}_{m,J+1})^T$ and let $\mathcal{Z}_m^{(k)} = (\mathcal{Z}_{m,1}^{(k)}, \mathcal{Z}_{m,2}^{(k)}, \ldots, \mathcal{Z}_{m,J+1}^{(k)})^T$ be the m-th interval and k-th step iterative solution of simplified Newton's method, which is given by

$$\mathcal{Z}_{m,j}^{(k+1)} = \mathcal{Z}_{m,j}^{(k)} - \frac{\mathcal{F}_{m,j}(\mathcal{Z}_m^{(k)})}{\lambda_m^{(k)}}, \quad j = 1,2,\cdots,J+1, \quad m = 0,1,\cdots,M-1, \tag{2.8}$$

where

$$\lambda_m^{(k)} = \max_{1 \le j \le J+1} \frac{\partial \mathcal{F}_{m,j}}{\partial \mathcal{Y}_{m,j}}, \quad \mathcal{Z}_0^{(0)} = \mathcal{Y}^0 \text{ and } \mathcal{Z}_m^{(0)} = \mathcal{Z}_{m-1}^{(m_k)}, \quad m = 1, \cdots, M-1.$$

Here, m_k is the maximum integer index of the iterative solution by the simplified Newton's method under a given tolerance, that is $\|\mathcal{Z}_m^{(k+1)} - \mathcal{Z}_m^{(k)}\|_2 \le \text{Tol}$.

3   Pricing models via transformation and truncation

With the optimal exercise boundary computed in Section 2, American put option pricing problem (2.1) becomes a parabolic problem on a known unbounded domain. In this section, artificial boundary condition and front-fixing transformation will be discussed for the Black-Scholes equation to reduce the unbounded domain to the desired rectangular bounded computational domain.

3.1   Artificial boundary condition

In this subsection, we shall present an artificial boundary condition [14, 24], which can truncate the infinite domain problem into a compact one and ensure the accuracy of calculation.

Observing equation (2.1), we can find that it is a backward equation with variational coefficients. Using the standard variable transforms [26]

$$P(S,t) = Ke^{-\alpha y - \beta \tau} \phi(y,\tau), \quad T - t = \tau, \quad S = Ke^y, \tag{3.1}$$

where α and β are constants to be determined, the Black-Scholes equation (2.1) with Dirichlet boundary conditions can be rewritten as a forward diffusion problem with constant coefficients in some variable domain

$$\begin{cases} \phi_\tau - \gamma \phi_{yy} + \nu \phi_y + \mu \phi = 0, & b(\tau) < y < +\infty, \quad 0 < \tau \le T, \\ \phi(y,0) = g(y,0), & b(0) \le y < +\infty, \\ \phi(b(\tau),\tau) = g(b(\tau),\tau), & 0 \le \tau \le T, \\ \lim_{y \to +\infty} e^{-\alpha y - \beta \tau} \phi(y,\tau) = 0, & 0 \le \tau \le T, \end{cases} \tag{3.2}$$

where the constants, the right hand function and the free boundary after transformation read as follows

$$\begin{gathered} \gamma = \frac{\sigma^2}{2}, \quad \nu = \gamma(1+2\alpha) + d - r, \quad \mu = r + \alpha(r-d) - \gamma\alpha(1+\alpha) - \beta, \\ g(y,\tau) = e^{\alpha y + \beta \tau}(1 - e^y)^+, \quad b(\tau) = \ln(B(T-\tau)/K). \end{gathered} \tag{3.3}$$

In the following, we collect some important facts about the optimal exercise boundary before and after the change of variables.

Lemma 3.1 (Property of the optimal exercise boundary, see [23,26]). *The free boundary $B(t)$ is a nondecreasing and bounded function with*

$$B_\infty^P \leq B(t) \leq B_T^P,$$

where $B_\infty^P = KX$ is the price of Permanent American put options, and

$$X = \frac{d-r+\gamma-\sqrt{(d-r+\gamma)^2+4r\gamma}}{d-r-\gamma-\sqrt{(d-r+\gamma)^2+4r\gamma}}, \qquad B_T^p = B(T) = \min\left(\frac{rK}{d},K\right).$$

Let $B = \ln(B_\infty^P/K) = \ln(X)$, then we get $B \leq b(\tau) \leq 0$, $\tau \in [0,T]$.

