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A Posteriori Error Estimation of Spectral and Spectral Element Methods for the Stokes/Darcy Coupled Problem

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Abstract. In this paper, we carry out an *a posteriori* error analysis of Legendre spectral approximations to the Stokes/Darcy coupled equations. The spectral approximations are based on a weak formulation of the coupled equations by using the Beavers-Joseph-Saffman interface condition. The main contribution of the paper consists of deriving a number of posteriori error indicators and their upper and lower bounds for the single domain case. An extension of the upper bounds to the multi-domain case in the spectral element framework is also given.

AMS subject classifications: 35Q35, 76D03

Key words: *A posteriori* error, Stokes/Darcy coupled equations, Spectral method, Spectral element method.

1 Introduction

The model of the Stokes equations coupled with the Darcy equations has been a subject of interest in a large variety of different fields, see, e.g. [9–12,14,16]. Recently, we have introduced a new formulation for the Stokes/Darcy coupled equations, subject respectively to the Beavers-Joseph-Saffman interface condition and an alternative matching interface condition [22]. Some spectral approximations are proposed and a priori error estimates are derived therein. In this paper we consider *a posteriori* error analysis for the above mentioned spectral approximations. The motivation of this consideration is that *a posteriori* error estimators are computable quantities in terms of the discrete solution, and can be used to measure the actual approximation errors without the knowledge of exact solutions. They are essential for designing algorithms with adaptive mesh refinement with minimal computational cost. On the other side, there are few work on *a posteriori* error analysis of the spectral method, and it is not clear if the adaptive strategy in the spectral method can be as efficient as in the finite element framework. Therefore this paper can be

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regarded as a step towards a better understanding about the adaptive spectral method, with particular attention to the Stokes/Darcy coupled equations.

Some *a posteriori* error analysis of the finite element approximation to the Stokes Darcy coupled equations have been carried in [1, 8]. The work [1] used the Lagrangian multiplier in their variational formulation, while [8] replaced the Darcy equations with a Poisson-like equation. In [1] the finite element subspaces consist of Bernardi-Raugel and Raviart-Thomas elements for the velocities, piecewise constants for the pressures, and continuous piecewise linear elements for the Lagrange multiplier defined on the interface. They have derived a residual-based *a posteriori* error estimate for the Stokes/Darcy coupled problem. The finite element spaces adopted in [8] are the Hood-Taylor element for the velocity and the pressure in the Stokes equation and conforming piecewise quadratic element for the Darcy pressure. The *a posteriori* error analysis was based on a suitable evaluation of the residual of the finite element solution.

In contrast to the lower order methods, *a posteriori* error estimation for high order methods such as spectral method is much less developed, although there exist a few papers on this topic for the elliptic problems (see, e.g., [3, 6, 13, 19]). The purpose of this work is to carry out an *a posteriori* error analysis for the spectral approximation of the Stokes/Darcy coupled equations. The analysis will be based on the formulation introduced in our previous work, which allows to extend the idea from [1,6,8,19] to derive the residual-based a posteriori error estimator in the framework of spectral element method.

The rest of the paper is organized as follows. In section 2 we briefly recall the formulation proposed in [22] for the Stokes/Darcy coupled problem. The core of the work is given by section 3 and section 4, where we develop the *a posteriori* error analysis. In section 3 we derive a residual-based *a posteriori* error estimate. The efficiency of this estimate is given in section 4. In section 5, we extend the results to the case of multi-domain in the framework of the spectral element method.

Throughout the paper we use the standard terminology for Sobolev spaces. In particular, If D is a bounded connected domain and $r \in \mathbb{R}$, then $|\cdot|_{r,D}$ and $\|\cdot\|_{r,D}$ stand for the semi-norm and norm in the Sobolev spaces $H^r(D)$, $[H^r(D)]^2$, and $[H^r(D)]^{2\times 2}$. In what follows, we will use c to mean a generic positive constant independent of any functions and of any discretization parameters. We also use the expression $A \lesssim B$ to mean that $A \leqslant cB$.

2 The Stokes/Darcy coupled problem

We are interested in the following Stokes/Darcy coupled equations in two dimensions:

$$\begin{cases} -\nabla\cdot(-p_sI+2\nu D(\mathbf{u}_s)) &= \mathbf{f}, \quad \text{in } \Omega_s, \\ \nabla\cdot\mathbf{u}_s &= 0, \quad \text{in } \Omega_s, \\ \mathbf{u}_s &= 0, \quad \text{on } \Gamma_s, \\ \mathbf{u}_d+\boldsymbol{\kappa}\nabla p_d &= 0, \quad \text{in } \Omega_d, \\ \nabla\cdot\mathbf{u}_d &= 0, \quad \text{in } \Omega_d, \\ \mathbf{u}_d\cdot\mathbf{n}_d &= 0, \quad \text{on } \Gamma_d, \end{cases} \tag{2.1}$$

where $\mathbf{u}_i=\mathbf{u}|_{\Omega_i}$ and $p_i=p|_{\Omega_i}, i=s,d$, with $\mathbf{u}$ and p denoting the velocity and pressure respectively,

$$D(\mathbf{u}_s)=\frac{1}{2}(\nabla\mathbf{u}_s+\nabla\mathbf{u}_s^T),$$

$\nu>0$ is the kinematic viscosity of the fluid, $\mathbf{f}$ is a given volumetric force, $\boldsymbol{\kappa}$ is defined by

$$\boldsymbol{\kappa}=\begin{pmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{pmatrix},$$

with κ_1 and κ_2 parameters associated to the kinematic viscosity of fluid, the permeability and porosity of the porous media in x and y direction respectively.

In (2.1), the computational domain Ω is assumed to be an open bounded subset of $\mathbb{R}^2$, with Lipschitz boundary $\partial\Omega$. Ω_s and Ω_d are respectively the fluid and porous media subdomains of Ω, such that $\Omega_s\cap\Omega_d=\emptyset$, $\bar{\Omega}_s\cup\bar{\Omega}_d=\bar{\Omega}$, $\Gamma_s=\partial\Omega_s\cap\partial\Omega$, $\Gamma_d=\partial\Omega_d\cap\partial\Omega$; see Fig. 1 for an example of such a domain. The unit normal vector $\mathbf{n}_s$ on the boundary Γ_s is chosen pointing outwards from Ω_s (similarly for the notation $\mathbf{n}_d$).

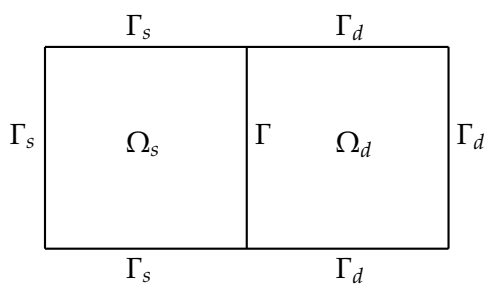

Figure 1 : The model computational domain of the coupled problem.

Mathematically, it is known that some suitable conditions on the interface $\Gamma:=\partial\Omega_s\cap\partial\Omega_d$ are needed to close Eq. (2.1). Here we consider the following matching conditions on the interface:

$$\begin{cases} \mathbf{u}_s\cdot\mathbf{n}_s=-\mathbf{u}_d\cdot\mathbf{n}_d \ \text{ on } \Gamma, \\ \big((p_sI-2\nu D(\mathbf{u}_s))\mathbf{n}_s\big)\cdot\mathbf{n}_s=p_d \ \text{ on } \Gamma, \\ \mathbf{u}_s\cdot\boldsymbol{\tau}=-\frac{\sqrt{\kappa}}{\alpha}2(D(\mathbf{u}_s)\mathbf{n}_s)\cdot\boldsymbol{\tau} \ \text{ on } \Gamma, \end{cases} \tag{2.2}$$

where $\tilde{\kappa}=\nu(\boldsymbol{\kappa}\boldsymbol{\tau})\cdot\boldsymbol{\tau}$ with $\boldsymbol{\tau}$ standing for the unit tangent vector on the boundary Γ, and α is a dimensionless constant which depends only on the structure of the porous media.

The first condition guarantees that the exchange of fluid between the two domains is conservative. The second one guarantees the balance of two driving forces. The third condition is usually called Beavers-Joseph-Saffman interface condition, which has been derived following the work of [16,17,21,22].

2.1 Weak formulation

To construct the weak formulation of (2.1)-(2.2), we need some basic notations. We use the notations L^2,H^1, H_0^1, and so on, to mean the usual Sobolev spaces. Let $(\mathbf{u},\mathbf{v})_\Omega=\int_\Omega \mathbf{uv}$, $\langle\mathbf{u},\mathbf{v}\rangle_\Gamma=\int_\Gamma \mathbf{uv}$, $\mathbb{L}^2(\Omega)=(L^2(\Omega))^2$, $\mathbb{H}^1(\Omega)=(H^1(\Omega))^2$, and $L_0^2(\Omega)=\{q:q\in L^2(\Omega),(q,1)_\Omega=0\}$. We introduce the following functional spaces:

$$\begin{aligned}
X_s&=\{\mathbf{v}_s:\mathbf{v}_s\in\mathbb{H}^1(\Omega_s),\mathbf{v}_s=0 \text{ on } \Gamma_s\},\\
X_d&=\{\mathbf{v}_d:\mathbf{v}_d\in\mathbb{L}^2(\Omega_d)\},\\
X&=X_s\times X_d,\\
M_s&=\{q_s:q_s\in L^2(\Omega_s)\},\\
M_d&=\{q_d:q_d\in H^1(\Omega_d)\},\\
M&=\{q\in L_0^2(\Omega):q=(q_s,q_d)\in M_s\times M_d\}.
\end{aligned}$$

The spaces X and M are respectively equipped with the norms

$$\|\mathbf{v}\|_X:=\|\mathbf{v}_s\|_{1,\Omega_s}+\|\mathbf{v}_d\|_{0,\Omega_d},\qquad \|q\|_M:=\|q_s\|_{0,\Omega_s}+\|q_d\|_{1,\Omega_d}.$$

Proceeding in the usual way (see, e.g. [22]), we find that the variational formulation of (2.1)-(2.2) reads: Find $(\mathbf{u},p)\in X\times M$ such that

$$\begin{aligned}
&a(\mathbf{u},\mathbf{v})+b(\mathbf{v},p)=F(\mathbf{v}), && \forall\mathbf{v}\in X,\\
&b(\mathbf{u},q)=0, && \forall q\in M,
\end{aligned}\tag{2.3}$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are two bilinear forms, defined respectively by