Next, we introduce the far field estimate, which plays a pivotal role to show the effectiveness of numerically solving the option pricing Eq. (2.1) on some truncated domains. The first result is due to Holmes and Yang [20], who have shown the estimate for American call options. For other numerical methods to truncate the unbounded domain we refer to [3] and [19].

Lemma 3.2 (see [20]). *For a given positive number $\varepsilon \in (0,1)$, we have*

$$P(S,t) \leq \varepsilon, \quad \forall\, S \geq Ke^Y, \quad 0 \leq t \leq T,$$

where

$$Y = -2.5\gamma T\alpha_0 + 0.5\sqrt{25\gamma^2 T^2\alpha_0^2 - 20\gamma T\log(\varepsilon/\sqrt{5K})}, \quad \alpha_0 = \frac{r-d}{\sigma^2} - \frac{1}{2}.$$

Define

$$L := Y - \log(X), \qquad a(\tau) := b(\tau) + L \tag{3.4}$$

and due to Lemma 3.1 and 3.2, we can easily see that

$$P(S,t) \leq \varepsilon, \quad \forall\, S \geq Ke^{a(T-t)}, \ 0 \leq t \leq T. \tag{3.5}$$

Given some sufficiently small threshold ε, we can derive corresponding upper bound Y for S (or equivalently $a(\tau)$ for y) from (3.5). In such a way, the bounded variable domain approximating problem (3.2) can be reformulated by

$$\begin{cases} \psi_\tau - \gamma\psi_{yy} + \nu\psi_y + \mu\psi = 0, & b(\tau) < y < a(\tau), \quad 0 < \tau \leq T, \\ \psi(y,0) = g(y,0), & b(0) \leq y \leq a(0), \\ \psi(b(\tau),\tau) = g(b(\tau),\tau), & 0 \leq \tau \leq T, \\ \psi(a(\tau),\tau) = 0, & 0 \leq \tau \leq T. \end{cases} \tag{3.6}$$

Note that two parallel free boundaries are introduced in (3.6).

3.2 Front-fixing transformation

In this subsection, we will firstly introduce the front-fixing method, which can transform the variable domain problem (3.6) into a rectangular bounded domain problem. And then, by some standard transformations, we can get a heat equation with homogeneous boundary conditions in $[-1,1]$.

By using the front-fixing transforms [34]

$$x = y - b(\tau), \qquad \varphi(x,\tau) = \psi(x + b(\tau),\tau), \tag{3.7}$$

we can reformulate variable domain problem (3.6) into the following problem in a rectangular domain:

$$\begin{cases} \varphi_\tau - \gamma \varphi_{xx} + (\nu - b'(\tau))\varphi_x + \mu\varphi = 0, & 0 < x < L, \quad 0 < \tau \leq T, \\ \varphi(x,0) = g(b(0) + x, 0), & 0 \leq x \leq L, \\ \varphi(0,\tau) = g(b(\tau),\tau), & 0 \leq \tau \leq T, \\ \varphi(L,\tau) = 0, & 0 \leq \tau \leq T. \end{cases} \tag{3.8}$$

Next, using the standard variable transforms

$$z = \frac{2x}{L} - 1, \qquad u(z,\tau) = \varphi\left(\frac{(1+z)L}{2},\tau\right) - \frac{1-z}{2} g(b(\tau),\tau), \tag{3.9}$$

problem (3.8) can be rewritten as the following equations with the spatial variable in $[-1,1]$:

$$\begin{cases} u_\tau - \frac{4\gamma}{L^2} u_{zz} + \frac{2(\nu - b'(\tau))}{L} u_z + \mu u = f(z,\tau), & -1 < z < 1, \quad 0 < \tau \leq T, \\ u(z,0) = g(b(0) + \frac{(1+z)L}{2},0) - \frac{1-z}{2} g(b(0),0), & -1 \leq z \leq 1, \\ u(-1,\tau) = 0, & 0 \leq \tau \leq T, \\ u(1,\tau) = 0, & 0 \leq \tau \leq T, \end{cases} \tag{3.10}$$

where

$$f(z,\tau) = \frac{z-1}{2} \frac{\mathrm{d}g(b(\tau),\tau)}{\mathrm{d}\tau} + \left(\frac{\nu - b'(\tau)}{L} + \frac{(z-1)\mu}{2}\right) g(b(\tau),\tau).$$

We can see that problem (3.10) is a heat equation with homogeneous boundary conditions in $[-1,1]$, which can be solved with high accuracy by numerical methods.

4 Spectral methods

In this section, numerical methods will be presented for pricing American options. In spatial direction, the Chebyshev spectral method [32] will be used to approximate equation (3.10), and a fourth-order Runge-Kutta method [31] will be presented in temporal direction. In additional, the stability of the semi-discrete spectral numerical scheme is established.

4.1 Notations

Before designing the spectral method for equation (3.10), we introduce some definitions, and recall some basic results to be used in the sequel.

First of all, we review the Chebyshev polynomials and their special properties. $T_n(z)=\cos(n\cos^{-1}(z))$, $n=0,1,\ldots,N$, are the Chebyshev polynomials, which satisfy the following three-term recurrence relation

$$T_{n+1}(z)=2zT_n(z)-T_{n-1}(z), \ \ n\geq 1,$$
$$T_0(z)=1, \qquad T_1(z)=z, \tag{4.1}$$

and the special property related to their derivatives

$$2T_n(z)=\frac{1}{n+1}T'_{n+1}(z)-\frac{1}{n-1}T'_{n-1}(z), \ \ n\geq 2,$$
$$T_0(z)=T'_1(z), \quad 2T_1(z)=\frac{1}{2}T'_2(z). \tag{4.2}$$

Setting $\omega(z)=(1-z^2)^{-1/2}$ as the Chebyshev weight function, we introduce the weighted inner product are weighted norm, respectively

$$(u,v)_\omega=\int_{-1}^1 uv\omega\mathrm{d}z, \qquad \|u\|_\omega^2=(u,u)_\omega.$$

And the weighted Sobolev spaces are defined by

$$L_\omega^2:=\{v:[-1,1]\to R\mid \|v\|_\omega<\infty\},$$
$$H_\omega^k:=\{v\in L_\omega^2\mid D^\alpha v\in L_\omega^2, 0<\alpha\leq k\},$$
$$H_{\omega,0}^k:=\{v\in H_\omega^k\mid v(-1)=v(1)=0\}.$$

4.2 Chebyshev spectral method

In this subsection, we mainly describe the Chebyshev spectral method for solving the equation (3.10) and prove the stability of this method.

Firstly, we approximate the unknown function $u(z,\tau)$ by Chebyshev polynomials

$$U(z,\tau)=\sum_{n=0}^N a_n(\tau)T_n(z). \tag{4.3}$$

Then the Chebyshev spectral method for problem (3.10) is to find $U\in P_0^N$ such that

$$(U_\tau,V)_\omega+\mathcal{A}(U,V)=(f,V)_\omega, \quad \forall\, V\in P_0^N, \ a.e., \ 0<\tau\leq T, \tag{4.4}$$

where the bilinear form is defined as

$$\mathcal{A}(U,V):=\frac{4\gamma}{L^2}\left(U_z,\frac{(V\omega)_z}{\omega}\right)_\omega+\frac{2(\nu-b'(\tau))}{L}(U_z,V)_\omega+\mu(U,V)_\omega,$$

and $P_0^N\subset H_{\omega,0}^1$ is the space consisting of polynomials with degree not exceeding N and vanish at the boundary.

Theorem 4.1. *Assume $b \in H^1([0,T])$, then the Chebyshev spectral method (4.4) is stable, i.e.*

$$\|U(\cdot,\tau)\|_\omega \leq C\Big(\|U(\cdot,0)\|_\omega + \int_0^\tau \|f(\cdot,s)\|_\omega ds\Big).$$

where C is a constant depending on T.