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}):=&\int_{\Omega_s}2\nu D(\mathbf{u}_s):D(\mathbf{v}_s)d\mathbf{x}+\int_{\Omega_d}(\boldsymbol{\kappa}^{-1}\mathbf{u}_d)\cdot\mathbf{v}_d d\mathbf{x}\\
&+\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}\int_\Gamma(\mathbf{u}_s\cdot\boldsymbol{\tau})(\mathbf{v}_s\cdot\boldsymbol{\tau})d\mathbf{x},\ \forall\mathbf{u},\mathbf{v}\in X\times X;\\
b(\mathbf{v},q):=&-\int_{\Omega_s}\nabla\cdot\mathbf{v}_s q_s d\mathbf{x}+\int_{\Omega_d}\mathbf{v}_d\cdot\nabla q_d d\mathbf{x}+\int_\Gamma(\mathbf{v}_s\cdot\mathbf{n}_s)q_d d\sigma,\ \forall\mathbf{v},q\in X\times M.
\end{aligned}\tag{2.4}$$

$F:X\to R$ is the linear functional:

$$F(\mathbf{v})=\int_{\Omega_s}\mathbf{f}\cdot\mathbf{v}_s d\mathbf{x},\ \ \forall\mathbf{v}\in X.$$

Theorem 2.1. The weak problem (2.3) admits a unique solution $(\mathbf{u},p)\in X\times M$. Moreover this solution satisfies:

$$\|\mathbf{u}\|_X+\|p\|_M\lesssim\|\mathbf{f}\|_{0,\Omega_s}.$$

Proof. The well-posedness of (2.3) follows from a straightforward application of the saddle point theory by verifying that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous, $a(\cdot,\cdot)$ is coercive, and $b(\cdot,\cdot)$ satisfies the following LBB compatibility condition [23]: there exists $\beta>0$ such that

$$\sup_{\mathbf{v}\in X}\frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_X}\geq\beta\|q\|_M,\qquad\forall q\in M.$$

□

Let

$$A((\mathbf{u},p),(\mathbf{v},q)):=a(\mathbf{u},\mathbf{v})+b(\mathbf{v},p)+b(\mathbf{u},q),\qquad\forall(\mathbf{u},p),(\mathbf{v},q)\in X\times M,$$

then we have

Lemma 2.1. There exists a constant $c>0$, such that

$$\inf_{(\mathbf{u},p)\in X\times M}\sup_{(\mathbf{v},q)\in X\times M}\frac{A((\mathbf{u},p),(\mathbf{v},q))}{(\|\mathbf{u}\|_X+\|p\|_M)(\|\mathbf{v}\|_X+\|q\|_M)}\geq c.$$

Proof. The proof is similar to Lemma 2.1 of [8]. □

Lemma 2.2. Let $(\mathbf{u},p)\in X\times M$ be the unique solution of (2.3), then

$$\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_s\cdot\boldsymbol{\tau})\boldsymbol{\tau}+p_d\mathbf{n}_s+(2\nu D(\mathbf{u}_s)-p_sI)\mathbf{n}_s=0\ \text{ on }\ \Gamma. \tag{2.5}$$

Proof. Taking $\mathbf{v}_d=0$ in the first equation of (2.3) , we get

$$\begin{aligned}&\int_{\Omega_s}2\nu D(\mathbf{u}_s):D(\mathbf{v}_s)d\mathbf{x}-\int_{\Omega_s}p_s\nabla\cdot\mathbf{v}_sd\mathbf{x}+\int_\Gamma(\mathbf{v}_s\cdot\mathbf{n}_s)p_ad\sigma+\int_\Gamma(\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}\mathbf{u}_s\cdot\boldsymbol{\tau})(\mathbf{v}_s\cdot\boldsymbol{\tau})d\sigma\\=&\int_{\Omega_s}\mathbf{f}\cdot\mathbf{v}_sd\mathbf{x}.\end{aligned}$$

Integrating by parts and rearranging the terms, we obtain that

$$\int_\Gamma\big(p_d\mathbf{n}_s+\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_s\cdot\boldsymbol{\tau})\boldsymbol{\tau}+(2\nu D(\mathbf{u}_s)-p_sI)\mathbf{n}_s\big)\cdot\mathbf{v}_sd\sigma=0,$$

from which we conclude (2.5). □

2.2 Spectral discretizations

We consider the spectral method to approximate coupled problems (2.3) . For ease of presentation, let's first assume that both Ω_s and Ω_d are rectangular domains. Define two discrete spaces:

$$X_N = X \cap (Q_N(\Omega_s)^2 \times Q_N(\Omega_d)^2), \qquad M_N = M \cap (Q_{N-2}(\Omega_s) \times Q_N(\Omega_d)),$$

where Q_N is the space of algebraic polynomials of degree less than or equal to N with respect to each single variable x or y. Then the spectral approximation to (2.3) reads: find $(\mathbf{u}_N, p_N) \in X_N \times M_N$, such that:

$$\begin{cases} a_N(\mathbf{u}_N, \mathbf{v}_N) + b_N(\mathbf{v}_N, p_N) = (\mathbf{f}, \mathbf{v}_N)^{GL}_{\Omega_s}, & \forall \mathbf{v}_N \in X_N, \\ b_N(\mathbf{u}_N, q_N) = 0, & \forall q_N \in M_N, \end{cases} \tag{2.6}$$

where $a_N(\cdot,\cdot)$, $b_N(\cdot,\cdot)$ are two bilinear forms, defined by:

$$\begin{aligned} a_N(\mathbf{u}_N, \mathbf{v}_N) &= (\boldsymbol{\kappa}^{-1}\mathbf{u}_{d,N}, \mathbf{v}_{d,N})^{GL}_{\Omega_d} + 2\nu(D(\mathbf{u}_{s,N}), D(\mathbf{v}_{s,N}))^{GL}_{\Omega_s} \\ &\quad + \frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}\langle \mathbf{u}_{s,N}\cdot\boldsymbol{\tau}, \mathbf{v}_{s,N}\cdot\boldsymbol{\tau}\rangle^{GL}_{\Gamma}, \\ b_N(\mathbf{v}_N, q_N) &= -(q_{s,N}, \nabla\cdot\mathbf{v}_{s,N})^{G}_{\Omega_s} + (\nabla q_{d,N}, \mathbf{v}_{d,N})^{GL}_{\Omega_d} \\ &\quad + \langle q_{d,N}, \mathbf{v}_{s,N}\cdot\mathbf{n}_s\rangle^{GL}_{\Gamma}, \end{aligned}$$

with $(\cdot,\cdot)^{GL}$, $(\cdot,\cdot)^{G}$ be evaluations of the continuous inner product $(\cdot,\cdot)$ by the Gauss-Lobatto and Gauss quadratures respectively.

Theorem 2.2. The spectral discrete problem (2.6) is well posed.

Proof. Similarly to the proof of Theorem 2.1, this theorem can be proved by directly verifying the continuity and coercivity of the form $a_N(\cdot,\cdot)$ in $X_N \times X_N$, the continuity of $b_N(\cdot,\cdot)$ in $X_N \times M_N$, and the following LBB condition (Lemma 3.1 of [22]): there exists a positive β_N, which may be dependent on N, such that

$$\inf_{q\in M_N}\sup_{\mathbf{v}\in X_N}\frac{b_N(\mathbf{v},q)}{\|\mathbf{v}\|_X\|q\|_M} \geq \beta_N.$$

□

Moreover, if we define the bilinear form $A_N(\cdot,\cdot)$ by

$$\begin{aligned} A_N((\mathbf{u}_N,p_N),(\mathbf{v}_N,q_N)) := a_N(\mathbf{u}_N,\mathbf{v}_N) + b_N(\mathbf{v}_N,p_N) + b_N(\mathbf{u}_N,q_N), \\ \forall (\mathbf{u}_N,p_N),(\mathbf{v}_N,q_N) \in X_N \times M_N. \end{aligned}$$

Then, as a consequence of the ellipticity property of the bilinear form $a_N(\cdot,\cdot)$ and the inf-sup condition on $b_N(\cdot,\cdot)$, we have the following result.

Lemma 2.3. It holds

$$\inf_{(\mathbf{u}_N,p_N)\in X_N\times M_N}\ \sup_{(\mathbf{v}_N,q_N)\in X_N\times M_N}\frac{A_N((\mathbf{u}_N,p_N),(\mathbf{v}_N,q_N))}{(\|\mathbf{u}_N\|_X+\|p_N\|_M)(\|\mathbf{v}_N\|_X+\|q_N\|_M)}\geq c_N,$$

where c_N is a positive constant depending on β_N.

3 A posteriori error estimation

This section is devoted to deriving *a posteriori* error estimation for the spectral approximation of the Stokes/Darcy coupled equations.

3.1 Upper bound estimation

Our analysis starts with the upper bound estimation for the error in terms of the error indicator. The analysis makes use of some known results on the polynomial approximation theory.

• *Polynomial approximation theory*

We first recall some well-known spectral projection operators, which will play an important role in the analysis of the error upper bound. Detailed proofs of the results presented in the following can be found in [2,5,20].

Let $\hat{\Omega}=(-1,1)^2$. For any $v\in L^2(\hat{\Omega})$, we define the orthogonal projection operator $\pi_N: L^2(\hat{\Omega})\to Q_N(\hat{\Omega})$ which satisfies

$$\int_{\hat{\Omega}}(v-\pi_N v)\phi d\mathbf{x}=0,\quad \forall\phi\in Q_N(\hat{\Omega}).$$

It is known (see [5]) that

$$\pi_N v(\mathbf{x})=\sum_{k,m=0}^{N}\hat{v}_{km}L_{km}(\mathbf{x}),\quad \text{with } \hat{v}_{km}=(k+\frac{1}{2})(m+\frac{1}{2})\int_{\hat{\Omega}}v(\mathbf{x})L_{km}(\mathbf{x})d\mathbf{x},$$

where $L_{km}(\mathbf{x})=L_k(x)L_m(y)$, L_k is the Legendre polynomial of degree k.
For any $v\in H^1(\hat{\Omega})$, $\pi_N^1: H^1(\hat{\Omega})\to Q_N(\hat{\Omega})$ is defined by

$$\int_{\hat{\Omega}}(\nabla(v-\pi_N^1 v)\cdot\nabla\phi+(v-\pi_N^1 v)\phi)d\mathbf{x}=0,\quad \forall\phi\in Q_N(\hat{\Omega}).$$

For any $v\in H_0^1(\hat{\Omega})$, $\pi_N^{1,0}: H_0^1(\hat{\Omega})\to Q_N^0(\hat{\Omega})$ is defined by

$$\int_{\hat{\Omega}}\nabla(v-\pi_N^{1,0}v)\cdot\nabla\phi d\mathbf{x}=0,\quad \forall\phi\in Q_N^0(\hat{\Omega}),$$

where $Q_N^0(\hat{\Omega})$ is the subspace of $Q_N(\hat{\Omega})$ of those polynomials that vanish on the boundary. For any $f\in C^0(\hat{\Omega})$, we denote by $I_N f\in Q_N(\hat{\Omega})$ the interpolation of f based on $(N+1)^2$ Legendre Gauss-Lobatto points.