Proof. Let $V = U$ in equation (4.4), we have

$$(U_\tau,U)_\omega + \frac{4\gamma}{L^2}\Big(U_z,\frac{(U\omega)_z}{\omega}\Big)_\omega + \frac{2(\nu - b'(\tau))}{L}(U_z,U)_\omega + \mu(U,U)_\omega = (f,U)_\omega. \qquad (4.5)$$

In [12], the following estimates

$$\Big(U_z,\frac{(U\omega)_z}{\omega}\Big)_\omega \geq \frac{1}{4}\|U_z\|_\omega^2, \qquad (4.6)$$

holds and there exists a constant $\delta > 0$ such that

$$\Big\|\frac{(U\omega)_z}{\omega}\Big\|_\omega \leq \delta\|U_z\|_\omega. \qquad (4.7)$$

Then we have the upper bound for the third term on the left hand side of (4.5) by (4.7) and Cauchy inequality with ε (Page 706 in [17]), as follows

$$\begin{aligned}
\Big|\frac{2(\nu - b'(\tau))}{L}(U_z,U)_\omega\Big| &= \Big|\frac{2(\nu - b'(\tau))}{L}\Big(U,\frac{(U\omega)_z}{\omega}\Big)_\omega\Big| \\
&\leq \Big|\frac{2(\nu - b'(\tau))}{L}\Big|\|U\|_\omega\Big\|\frac{(U\omega)_z}{\omega}\Big\|_\omega \leq \Big|\frac{2\delta(\nu - b'(\tau))}{L}\Big|\|U\|_\omega\|U_z\|_\omega \\
&\leq \frac{\delta^2(\nu - b'(\tau))^2}{\gamma}\|U\|_\omega^2 + \frac{\gamma}{L^2}\|U_z\|_\omega^2.
\end{aligned} \qquad (4.8)$$

Substituting (4.6) and (4.8) into (4.5), it obtains

$$\frac{1}{2}\frac{d\|U\|_\omega^2}{d\tau} \leq \frac{\delta^2(\nu - b'(\tau))^2}{\gamma}\|U\|_\omega^2 + |\mu|\|U\|_\omega^2 + \|f\|_\omega\|U\|_\omega.$$

For simplicity, we let $C(\tau) = \frac{\delta^2(\nu - b'(\tau))^2}{\gamma} + |\mu|$, the above inequality is simplified as

$$\frac{d\|U\|_\omega}{d\tau} \leq C(\tau)\|U\|_\omega + \|f\|_\omega.$$

From the assumption on function b, it indicates that $C(\tau)$ is integrable over $(0,T]$. Then using Gronwall inequality, we see that

$$\|U(\cdot,\tau)\|_\omega \leq e^{\int_0^\tau C(s)ds}\Big(\|U(\cdot,0)\|_\omega + \int_0^\tau \|f(\cdot,s)\|_\omega ds\Big),$$

which obtains the conclusion. $\square$

4.3 Implementation procedure

In this subsection, we mainly describe the implementation procedure of Chebyshev spectral method [31, 32] for solving equation (3.10).

Let z_n be the Chebyshev-Gauss-Lobatto points $z_n = \cos(\pi n / N)$, $0 \le n \le N$, and $\tau_m = mT/M$, $0 \le m \le M$ be the temporal partition grids. Notice that $\{a_n^m = a_n(\tau_m)\}$ can be expressed explicitly by $\{U^m(z_n) = U(x_n, \tau_m)\}$ from (4.3), that is

$$a_n^m = \frac{2}{Nc_n} \sum_{i=0}^{N} c_i^{-1} U^m(z_i) \cos(\pi i n / N), \qquad 0 \le n \le N, \tag{4.9}$$

where c_0, $c_N = 2$ and $c_n = 1$, for $0 < n < N$.