Lemma 3.1. [2,5,7,15] For any $v\in H^m(\hat{\Omega}),m\geq 0$, we have

$$\|v-\pi_N v\|_{l,\hat{\Omega}}\lesssim N^{\sigma(l)-m}\|v\|_{m,\hat{\Omega}},\quad 0\leq l\leq m,$$

where $\sigma(l)=0$ if $l=0$, and $\sigma(l)=2l-\frac{1}{2}$ if $l>0$. Furthermore, for any $v\in H^1(\hat{\Omega})$, we have

$$\|v-\pi_N v\|_{0,\partial\hat{\Omega}}\lesssim N^{-\frac{1}{2}}\|v\|_{1,\hat{\Omega}}.$$

If $v\in H^m(\hat{\Omega})$, $m\geq 1$, then

$$\|v-\pi_N^1 v\|_{\mu,\hat{\Omega}}\lesssim N^{\mu-m}\|v\|_{m,\hat{\Omega}},\quad \mu=0,1.$$

If $v\in H_0^1(\hat{\Omega})\cap H^m(\hat{\Omega})$, $m\geq 1$, then

$$\|v-\pi_N^{1,0} v\|_{0,\partial\hat{\Omega}}\lesssim N^{-\frac{1}{2}}\|v\|_{1,\hat{\Omega}} \tag{3.1a}$$

$$\|v-\pi_N^{1,0} v\|_{\mu,\hat{\Omega}}\lesssim N^{\mu-m}\|v\|_{m,\hat{\Omega}},\quad \mu=0,1. \tag{3.1b}$$

Remark 3.1. The projection operator $\pi_N^{1,0}$ can also be defined in the space $H^1_{\Gamma_D}(\Omega)=\{v:v\in H^1(\Omega),v|_{\Gamma_D}=0,\Gamma_D\subset\partial\Omega\}$, and the estimates (3.1a) and (3.1b) still hold for $v\in H^1_{\Gamma_D}(\Omega)\cap H^m(\Omega)$.

Lemma 3.2. [4] If $f\in H^m(\hat{\Omega})$, $m\geq 2$, then

$$\|f-I_N f\|_{0,\hat{\Omega}}\lesssim N^{-m}\|f\|_{m,\hat{\Omega}}, \tag{3.2}$$

where I_N is the Lagrange interpolation operator.

- *Upper bound*

In the following theorem, we give the main result of the a posteriori error estimation for the Stokes/Darcy coupled equations.

Theorem 3.1. Assume that the data $\mathbf{f}$ belongs to $\mathbb{H}^r(\Omega_s)$, $r>1$. Let $(\mathbf{u},p)$ be the solution of (2.3), $(\mathbf{u}_N,p_N)$ be the solution of (2.6). Then it holds

$$\|\mathbf{u}-\mathbf{u}_N\|_X+\|p-p_N\|_M\lesssim(\eta+N^{-r}\|\mathbf{f}\|_{r,\Omega_s}), \tag{3.3}$$

where

$$\eta=\Big(\sum_{i=1}^{7}\eta_i^2\Big)^{1/2} \tag{3.4}$$

with

$$\eta_1=\Big(N^{-2}\int_{\Omega_s}(\mathbf{f}_N+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I))^2d\mathbf{x}\Big)^{1/2},\qquad \eta_2=\Big(\int_{\Omega_s}(\nabla\cdot\mathbf{u}_{s,N})^2d\mathbf{x}\Big)^{1/2},$$
$$\eta_3=\Big(N^{-2}\int_{\Omega_d}(\nabla\cdot\mathbf{u}_{d,N})^2d\mathbf{x}\Big)^{1/2},\qquad \eta_4=\Big(\int_{\Omega_d}(\boldsymbol{\kappa}^{-1}\mathbf{u}_{d,N}+\nabla p_{d,N})^2d\mathbf{x}\Big)^{1/2},$$
$$\eta_5=\Big(N^{-1}\int_{\Gamma}(\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_{s,N}\cdot\boldsymbol{\tau})\boldsymbol{\tau}+p_{d,N}\mathbf{n}_s+(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I)\mathbf{n}_s)^2d\sigma\Big)^{1/2},$$
$$\eta_6=\Big(N^{-1}\int_{\Gamma}(\mathbf{u}_{s,N}\cdot\mathbf{n}_s+\mathbf{u}_{d,N}\cdot\mathbf{n}_d)^2d\sigma\Big)^{1/2},\qquad \eta_7=\Big(N^{-1}\int_{\Gamma_d}(\mathbf{u}_{d,N}\cdot\mathbf{n}_d)^2d\sigma\Big)^{1/2},$$

and $\mathbf{f}_N=I_N\mathbf{f}$.

Proof. Let $e_{\mathbf{u}}=\mathbf{u}-\mathbf{u}_N$, $e_p=p-p_N$, it follows from Lemma 2.1 that there exists $(\mathbf{v},q)\in X\times M$ such that

$$\begin{aligned}&c(\|e_{\mathbf{u}}\|_X+\|e_p\|_M)(\|\mathbf{v}\|_X+\|q\|_M)\\ \le&A((e_{\mathbf{u}},e_p),(\mathbf{v},q))=a(e_{\mathbf{u}},\mathbf{v})+b(\mathbf{v},e_p)+b(e_{\mathbf{u}},q).\end{aligned}\tag{3.5}$$

Next, we estimate the terms on the right-hand side. For the last term, we have

$$b(e_{\mathbf{u}},q)=-b(\mathbf{u}_N,q).\tag{3.6}$$

Defining

$$\tilde{\pi}_Nq=\begin{cases}\pi_{N-2}q_s, & if\quad \mathbf{x}\in\Omega_s,\\ \pi_{N-1}q_d, & if\quad \mathbf{x}\in\Omega_d,\end{cases}$$

then $\tilde{\pi}_Nq\in M_N$, and

$$b_N(\mathbf{u}_N,\tilde{\pi}_Nq)=0.$$

Thus

$$b(\mathbf{u}_N,q)=b(\mathbf{u}_N,q-\tilde{\pi}_Nq)+(b-b_N)(\mathbf{u}_N,\tilde{\pi}_Nq).\tag{3.7}$$

Using (3.6), (3.7) and the exactitude of the Gauss-Lobatto quadratures, we obtain

$$\begin{aligned}b(e_{\mathbf{u}},q)=&\int_{\Omega_s}\nabla\cdot\mathbf{u}_{s,N}(q_s-\pi_{N-2}q_s)d\mathbf{x}-\int_{\Omega_d}\mathbf{u}_{d,N}\cdot\nabla(q_d-\pi_{N-1}q_d)d\mathbf{x}\\ &-\int_{\Gamma}(\mathbf{u}_{s,N}\cdot\mathbf{n}_s)(q_d-\pi_{N-1}q_d)d\sigma.\end{aligned}$$

Integrating by parts leads to

$$\begin{aligned}b(e_{\mathbf{u}},q)=&\int_{\Omega_s}\nabla\cdot\mathbf{u}_{s,N}(q_s-\pi_{N-2}q_s)d\mathbf{x}+\int_{\Omega_d}\nabla\cdot\mathbf{u}_{d,N}(q_d-\pi_{N-1}q_d)d\mathbf{x}\\ &-\int_{\Gamma}(\mathbf{u}_{s,N}\cdot\mathbf{n}_s+\mathbf{u}_{d,N}\cdot\mathbf{n}_d)(q_d-\pi_{N-1}q_d)d\sigma\\ &-\int_{\Gamma_d}(\mathbf{u}_{d,N}\cdot\mathbf{n}_d)(q_d-\pi_{N-1}q_d)d\sigma.\end{aligned}$$

By using Lemma 3.1, we get

$$b(e_{\mathbf{u}},q)\lesssim(\|\nabla\cdot\mathbf{u}_{s,N}\|_{0,\Omega_s}+N^{-1}\|\nabla\cdot\mathbf{u}_{d,N}\|_{0,\Omega_d} \\ +N^{-\frac{1}{2}}\|\mathbf{u}_{s,N}\cdot\mathbf{n}_s+\mathbf{u}_{d,N}\cdot\mathbf{n}_d\|_{0,\Gamma}+N^{-\frac{1}{2}}\|\mathbf{u}_{d,N}\cdot\mathbf{n}_d\|_{0,\Gamma_d})\|q\|_M. \tag{3.8}$$

It remains to evaluate the first two terms on the right-hand side of (3.5). We deduce from (2.3) and (2.6) that, for all $\mathbf{v}_N\in X_N$,

$$a(e_{\mathbf{u}},\mathbf{v})+b(\mathbf{v},e_p)=a(e_{\mathbf{u}},\mathbf{v}-\mathbf{v}_N)+b(\mathbf{v}-\mathbf{v}_N,e_p)+a_N(\mathbf{u}_N,\mathbf{v}_N)-a(\mathbf{u}_N,\mathbf{v}_N) \\ +b_N(\mathbf{v}_N,p_N)-b(\mathbf{v}_N,p_N)+(\mathbf{f},\mathbf{v}_N)-(\mathbf{f},\mathbf{v}_N)_N^{GL}.$$