The first order derivative of the unknown function $u(z, \tau_m)$ can be approximated by

$$\frac{dU^m(z)}{dz} = \sum_{n=0}^{N-1} a_n^{(m,1)} T_n(z). \tag{4.10}$$

Using the property (4.2) of Chebyshev polynomials, we obtain

$$\begin{aligned}
\frac{dU^m(z)}{dz} &= \sum_{n=0}^{N-1} a_n^{(m,1)} T_n(z) \\
&= a_0^{(m,1)} T_1'(z) + \frac{1}{4} a_1^{(m,1)} T_2'(z) + \frac{1}{2} \sum_{n=2}^{N} a_n^{(m,1)} \left[\frac{1}{n+1} T_{n+1}'(z) - \frac{1}{n-1} T_{n-1}'(z) \right] \\
&= a_0^{(m,1)} T_1'(z) + \sum_{n=2}^{N} \frac{1}{2n} a_{n-1}^{(m,1)} T_n'(z) - \sum_{n=1}^{N-1} \frac{1}{2n} a_{n+1}^{(m,1)} T_n'(z) \\
&= \sum_{n=1}^{N} \frac{1}{2n} \left(c_{n-1} a_{n-1}^{(m,1)} - a_{n+1}^{(m,1)} \right) T_n'(z),
\end{aligned} \tag{4.11}$$

where $a_N^{(m,1)} = a_{N+1}^{(m,1)} = 0$. From Eq. (4.3), we know that

$$\frac{dU^m(z)}{dz} = \sum_{n=1}^{N} a_n^m T_n'(z). \tag{4.12}$$

By comparing the coefficients of (4.11) and (4.12), we obtain

$$c_n a_n^{(m,1)} = a_{n+2}^{(m,1)} + 2(n+1) a_{n+1}^m, \qquad n = N-1, \cdots, 0.$$

In a similar way, we get the second-order derivatives in the following form

$$\frac{d^2 U^m(z)}{dz^2} = \sum_{n=0}^{N-1} a_n^{(m,2)} T_n(z), \tag{4.13}$$

then we have

$$c_n a_n^{(m,2)} = a_{n+2}^{(m,2)} + 2(n+1)a_{n+1}^{(m,1)}, \quad n = N-2, \cdots, 0$$
$$a_{N+1}^{(m,2)} = a_N^{(m,2)} = a_{N-1}^{(m,2)} = 0.$$

Therefore, replacing the unknown function and its derivatives by their corresponding approximations (4.3), (4.11) and (4.13) in the equation (3.10) produces a ODE systems about $a_n(\tau)$. And the fourth-order Runge-Kutta (RK4) method [31] is utilized with equation (4.9) to solve the resulting ODEs problem as follows

$$\begin{cases} U^{m+1} = U^m + \frac{\Delta\tau}{6}(K_1 + 2K_2 + 2K_3 + K_4), \\ K_1 = F(U^m, z, \tau_m), \quad\quad\quad K_2 = F(U^m + \frac{\Delta\tau}{2}K_1, z, \tau_m + \frac{\Delta\tau}{2}), \\ K_3 = F(U^m + \frac{\Delta\tau}{2}K_2, z, \tau_m + \frac{\Delta\tau}{2}), \quad K_4 = F(U^m + \Delta\tau K_3, z, \tau_{m+1}), \end{cases} \quad (4.14)$$

where $U^m = (U^m(z_1), \dots, U^m(z_N))$ and

$$F(u, z, \tau) = \frac{4\gamma}{L^2}u_{zz} - \frac{2(\nu - b'(\tau))}{L}u_z - \mu u + f(z, \tau).$$

5 Numerical experiments

In this section, we present numerical simulations to verify the theoretic analysis in Section 2 and Section 4, and check the efficiency of the collocation method and spectral method for American option pricing problems.

Here, we consider the model of one-year ($T=1$) American put options, set $\sigma = 0.2$, $K = 10$ in equation (2.1), and vary parameters r and d for following three cases

- Case I: $r < d$ with $r = 0.005$ and $d = 0.01$;

- Case II: $r = d$ with $r = d = 0.01$;

- Case III: $r > d$ with $r = 0.05$ and $d = 0.01$.

Before verifying the efficiency of our constructive method, we firstly declare that $\varepsilon = 10^{-6}$ is chosen in Theorem 3.2, then take the solution obtained by binomial method [23] with 40000 points in temporal direction as the numerical approximation of the exact solution.

In the following examples, we show the efficiency of the collocation method for solving the optimal exercise boundary.

Example 5.1. The Eq. (2.2) which satisfied by the optimal exercise boundary with above three cases is considered. We take $J = 3$ and Tol $= 10^{-6}$ in Eq. (2.8), and let $M = 25$ in temporal direction. Then the approximation of optimal exercise boundary can be obtained by

$$B(t_{m,j}) = B(T - \tau_{m,j}) = e^{y(\tau_{m,j})}, \quad j = 1, 2, \cdots, J+1, \quad m = 0, 1, \cdots, M-1.$$
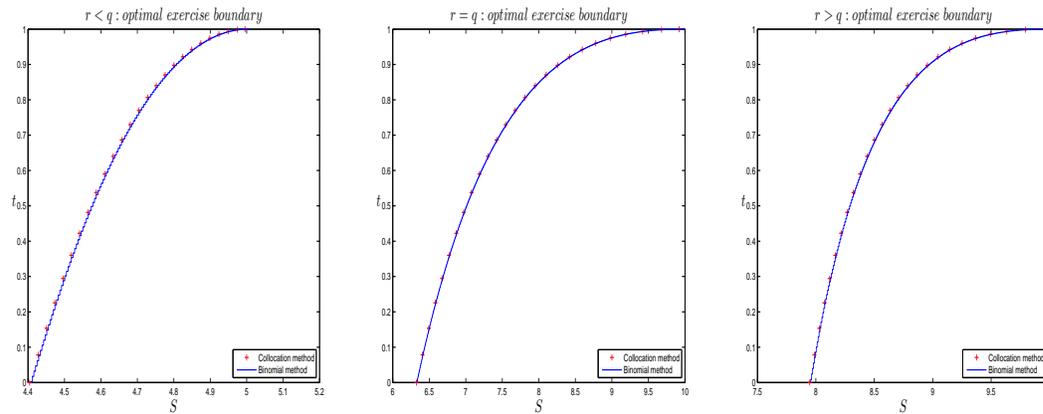
Figure 1: The optimal exercise boundary computed by collocation method with $M=25$ and the binomial method with 40000 points for three cases. Case I: $r<d$ (left); Case II: $r=d$ (middle); Case III: $r>d$ (right).

Table 1: L_2 error and time cost about optimal exercise boundary computed by collocation method (CM) with $M=25$ and binomial method (BM) with $M=10000$ for three cases.

Case	Method	M	Error/10^{-4}	Time/s
$r<d$	CM	25	3.579464	6.037314
	BM	10000	4.690713	63.972481
$r=d$	CM	25	6.811391	6.316974
	BM	10000	9.707295	48.909902
$r>d$	CM	25	6.566997	5.792954
	BM	10000	8.645543	25.168727

The results of the collocation method for these three cases are shown in Fig. 1 with notation "+", and the solid lines represent the optimal exercise boundary computed by the binomial method with 40000 points in temporal direction. From Fig. 1, we can see that the optimal exercise boundary computed by the collocation method approximates the exact solution well. We also give a comparison between the collocation method and binomial method in Table 5.1, from which, we can find that the collocation method is much faster than binomial method in the same accuracy. The numerical results confirm the general fact that binomial method needs more computational cost to guarantee the accuracy.

With the optimal exercise boundary obtained by the collocation method, the free boundary problem (2.1) is converted into a regular parabolic problem (3.10), which will be solved by Chebyshev spectral method.