By using the definition (2.4), we get, for all $\mathbf{v}_N\in X_N$,

$$\begin{aligned}&a(e_{\mathbf{u}},\mathbf{v}-\mathbf{v}_N)+b(\mathbf{v}-\mathbf{v}_N,e_p)\\ =&\int_{\Omega_s}(2\nu D(\mathbf{u}_s-\mathbf{u}_{s,N})):D(\mathbf{v}_s-\mathbf{v}_{s,N})d\mathbf{x}+\int_{\Omega_d}(\boldsymbol{\kappa}^{-1}(\mathbf{u}_d-\mathbf{u}_{d,N}))\cdot(\mathbf{v}_d-\mathbf{v}_{d,N})d\mathbf{x}\\ &+\int_\Gamma(\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_s-\mathbf{u}_{s,N})\cdot\boldsymbol{\tau})((\mathbf{v}_s-\mathbf{v}_{s,N})\cdot\boldsymbol{\tau})d\sigma-\int_{\Omega_s}(p_s-p_{s,N})\nabla\cdot(\mathbf{v}_s-\mathbf{v}_{s,N})d\mathbf{x}\\ &+\int_{\Omega_d}(\mathbf{v}_d-\mathbf{v}_{d,N})\cdot\nabla(p_d-p_{d,N})d\mathbf{x}+\int_\Gamma(p_d-p_{d,N})((\mathbf{v}_s-\mathbf{v}_{s,N})\cdot\mathbf{n}_s)d\sigma.\end{aligned}$$

Applying the integration by parts and (2.5) leads to

$$\begin{aligned}&a(e_{\mathbf{u}},\mathbf{v}-\mathbf{v}_N)+b(\mathbf{v}-\mathbf{v}_N,e_p)\\ =&\int_{\Omega_s}(\mathbf{f}_N+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I))\cdot(\mathbf{v}_s-\mathbf{v}_{s,N})d\mathbf{x}+\int_{\Omega_s}(\mathbf{f}-\mathbf{f}_N)\cdot(\mathbf{v}_s-\mathbf{v}_{s,N})d\mathbf{x}\\ &-\int_\Gamma(\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_{s,N}\cdot\boldsymbol{\tau})\boldsymbol{\tau}+p_{d,N}\mathbf{n}_s+(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I)\mathbf{n}_s)\cdot(\mathbf{v}_s-\mathbf{v}_{s,N})d\sigma\\ &-\int_{\Omega_d}(\boldsymbol{\kappa}^{-1}\mathbf{u}_{d,N}+\nabla p_{d,N})\cdot(\mathbf{v}_d-\mathbf{v}_{d,N})d\mathbf{x},\ \forall\mathbf{v}_N\in X_N.\end{aligned}\tag{3.9}$$

Moreover, it is known [20] that

$$|(\mathbf{f},\mathbf{v}_N)-(\mathbf{f},\mathbf{v}_N)_N^{GL}|\lesssim N^{-r}\|\mathbf{f}\|_{r,\Omega_s}\|\mathbf{v}_N\|_{1,\Omega_s},\qquad r\geq 2. \tag{3.10}$$

If we take

$$\mathbf{v}_N=\bar{\pi}_N\mathbf{v}:=\begin{cases}\pi_{N-1}^{1,0}\mathbf{v}_s, & if\quad \mathbf{x}\in\Omega_s,\\ \pi_{N-1}\mathbf{v}_d, & if\quad \mathbf{x}\in\Omega_d,\end{cases}$$

then $\mathbf{v}_N\in X_N$, and

$$a_N(\mathbf{u}_N,\mathbf{v}_N)-a(\mathbf{u}_N,\mathbf{v}_N)=0,\qquad b_N(\mathbf{v}_N,p_N)-b(\mathbf{v}_N,p_N)=0. \tag{3.11}$$

Combining (3.2), (3.5), (3.8), (3.9), (3.10), and (3.11), we deduce

$$
\begin{aligned}
&(\|e_{\mathbf{u}}\|_X+\|e_p\|_M)(\|\mathbf{v}\|_X+\|q\|_M)\\
\lesssim&\|\mathbf{f}_N+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I)\|_{0,\Omega_s}\|\mathbf{v}_s-\pi^{1,0}_{N-1}\mathbf{v}_s\|_{0,\Omega_s}\\
&+\|\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_{s,N}\cdot\boldsymbol{\tau})\boldsymbol{\tau}+p_{d,N}\mathbf{n}_s+(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I)\mathbf{n}_s\|_{0,\Gamma}\|\mathbf{v}_s-\pi^{1,0}_{N-1}\mathbf{v}_s\|_{0,\Gamma}\\
&+\|\boldsymbol{\kappa}^{-1}\mathbf{u}_{d,N}+\nabla p_{d,N}\|_{0,\Omega_d}\|\mathbf{v}_d-\pi_{N-1}\mathbf{v}_d\|_{0,\Omega_d}+N^{-r}\|\mathbf{f}\|_{r,\Omega_s}\|\pi^{1,0}_{N-1}\mathbf{v}_s\|_{1,\Omega_s}\\
&+\big(\|\nabla\cdot\mathbf{u}_{s,N}\|_{0,\Omega_s}+N^{-1}\|\nabla\cdot\mathbf{u}_{d,N}\|_{0,\Omega_d}+N^{-\frac{1}{2}}\|\mathbf{u}_{s,N}\cdot\mathbf{n}_s+\mathbf{u}_{d,N}\cdot\mathbf{n}_d\|_{0,\Gamma}\\
&+N^{-\frac{1}{2}}\|\mathbf{u}_{d,N}\cdot\mathbf{n}_d\|_{0,\Gamma_d}\big)\|q\|_M.
\end{aligned}
$$

Using Lemma 3.1 and some rearrangement, we obtain

$$
\begin{aligned}
&(\|e_{\mathbf{u}}\|_X+\|e_p\|_M)(\|\mathbf{v}\|_X+\|q\|_M)\\
\lesssim&\Big\{N^{-1}\|\mathbf{f}_N+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I)\|_{0,\Omega_s}+\|\nabla\cdot\mathbf{u}_{s,N}\|_{0,\Omega_s}\\
&+N^{-\frac{1}{2}}\Big\|\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_{s,N}\cdot\boldsymbol{\tau})\boldsymbol{\tau}+p_{d,N}\mathbf{n}_s+(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I)\mathbf{n}_s\Big\|_{0,\Gamma}\\
&+\|\boldsymbol{\kappa}^{-1}\mathbf{u}_{d,N}+\nabla p_{d,N}\|_{0,\Omega_d}+N^{-r}\|\mathbf{f}\|_{r,\Omega_s}+N^{-1}\|\nabla\cdot\mathbf{u}_{d,N}\|_{0,\Omega_d}+N^{-\frac{1}{2}}\|\mathbf{u}_{d,N}\cdot\mathbf{n}_d\|_{0,\Gamma_d}\\
&+N^{-\frac{1}{2}}\|\mathbf{u}_{s,N}\cdot\mathbf{n}_s+\mathbf{u}_{d,N}\cdot\mathbf{n}_d\|_{0,\Gamma}\Big\}(\|\mathbf{v}\|_X+\|q\|_M).
\end{aligned}
$$

The desired estimate (3.3) is then obtained by dividing both sides by $\|\mathbf{v}\|_X+\|q\|_M$. □

3.2 Lower bound estimation

Now it is interesting to see whether η is also a lower bound of $\|\mathbf{u}-\mathbf{u}_N\|_X+\|p-p_N\|_M$. Unfortunately, it is not true due to the poor inverse inequalities for polynomials. Nevertheless, inspired by the existing results on the *a posteriori* error estimation for hp-FEM of elliptic equations [19], we are able to derive a lower bound for the error $\|\mathbf{u}-\mathbf{u}_N\|_X+\|p-p_N\|_M$ in term of the modified indicator η_θ, $\theta\in[0,1]$, to be defined below.

The estimation of the lower bound makes use of some polynomial inverse inequalities in weighted Sobolev spaces, which we recall below.

- *Polynomial inverse estimates*

Let $\Lambda=(-1,1)$, we define the weight function $\Phi_\Lambda=(1-x^2)$. Then there holds the following inverse estimates.

Lemma 3.3. [3, 19] Let $-1<\alpha<\beta$, $\delta\in[0,1]$. Then the following estimates hold for all

polynomials $q_N \in Q_N(\Lambda)$,

$$\begin{aligned}
&\int_\Lambda q_N'(x)^2 \Phi_\Lambda(x)dx \lesssim N^2 \int_\Lambda q_N^2(x)dx,\\
&\int_\Lambda q_N^2(x)\Phi_\Lambda^\alpha(x)dx \lesssim N^{2(\beta-\alpha)} \int_\Lambda q_N^2(x)\Phi_\Lambda^\beta(x)dx,\\
&\int_\Lambda q_N'(x)^2 \Phi_\Lambda^{2\delta}(x)dx \lesssim N^{2(2-\delta)} \int_\Lambda q_N^2(x)\Phi_\Lambda^\delta(x)dx,\\
&\int_0^1 q_N'(x)^2 dx \lesssim N^2 \int_0^1 q_N^2(x)\Phi_\Lambda^{-1}(x)dx, \quad \text{if additionally } q_N(\pm 1)=0.
\end{aligned} \tag{3.12}$$

The generalization of the above inequalities to 2-dimensional case can be done by introducing the distance function as follows:

$$\Phi_{\hat\Omega}(\mathbf{x}) := dist(\mathbf{x}, \partial\hat\Omega).$$

Then we have some similar inverse inequalities in 2D, as stated in the following lemma.

Lemma 3.4. [19] Let $\alpha,\beta \in R$ satisfy $-1<\alpha<\beta$ and $\delta\in[0,1]$. Then it holds for all polynomials $q_N \in Q_N(\hat\Omega)$,

$$\int_{\hat\Omega} |\nabla q_N(\mathbf{x})|^2 \Phi_{\hat\Omega}(\mathbf{x})d\mathbf{x} \lesssim N^2 \int_{\hat\Omega} q_N^2(\mathbf{x})d\mathbf{x},$$

$$\int_{\hat\Omega} q_N^2(\mathbf{x})\Phi_{\hat\Omega}^\alpha(\mathbf{x})d\mathbf{x} \lesssim N^{2(\beta-\alpha)} \int_{\hat\Omega} q_N^2(\mathbf{x})\Phi_{\hat\Omega}^\beta(\mathbf{x})d\mathbf{x}, \tag{3.13a}$$

$$\int_{\hat\Omega} |\nabla q_N(\mathbf{x})|^2 \Phi_{\hat\Omega}^{2\delta}(\mathbf{x})d\mathbf{x} \lesssim N^{2(2-\delta)} \int_{\hat\Omega} q_N^2(\mathbf{x})\Phi_{\hat\Omega}^\delta(\mathbf{x})d\mathbf{x}. \tag{3.13b}$$

If additionally $q_N=0$ on $\partial\hat\Omega$, then

$$\int_{\hat\Omega} |\nabla q_N(\mathbf{x})|^2 d\mathbf{x} \lesssim N^2 \int_{\hat\Omega} q_N^2(\mathbf{x})\Phi_{\hat\Omega}^{-1}(\mathbf{x})d\mathbf{x}.$$

We will also need a known polynomial lifting result for the extension from an edge to the domain.

Lemma 3.5. [19] Let $\hat e$ be an edge of $\hat\Omega$, then for every univariate polynomial $q_{\hat e}$ of degree N on $\hat e$, $\varepsilon\in(0,1]$, and $\alpha\in(1/2,1]$, there exists an extension $q_{\hat\Omega}\in H^1(\hat\Omega)$ such that

$$\begin{aligned}
&q_{\hat\Omega}|_{\hat e} = q_{\hat e}\Phi_\Lambda^\alpha \quad \text{and} \quad q_{\hat\Omega}|_{\partial\hat\Omega\setminus\hat e}=0,\\
&\|q_{\hat\Omega}\|_{0,\hat\Omega}^2 \le c_\alpha \varepsilon \|q_{\hat e}\Phi_\Lambda^{\frac{\alpha}{2}}\|_{0,\hat e}^2,\\
&\|\nabla q_{\hat\Omega}\|_{0,\hat\Omega}^2 \le c_\alpha(\varepsilon N^{2(2-\alpha)}+\varepsilon^{-1})\|q_{\hat e}\Phi_\Lambda^{\frac{\alpha}{2}}\|_{0,\hat e}^2,
\end{aligned}$$

where c_α is a constant depending only on α.

• *Lower bound*

Let F_i $(i=s,d)$ be the mappings from $\hat{\Omega}$ to the element Ω_i, F_Γ and F_{Γ_d} be the mappings from Λ to Γ and Γ_d respectively. We define the following weight functions:

$$\Phi_s=\Phi_{\hat{\Omega}}\circ F_s^{-1},\quad \Phi_d=\Phi_{\hat{\Omega}}\circ F_d^{-1},\quad \Phi_\Gamma=\Phi_\Lambda\circ F_\Gamma^{-1},\quad \Phi_{\Gamma_d}=\Phi_\Lambda\circ F_{\Gamma_d}^{-1}$$

and the following error indicator $\eta_\theta,\theta\in[0,1]$:

$$\eta_\theta=\Big(\sum_{i=1}^{7}\eta_{\theta,i}^2\Big)^{1/2},$$

where

$$\begin{aligned}
\eta_{\theta,1}&=\Big(N^{-2}\int_{\Omega_s}\big(I_N\mathbf{f}+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I)\big)^2\Phi_s^\theta d\mathbf{x}\Big)^{1/2},\\
\eta_{\theta,2}&=\Big(\int_{\Omega_s}(\nabla\cdot\mathbf{u}_{s,N})^2d\mathbf{x}\Big)^{1/2},\\
\eta_{\theta,3}&=\Big(N^{-2}\int_{\Omega_d}(\nabla\cdot\mathbf{u}_{d,N})^2\Phi_d^\theta d\mathbf{x}\Big)^{1/2},\\
\eta_{\theta,4}&=\Big(\int_{\Omega_d}(\boldsymbol{\kappa}^{-1}\mathbf{u}_{d,N}+\nabla p_{d,N})^2d\mathbf{x}\Big)^{1/2},\\
\eta_{\theta,5}&=\Big(N^{-1}\int_{\Gamma}\Big(\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_{s,N}\cdot\boldsymbol{\tau})\boldsymbol{\tau}+p_{d,N}\mathbf{n}_s+\big(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I\big)\mathbf{n}_s\Big)^2\Phi_\Gamma^\theta d\sigma\Big)^{1/2},\\
\eta_{\theta,6}&=\Big(N^{-1}\int_{\Gamma}(\mathbf{u}_{s,N}\cdot\mathbf{n}_s+\mathbf{u}_{d,N}\cdot\mathbf{n}_d)^2\Phi_\Gamma^\theta d\sigma\Big)^{1/2},\\
\eta_{\theta,7}&=\Big(N^{-1}\int_{\Gamma_d}(\mathbf{u}_{d,N}\cdot\mathbf{n}_d)^2\Phi_{\Gamma_d}^\theta d\sigma\Big)^{1/2}.
\end{aligned}$$

Using the definition of the weight functions and the shape regularity, we can apply Lemma 3.4 to bound η_i by $\eta_{\theta,i}$. Indeed, by setting $\alpha=0$ and $\beta=\theta$ in (3.13a) of Lemma 3.4, we have $\eta_i\lesssim N^\theta\eta_{\theta,i}$.

The following lemmas are the main results about the lower bound estimates.

Lemma 3.6. Let $\theta\in[0,1]$, $\zeta>0$, then we have

$$\eta_{\theta,1}\lesssim N^{1-\theta}(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|p_s-p_{s,N}\|_{0,\Omega_s})+N^{\max\{-1,\zeta-\theta-\frac{1}{2}\}}\|\mathbf{f}-I_N\mathbf{f}\|_{0,\Omega_s}.$$

Proof. Let $\mathbf{v}=\big(I_N\mathbf{f}+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I)\big)\Phi_s^\theta$, then we have

$$\begin{aligned}
\|\mathbf{v}\Phi_s^{-\frac{\theta}{2}}\|_{0,\Omega_s}^2&=\int_{\Omega_s}\big(I_N\mathbf{f}+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I)\big)\cdot\mathbf{v}d\mathbf{x}\\
&=\int_{\Omega_s}\big(\mathbf{f}+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I)\big)\cdot\mathbf{v}d\mathbf{x}-\int_{\Omega_s}(\mathbf{f}-I_N\mathbf{f})\cdot\mathbf{v}d\mathbf{x}.
\end{aligned}$$

Using $-\nabla\cdot(-p_sI+2\nu D(\mathbf{u}_s))=\mathbf{f}$, we obtain

$$\|\mathbf{v}\Phi_s^{-\frac{\theta}{2}}\|_{0,\Omega_s}^2=\int_{\Omega_s}-\nabla\cdot\big(2\nu D(\mathbf{u}_s-\mathbf{u}_{s,N})+(p_s-p_{s,N})I\big)\cdot\mathbf{v}d\mathbf{x}-\int_{\Omega_s}(\mathbf{f}-I_N\mathbf{f})\cdot\mathbf{v}d\mathbf{x}.$$

Integrating by parts gives

$$\begin{aligned}\|\mathbf{v}\Phi_s^{-\frac{\theta}{2}}\|_{0,\Omega_s}^2=&\int_{\Omega_s}\big(2\nu D(\mathbf{u}_s-\mathbf{u}_{s,N})-(p_s-p_{s,N})I\big)\cdot\nabla\mathbf{v}d\mathbf{x}-\int_{\Omega_s}(\mathbf{f}-I_N\mathbf{f})\cdot\mathbf{v}d\mathbf{x}\\ \le&\big(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|p_s-p_{s,N}\|_{0,\Omega_s}\big)|\mathbf{v}|_{1,\Omega_s}+\|(\mathbf{f}-I_N\mathbf{f})\Phi_s^{\frac{\theta}{2}}\|_{0,\Omega_s}\|\mathbf{v}\Phi_s^{-\frac{\theta}{2}}\|_{0,\Omega_s}.\end{aligned}$$

• When $\theta>\frac{1}{2}$, we use the inverse estimates (3.13b) to estimate the H^1-semi norm of $\mathbf{v}$:

$$\begin{aligned}|\mathbf{v}|_{1,\Omega_s}^2\le&2\int_{\Omega_s}\Phi_s^{2\theta}(\nabla(I_N\mathbf{f}+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I)))^2d\mathbf{x}\\ &+2\int_{\Omega_s}(I_N\mathbf{f}+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I))^2(\nabla\Phi_s^{\theta})^2d\mathbf{x}\\ \lesssim&N^{2(2-\theta)}\int_{\Omega_s}\Phi_s^{\theta}(I_N\mathbf{f}+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I))^2d\mathbf{x}\\ &+\int_{\Omega_s}\Phi_s^{2\theta-2}(I_N\mathbf{f}+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I))^2d\mathbf{x}\\ \lesssim&N^{2(2-\theta)}\|\mathbf{v}\Phi_s^{-\frac{\theta}{2}}\|_{0,\Omega_s}^2.\end{aligned}$$

(Note that Lemma 3.4 is applicable since $2\theta-2>-1$ for $\theta>\frac{1}{2}$). Thus, we have

$$\|\mathbf{v}\Phi_s^{-\frac{\theta}{2}}\|_{0,\Omega_s}\lesssim N^{2-\theta}(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|p_s-p_{s,N}\|_{0,\Omega_s})+\|\mathbf{f}-I_N\mathbf{f}\|_{0,\Omega_s}.$$

(In the above estimation, we have used the fact that the weighted function Φ_s is bounded). By definition, we have $\eta_{\theta,1}=N^{-1}\|\mathbf{v}\Phi_s^{-\frac{\theta}{2}}\|_{0,\Omega_s}$. Therefore

$$\eta_{\theta,1}\lesssim N^{1-\theta}\big(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|p_s-p_{s,N}\|_{0,\Omega_s}\big)+N^{-1}\|\mathbf{f}-I_N\mathbf{f}\|_{0,\Omega_s}. \tag{3.14}$$

• When $0\le\theta\le\frac{1}{2}$, using first (3.13a) then (3.14), we get: for $\beta>\frac{1}{2}$,

$$\begin{aligned}\eta_{\theta,1}\lesssim&N^{\beta-\theta}\eta_{\beta,1}\\ \lesssim&N^{\beta-\theta}\{N^{1-\beta}(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|p_s-p_{s,N}\|_{0,\Omega_s})+N^{-1}\|\mathbf{f}-I_N\mathbf{f}\|_{0,\Omega_s}\}.\end{aligned}$$

By setting $\beta=\frac{1}{2}+\zeta$, $\zeta>0$, we obtain

$$\eta_{\theta,1}\lesssim N^{1-\theta}(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|p_s-p_{s,N}\|_{0,\Omega_s})+N^{\zeta-\theta-\frac{1}{2}}\|\mathbf{f}-I_N\mathbf{f}\|_{0,\Omega_s}. \tag{3.15}$$

Combining (3.14) and (3.15) completes the proof. □

Lemma 3.7. It holds $\eta_{\theta,2} \lesssim \|\mathbf{u}_s - \mathbf{u}_{s,N}\|_{1,\Omega_s}$.