Example 5.2. We consider to solve Eq. (3.10) by using spectral method for the three cases $r<d$, $r=d$ and $r>d$. In spatial direction approximations, $N=43$, $N=33$ and $N=15$ are taken as the highest degrees of Chebyshev polynomials, respectively. And in temporal direction, we choose $M=10000$, and use RK4 to solve (3.10). Then we can obtain the
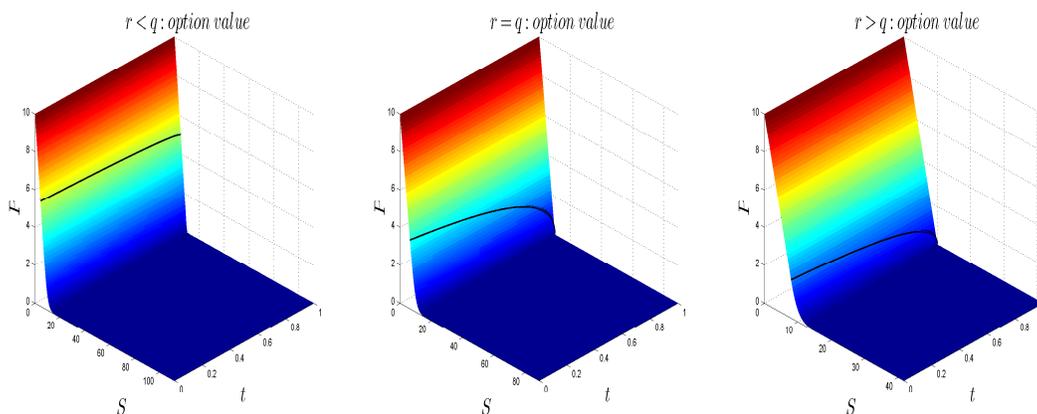
Figure 2: The option values computed by the Chebyshev spectral method. Case I: $r<d$ (left); Case II: $r=d$ (middle); Case III: $r>d$ (right).

approximations of the option values by

$$P(S_n,\tau_m) = Ke^{-\alpha(b(\tau_m)+\frac{L(1+z_n)}{2})-\beta\tau_m}\left(u(z_n,\tau_m)+\frac{1-z_n}{2}g(b(\tau_m),\tau_m)\right),$$

$$S_n = Ke^{b(\tau_m)+\frac{L(1+z_n)}{2}}, \quad n=0,1,\cdots,N+1, \ m=1,\cdots,M+1,$$

where $\alpha=\alpha_0$, $\beta=\gamma\alpha^2+r$.

Fig. 2 lists the results of the spectral method for these three cases. From the numerical simulations, we can see that the option values computed by the spectral method is a good approximation of the exact solution. The comparison between the spectral method and finite element method is also made in Table 5.2, from which, we can notice that the spectral method is faster than finite element method with the same accuracy, especially the case of $r>d$. The results about option values in Table 5.2 also show that the bigger ratio r/d is, the less time consuming and the more accurate option values are simulated by spectral method. This observation confirms with the truth that the spectral method works more effectively for smoother functions, in fact, with r/d being bigger, the initial value becomes smoother. In general, we can conclude that spectral method is an effective method for American option pricing problems.

6 Conclusions

In this paper, we introduce a collocation method based on the graded meshes for solving the nonlinear Volterra integral equation, which is derived for the optimal exercise boundary arising in pricing American option. With this boundary, we reformulated the Black-Scholes equation into a one-dimensional parabolic problem in $[-1,1]$ by the artificial boundary condition truncation, front-fixing and the stretching transformations. The

Table 2: L_2 error, L_∞ error and time cost about option values computed by spectral method (SM) and finite element method (FEM) at $t=0$ for three cases.

$r<d$	Method	N	L_2 Error$/10^{-2}$	L_∞ Error$/10^{-2}$	Time/s
	SM	43	1.413937	1.284660	12.634272
	FEM	800	1.414865	1.284730	12.739317
$r=d$	Method	N	L_2 Error$/10^{-3}$	L_∞ Error$/10^{-3}$	Time/s
	SM	33	5.173144	5.576551	9.060907
	FEM	600	5.175996	5.582139	10.933538
$r>d$	Method	N	L_2 Error$/10^{-4}$	L_∞ Error$/10^{-4}$	Time/s
	SM	15	1.809718	3.464621	4.363159
	FEM	400	1.901018	3.623287	8.393993

resulting problem for the valuation of American option is solved by spectral method coupled with Runge-Kutta method. We also give the stability analysis of spectral method, and numerical simulations are provided to verify the efficiency of these numerical algorithms.

Acknowledgments

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