Proof. Note that $\nabla\cdot\mathbf{u}_s = 0$ in Ω_s, then

$$\eta_{\theta,2}^2 = \int_{\Omega_s} (\nabla\cdot\mathbf{u}_{s,N})^2 d\mathbf{x} = \int_{\Omega_s} (\nabla\cdot\mathbf{u}_{s,N} - \nabla\cdot\mathbf{u}_s)^2 d\mathbf{x} \lesssim \|\mathbf{u}_s - \mathbf{u}_{s,N}\|_{1,\Omega_s}^2.$$

□

Lemma 3.8. It holds $\eta_{\theta,3} \lesssim N^{1-\theta}\|\mathbf{u}_d - \mathbf{u}_{d,N}\|_{0,\Omega_d}$.

Proof. Defining $v = (\nabla\cdot\mathbf{u}_{d,N})\Phi_d^\theta$, $\theta\in[0,1]$, and noting that $\nabla\cdot\mathbf{u}_d = 0$ in Ω_d, then we find

$$\|v\Phi_d^{-\frac{\theta}{2}}\|_{0,\Omega_d}^2 = \int_{\Omega_d} (\nabla\cdot\mathbf{u}_{d,N}) v d\mathbf{x} = \int_{\Omega_d} (\nabla\cdot\mathbf{u}_{d,N} - \nabla\cdot\mathbf{u}_d) v d\mathbf{x}.$$

Integrating by parts, and using Cauchy-Schwarz inequality, we deduce

$$\|v\Phi_d^{-\frac{\theta}{2}}\|_{0,\Omega_d}^2 = \int_{\Omega_d} (\mathbf{u}_{d,N} - \mathbf{u}_d)\cdot\nabla v d\mathbf{x} \le \|\mathbf{u}_{d,N} - \mathbf{u}_d\|_{0,\Omega_d} |v|_{1,\Omega_d}. \tag{3.16}$$

Now we want to bound the H^1-semi norm of v.
• When $\theta > \frac{1}{2}$, using the inverse estimate (3.13a) of Lemma 3.4 and the affine transformation from the reference element Ω to Ω_d, we can get:

$$\begin{aligned}
|v|_{1,\Omega_d}^2 &= \int_{\Omega_d} |\nabla(\nabla\cdot\mathbf{u}_{d,N}\Phi_d^\theta)|^2 d\mathbf{x} \\
&\le 2\int_{\Omega_d} |\nabla(\nabla\cdot\mathbf{u}_{d,N})|^2 \Phi_d^{2\theta} d\mathbf{x} + 2\int_{\Omega_d} (\nabla\cdot\mathbf{u}_{d,N})^2 |\nabla\Phi_d^\theta|^2 d\mathbf{x} \\
&\le 2N^{2(2-\theta)} \int_{\Omega_d} (\nabla\cdot\mathbf{u}_{d,N})^2 \Phi_d^\theta d\mathbf{x} + c\int_{\Omega_d} (\nabla\cdot\mathbf{u}_{d,N})^2 \Phi_d^{2(\theta-1)} d\mathbf{x} \\
&\lesssim N^{2(2-\theta)} \|v\Phi_d^{-\frac{\theta}{2}}\|_{0,\Omega_d}^2.
\end{aligned} \tag{3.17}$$

In the above estimation, in order to use Lemma 3.4, we let $\theta > \frac{1}{2}$, so that $2\theta - 2 > -1$.
As by definition $\eta_{\theta,3} = N^{-1}\|v\Phi_d^{-\frac{\theta}{2}}\|_{0,\Omega_d}$, inserting (3.17) into (3.16), we obtain

$$\eta_{\theta,3} \lesssim N^{1-\theta}\|\mathbf{u}_{d,N} - \mathbf{u}_d\|_{0,\Omega_d}.$$

• When $0 \le \theta \le \frac{1}{2}$, using (3.13a) and (3.14): for $\beta > \frac{1}{2}$, we have

$$\eta_{\theta,3} \lesssim N^{\beta-\theta}\eta_{\beta,3} \lesssim N^{1-\theta}\|\mathbf{u}_d - \mathbf{u}_{d,N}\|_{0,\Omega_d}.$$

This completes the proof.

□

Lemma 3.9. For the error indicator $\eta_{\theta,4}$, we have

$$\eta_{\theta,4} \lesssim \|\mathbf{u}_d - \mathbf{u}_{d,N}\|_{0,\Omega_d} + \|p_d - p_{d,N}\|_{1,\Omega_d}.$$

Proof. Note that $\boldsymbol{k}^{-1}\mathbf{u}_d + \nabla p_d = 0$ in Ω_d, then

$$\begin{aligned}\eta_{\theta,4}^2 &= \int_{\Omega_d} (\boldsymbol{\kappa}^{-1}\mathbf{u}_{d,N} + \nabla p_{d,N})^2 d\mathbf{x} \\ &= \int_{\Omega_d} (\boldsymbol{\kappa}^{-1}(\mathbf{u}_d - \mathbf{u}_{d,N}) + \nabla(p_d - p_{d,N}))^2 d\mathbf{x} \\ &\lesssim (\|\mathbf{u}_d - \mathbf{u}_{d,N}\|_{0,\Omega_d} + \|p_d - p_{d,N}\|_{1,\Omega_d})^2.\end{aligned}$$

□

Lemma 3.10. For any $\theta \in [0,1], \zeta > 0$, we have

$$\begin{aligned}\eta_{\theta,5} \lesssim\ & N^{\max\{\frac{1}{2}+\zeta-\theta,0\}} \big\{ N^{\frac{1}{2}} (\|\mathbf{u}_s - \mathbf{u}_{s,N}\|_{1,\Omega_s} + \|p_s - p_{s,N}\|_{0,\Omega_s} + \|p_d - p_{d,N}\|_{1,\Omega_d}) \\ & + (N^{-\frac{3}{2}} + N^{\zeta-2}) \|\mathbf{f} - I_N\mathbf{f}\|_{0,\Omega_s} \big\}.\end{aligned}$$

Proof. Let

$$\mathbf{g}_\theta := \big((\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}} \mathbf{u}_{s,N} \cdot \boldsymbol{\tau}) \boldsymbol{\tau} + p_{d,N}\mathbf{n}_s + (2\nu D(\mathbf{u}_{s,N}) - p_{s,N} I)\mathbf{n}_s \big) \Phi_\Gamma^{\frac{\theta}{2}}.$$

By using Lemma 3.5 and the affine transformations from the reference element $\hat{\Omega}$ to Ω_s and Ω_d respectively, we can construct a function $\mathbf{w}_\theta \in H_0^1(\Omega)$, such that $\mathbf{w}_\theta|_\Gamma = \mathbf{g}_\theta \Phi_\Gamma^{\frac{\theta}{2}}$, and for $\theta \in (\frac{1}{2}, 1]$, $\varepsilon \in (0,1]$:

$$\|\mathbf{w}_\theta\|_{0,\Omega_s} \le c_\theta \varepsilon^{\frac{1}{2}} \|\mathbf{g}_\theta\|_{0,\Gamma}, \quad \|\nabla \mathbf{w}_\theta\|_{0,\Omega_s} \le c_\theta (\varepsilon N^{2(2-\theta)} + \varepsilon^{-1})^{\frac{1}{2}} \|\mathbf{g}_\theta\|_{0,\Gamma}, \tag{3.18}$$

where c_θ depends on θ. Using (2.5), we obtain

$$\begin{aligned}\|\mathbf{g}_\theta\|_{0,\Gamma}^2 = \int_\Gamma \mathbf{w}_\theta \cdot (\Phi_\Gamma^{-\frac{\theta}{2}} \mathbf{g}_\theta) d\sigma &= \int_\Gamma \mathbf{w}_\theta \cdot ((2\nu D(\mathbf{u}_{s,N} - \mathbf{u}_s) - (p_{s,N} - p_s) I)\mathbf{n}_s) d\sigma \\ &\quad + \int_\Gamma \mathbf{w}_\theta \cdot \Big((\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}} (\mathbf{u}_{s,N} - \mathbf{u}_s) \cdot \boldsymbol{\tau}) \boldsymbol{\tau} + (p_{d,N} - p_d)\mathbf{n}_s \Big) d\sigma.\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}\|\mathbf{g}_\theta\|_{0,\Gamma}^2 &= \int_{\Omega_s} \nabla \mathbf{w}_\theta : (2\nu D(\mathbf{u}_{s,N} - \mathbf{u}_s) - (p_{s,N} - p_s) I) d\mathbf{x} \\ &\quad + \int_{\Omega_s} \mathbf{w}_\theta \cdot \big(\nabla \cdot (2\nu D(\mathbf{u}_{s,N} - \mathbf{u}_s) - (p_{s,N} - p_s) I) \big) d\mathbf{x} \\ &\quad + \int_\Gamma \mathbf{w}_\theta \cdot \Big((\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}} (\mathbf{u}_{s,N} - \mathbf{u}_s) \cdot \boldsymbol{\tau}) \boldsymbol{\tau} + (p_{d,N} - p_d)\mathbf{n}_s \Big) d\sigma.\end{aligned}$$

Plugging the first equation of (2.1) into the above equation gives

$$\begin{aligned}\|\mathbf{g}_\theta\|_{0,\Gamma}^2 = &\int_{\Omega_s} \nabla \mathbf{w}_\theta : (2\nu D(\mathbf{u}_{s,N}-\mathbf{u}_s)-(p_{s,N}-p_s)I)d\mathbf{x}\\ &+\int_{\Omega_s} \mathbf{w}_\theta\cdot(\mathbf{f}+\nabla\cdot(2\nu D(\mathbf{u}_{s,N})-p_{s,N}I))d\mathbf{x}\\ &+\int_\Gamma \mathbf{w}_\theta\cdot\Big((\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_{s,N}-\mathbf{u}_s)\cdot\boldsymbol{\tau})\boldsymbol{\tau}+(p_{d,N}-p_d)\mathbf{n}_s\Big)d\sigma.\end{aligned}$$

Then by using Cauchy-Schwarz inequality and Lemma 3.6 for $\theta=0$, we obtain

$$\begin{aligned}\|\mathbf{g}_\theta\|_{0,\Gamma}^2 \lesssim &\|\nabla\mathbf{w}_\theta\|_{0,\Omega_s}(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|p_s-p_{s,N}\|_{0,\Omega_s})\\ &+\|\mathbf{w}_\theta\|_{0,\Omega_s}\big(N(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|p_s-p_{s,N}\|_{0,\Omega_s})+(1+N^{\zeta-\frac{1}{2}})\|\mathbf{f}-I_N\mathbf{f}\|_{0,\Omega_s}\big)\\ &+\|\mathbf{w}_\theta\|_{1,\Omega_s}(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|p_d-p_{d,N}\|_{1,\Omega_d}),\end{aligned}$$

where $\zeta>0$ is from Lemma 3.6. In order to apply Lemma 3.5, we distinguish two cases.
• Case $\theta>\frac{1}{2}$. In this case Lemma 3.5 is directly applicable, which, together with the affine equivalence and (3.18) with $\varepsilon=N^{-2}$, yields

$$\begin{aligned}\|\mathbf{g}_\theta\|_{0,\Gamma} \lesssim &N(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|p_s-p_{s,N}\|_{0,\Omega_s}+\|p_d-p_{d,N}\|_{1,\Omega_d})\\ &+(N^{-1}+N^{\zeta-\frac{3}{2}})\|\mathbf{f}-I_N\mathbf{f}\|_{0,\Omega_s}.\end{aligned}$$

Then by using the relationship $\eta_{\theta,5}=N^{-\frac{1}{2}}\|\mathbf{g}_\theta\|_{0,\Gamma}$, we get:

$$\begin{aligned}\eta_{\theta,5} \lesssim &N^{\frac{1}{2}}(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|p_s-p_{s,N}\|_{0,\Omega_s}+\|p_d-p_{d,N}\|_{1,\Omega_d})\\ &+(N^{-\frac{3}{2}}+N^{\zeta-2})\|\mathbf{f}-I_N\mathbf{f}\|_{0,\Omega_s}.\end{aligned} \tag{3.19}$$

• Case $0\le\theta\le\frac{1}{2}$. Let $\beta=\frac{1}{2}+\zeta,\zeta>0$. According to (3.13a) we have $\eta_{\theta,5}\lesssim N^{\beta-\theta}\eta_{\beta,5}$. Applying (3.19) to $\eta_{\beta,5}$, we obtain

$$\begin{aligned}\eta_{\theta,5} \lesssim &N^{\frac{1}{2}+\zeta-\theta}\{N^{\frac{1}{2}}(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|p_s-p_{s,N}\|_{0,\Omega_s}+\|p_d-p_{d,N}\|_{1,\Omega_d})\\ &+(N^{-\frac{3}{2}}+N^{\zeta-2})\|\mathbf{f}-I_N\mathbf{f}\|_{0,\Omega_s}\}.\end{aligned} \tag{3.20}$$

Finally, combining (3.19) and (3.20) leads to the desired estimate. □

Similarly, we will be able to derive some estimates for $\eta_{\theta,6}$ and $\eta_{\theta,7}$. These estimates are stated in the following lemma without a detailed proof.

Lemma 3.11. For any $\theta\in[0,1],\zeta>0$, we have

$$\begin{aligned}&\eta_{\theta,6} \lesssim N^{\max\{1+\zeta-\theta,\frac{1}{2}\}}(\|\mathbf{u}_s-\mathbf{u}_{s,N}\|_{1,\Omega_s}+\|\mathbf{u}_d-\mathbf{u}_{d,N}\|_{0,\Omega_d}),\\ &\eta_{\theta,7} \lesssim N^{\max\{1+\zeta-\theta,\frac{1}{2}\}}\|\mathbf{u}_d-\mathbf{u}_{d,N}\|_{0,\Omega_d}.\end{aligned}$$

By collecting the above results, we are now in a position to give the main result of this section, i.e., the error lower bound of the numerical solution, in the following theorem.

Theorem 3.2. For any $\theta\in[0,1],\zeta>0$, it holds

$$\eta_\theta \lesssim N^{\max\{1+\zeta-\theta,\frac{1}{2}\}}(\|\mathbf{u}-\mathbf{u}_N\|_X+\|p-p_N\|_M)+N^{\max\{2\zeta-\theta-\frac{3}{2},\zeta-\theta-\frac{1}{2},-1\}}\|\mathbf{f}-I_N\mathbf{f}\|_{0,\Omega_s}.$$

Proof. It is a direct consequence of Lemma 3.6 to Lemma 3.11. □

Remark 3.2. Compared to the well known classical inverse inequality (see, e.g., [5]), the weighted inverse inequalities given in the lemma 3.3 have much weaker powers on the polynomial degree N. These better inequalities have allowed us to derive shaper lower bounds for the error indicators $\eta_{\theta,i}$. Obviously, the bigger is θ the shaper is the estimate. On the other side bigger θ means heavier weight. In general there does not exist optimal choice of θ and ζ. It depends on both the boundary distance function and solution regularity.

Remark 3.3. By using the inequality $\eta\lesssim N^\theta\eta_\theta$, we can easily derive the lower bound of the error with the indicator η.

4 Numerical tests

We now carry out some numerical tests to investigate the behavior of the numerical solution with respect to polynomial degree N. The main purpose is to verify the error indicators provided in the previous section.

The computational domain is $\Omega=(-1,3)\times(-1,1)$ with $\Omega_s=(-1,1)\times(-1,1)$, $\Omega_d=(1,3)\times(-1,1)$. By using the nodal basis for the discrete spaces X_N and M_N, the spectral approximation (2.6) results in a discrete saddle point problem. This problem is then split, by applying the Uzawa algorithm, into two positive definite symmetric systems: one for the pressure and another for the velocity. The pressure system is solved by an inner/outer preconditioned conjugate gradient iteration. We consider the following exact analytical solution:

$$\mathbf{u}=\begin{pmatrix}\cos(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y)\\ -\sin(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)\end{pmatrix},\quad p_s=\cos(\frac{\pi}{2}x),\quad p_d=\nu\frac{\pi}{2}\sin(\frac{\pi}{2}y)\sin(\frac{\pi}{2}x)+\cos(\frac{\pi}{2}x).$$

In Fig. 1, we plot the error estimators as functions of polynomial degree N. The result presented in this figure indicates the exponential decay rate of the errors, as in this semi-log representation the error curves are all straight lines. Note that the error indicator η_4 is vanishing up to the machine precision for all polynomial degree N. Indeed, from the spectral approximation (2.6) it can be easily verified that $\boldsymbol{\kappa}^{-1}\mathbf{u}_{s,N}+\nabla p_{d,N}$ is identically zero for any N.

In Fig. 2, we compare the lower error indicator η and the sum of the velocity error and pressure error versus the polynomial degree N. We observe that the two error curves are very close to each other and nearly have the same slope. This is an implication that the lower bound of the velocity and pressure error given in Theorem 3.2 is optimal for this test problem.

5 Spectral element method

In this section, we extend the above estimates to the spectral element method. The domain Ω is split into a number of subdomains as follows:

$$\bar{\Omega}_s=\bigcup_{k=1}^{K_s}\bar{\Omega}_s^k,\quad \Omega_s^k\cap\Omega_s^l=\emptyset,\qquad \text{if } k\neq l,$$
$$\bar{\Omega}_d=\bigcup_{k=1}^{K_d}\bar{\Omega}_d^k,\quad \Omega_d^k\cap\Omega_d^l=\emptyset,\qquad \text{if } k\neq l,$$
$$\Gamma=\partial\Omega_s\cap\partial\Omega_d=\bigcup_{k=1}^{K_\Gamma}\Gamma^k,\quad \text{with } \Gamma^k \text{ an entire side of some element } \Omega_s^k,$$

where K_s and K_d stand for the element numbers in the Stokes and Darcy domain, respectively, K_Γ is the number of the sides that Γ contains. Denote the triangulation by

$$\mathcal{T}=\mathcal{T}_s\cup\mathcal{T}_d,\quad \mathcal{T}_s=\{\bar{\Omega}_s^k\}_{k=1}^{K_s},\quad \mathcal{T}_d=\{\bar{\Omega}_d^k\}_{k=1}^{K_d}.$$

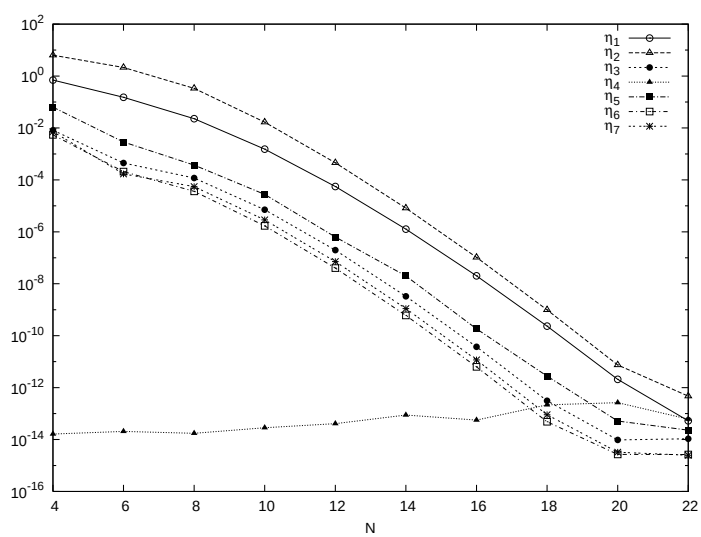

Figure 1: Error estimators versus the polynomial degree N .

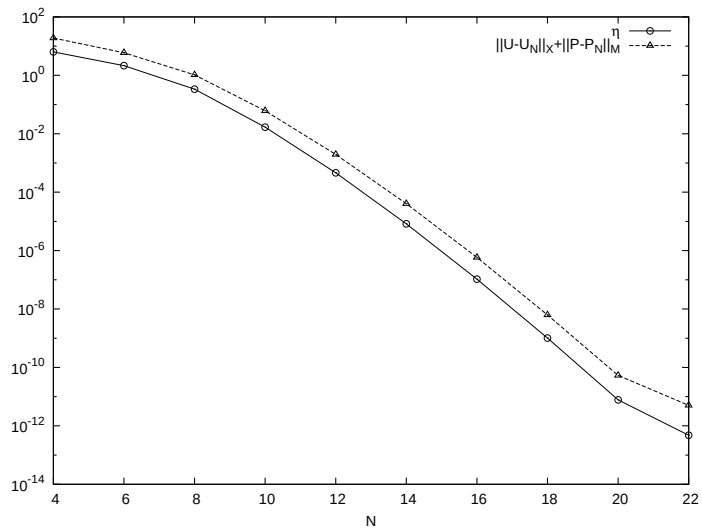

Figure 2: Total error indicator and the error of the velocity and pressure solution as functions of N.

We intend to derive a posteriori error estimate for the spectral element method based on the triangulation $\mathcal{T}$. We denote by $\hat{\Omega}$ the reference square, and suppose that each element T in $\mathcal{T}$ is the image of $\hat{\Omega}$ under an affine map $F_T:\hat{\Omega}\to T$.

We denote by $\mathcal{E}(T)$ the set of the edges of element T, and let

$$\mathcal{E}(\mathcal{T})=\bigcup_{T\in\mathcal{T}}\mathcal{E}(T),\quad \mathcal{E}(\Gamma)=\{e\in\mathcal{E}(\mathcal{T}):e\subset\Gamma\},\quad \mathcal{E}(\Omega_i):=\{e\in\mathcal{E}(\mathcal{T}):e\subset\Omega_i\},\quad i=s,d.$$

In what follows, $h_T:=\text{diam}\ T$, and h_e stands for the length of the edge e in $\mathcal{E}(\mathcal{T})$.

We define the piecewise polynomial spaces as follows:

$$\begin{aligned}
&P_{N,K}(\Omega)=\{u\in C(\bar{\Omega}):u\circ F_T\in Q_N(\hat{\Omega}),\forall T\in\mathcal{T}\},\\
&P_{N,K_f}(\Omega_s)=\{\varphi\in L^2(\Omega_s):\varphi\circ F_T\in Q_N(\hat{\Omega}),\forall T\in\mathcal{T}_s\},\\
&P_{N,K_d}(\Omega_d)=\{\varphi\in L^2(\Omega_d):\varphi\circ F_T\in Q_N(\hat{\Omega}),\forall T\in\mathcal{T}_d\}.
\end{aligned}$$

Then let

$$\begin{aligned}
&X_\delta=X\cap(P_{N,K_s}(\Omega_s)^2\times P_{N,K_d}(\Omega_d)^2),\\
&M_\delta=M\cap(P_{N-2,K_s}(\Omega_s)\times P_{N,K_d}(\Omega_d)).
\end{aligned}$$

The spectral element approximation to the problem (2.3) reads: find $(\mathbf{u}_\delta,p_\delta)\in X_\delta\times M_\delta$, such that

$$\begin{cases}
a_\delta(\mathbf{u}_\delta,\mathbf{v}_\delta)+b_\delta(\mathbf{v}_\delta,p_\delta)=\sum_{k=1}^{K_s}(\mathbf{f},\mathbf{v}_\delta)^{GL}_{\Omega_s^k}, & \forall\mathbf{v}_\delta\in X_\delta,\\
b_\delta(\mathbf{u}_\delta,q_\delta)=0, & \forall q_\delta\in M_\delta,
\end{cases}\tag{5.1}$$

where the bilinear forms $a_\delta(\cdot,\cdot)$ and $b_\delta(\cdot,\cdot)$ are defined respectively by:

$$a_\delta(\mathbf{u}_\delta,\mathbf{v}_\delta)=\sum_{k=1}^{K_d}(\boldsymbol{\kappa}^{-1}\mathbf{u}_{d,\delta},\mathbf{v}_{d,\delta})^{GL}_{\Omega_d^k}+\sum_{k=1}^{K_s}2\nu(D(\mathbf{u}_{s,\delta}),D(\mathbf{v}_{s,\delta}))^{GL}_{\Omega_s^k}+\sum_{k=1}^{K_\Gamma}\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}\langle\mathbf{u}_{s,\delta}\cdot\boldsymbol{\tau},\mathbf{v}_{s,\delta}\cdot\boldsymbol{\tau}\rangle^{GL}_{\Gamma^k},$$

$$b_\delta(\mathbf{v}_\delta,q_\delta)=-\sum_{k=1}^{K_s}(q_{s,\delta},\nabla\cdot\mathbf{v}_{s,\delta})^{G}_{\Omega_s^k}+\sum_{k=1}^{K_d}(\nabla q_{d,\delta},\mathbf{v}_{d,\delta})^{GL}_{\Omega_d^k}+\sum_{k=1}^{K_\Gamma}\langle q_{d,\delta},\mathbf{v}_{s,\delta}\cdot\mathbf{n}_s\rangle^{GL}_{\Gamma^k}.$$

We first present a lemma which is simplification of the well-known results for Clément-type and Scott-Zhang type quasi interpolation operators [18] in the spectral element context.

Lemma 5.1. There exist a bounded linear operator $I_\delta:L^1(\Omega)\to P_{N,K}(\Omega)$, a bounded linear operator $I_\delta^0:H^1_{\Gamma_D}(\hat{\Omega})\cap C^0(\Omega)\to P_{N,K}(\Omega)\cap H^1_{\Gamma_D}(\Omega)$, such that for all $u\in H^1(\Omega)$, all elements $T\in\mathcal{T}$, and all edges $e\in\mathcal{E}(\mathcal{T})$, it holds

$$\begin{aligned}
&\|u-I_\delta u\|_{L^2(T)}+\frac{h_T}{N}\|\nabla(u-I_\delta u)\|_{L^2(T)}\lesssim\frac{h_T}{N}\|\nabla u\|_{L^2(\Omega_T)},\\
&\|u-I_\delta u\|_{L^2(e)}\lesssim\sqrt{\frac{h_e}{N}}\|\nabla u\|_{L^2(\Omega_e)},\\
&\|u-I_\delta^0 u\|_{L^2(T)}+\frac{h_T}{N}\|\nabla(u-I_\delta^0 u)\|_{L^2(T)}\lesssim\frac{h_T}{N}\|\nabla u\|_{L^2(\Omega_T)},\\
&\|u-I_\delta^0 u\|_{L^2(e)}\lesssim\sqrt{\frac{h_e}{N}}\|\nabla u\|_{L^2(\Omega_e)},
\end{aligned}$$

where Ω_T, Ω_e are patches covering T and e with a few layers, respectively.

Next we derive an *a posteriori* error estimate for the solution of problem (5.1). To this end, we define for each $T\in\mathcal{T}_s$ the *a posteriori* error indicator

$$\begin{aligned}
\eta_{s,T}^2=&\frac{h_T^2}{N^2}\|\mathbf{f}+\nabla\cdot(2\nu D(\mathbf{u}_{s,\delta}))-\nabla p_{s,\delta}\|_{0,T}^2+\|\nabla\cdot\mathbf{u}_{s,\delta}\|_{0,T}^2\\
&+\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}(\Omega_s)}\frac{h_e}{N}\|[2\nu D(\mathbf{u}_{s,\delta})\mathbf{n}_s-p_{s,\delta}\mathbf{n}_s]\|_{0,e}^2\\
&+\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}(\Gamma)}\frac{h_e}{N}\|2\nu D(\mathbf{u}_{s,\delta})\mathbf{n}_s-p_{d,\delta}\mathbf{n}_d+p_{s,\delta}\mathbf{n}_s+\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_{s,\delta}\cdot\boldsymbol{\tau})\boldsymbol{\tau}\|_{0,e}^2,
\end{aligned}$$

where $[\varphi]$ denotes the jump of the function φ across the edge. Similarly, for each $T\in\mathcal{T}_d$, we set

$$\begin{aligned}
\eta_{d,T}^2=&\|\boldsymbol{\kappa}^{-1}\mathbf{u}_{d,\delta}+\nabla p_{d,\delta}\|_{0,T}^2+\frac{h_T^2}{N^2}\|\nabla\cdot\mathbf{u}_{d,\delta}\|_{0,T}^2+\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}(\Omega_d)}\frac{h_e}{N}\|[\mathbf{u}_{d,\delta}\cdot\mathbf{n}_d]\|_{0,e}^2\\
&+\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}(\Gamma)}\frac{h_e}{N}\|\mathbf{u}_{s,\delta}\cdot\mathbf{n}_s+\mathbf{u}_{d,\delta}\cdot\mathbf{n}_d\|_{0,e}^2.
\end{aligned}$$

Theorem 5.1. Assume that the data $\mathbf{f}\in\mathbb{H}^r(\Omega_s)$, $r>1$. Let $(\mathbf{u},\mathrm{p})$ and $(\mathbf{u}_\delta,p_\delta)$ be the solutions of (2.3) and (5.1) respectively. Then we have

$$\|\mathbf{u}-\mathbf{u}_\delta\|_X+\|p-p_\delta\|_M\lesssim\eta+\sum_{T\in\mathcal{T}_s}h_T^{\min\{r,N+1\}}N^{-r}\|\mathbf{f}\|_{r,\Omega_s}, \tag{5.2}$$

where $\eta:=\{\sum_{T\in\mathcal{T}_s}\eta_{s,T}^2+\sum_{T\in\mathcal{T}_d}\eta_{d,T}^2\}^{\frac{1}{2}}$.

Proof. Similar to the single domain spectral method in Theorem 3.1, we denote $e_{\mathbf{u}}=\mathbf{u}-\mathbf{u}_\delta$, $e_p=p-p_\delta$, then there exists $\mathbf{v}\in X,q\in M$ such that

$$\begin{aligned}&(\|e_{\mathbf{u}}\|_X+\|e_p\|_M)(\|\mathbf{v}\|_X+\|q\|_M)\\ \lesssim&A((e_{\mathbf{u}},e_p);(\mathbf{v},q))=a(e_{\mathbf{u}},\mathbf{v})+b(\mathbf{v},e_p)+b(e_{\mathbf{u}},q).\end{aligned} \tag{5.3}$$

We estimate the terms on the right sides. Setting

$$\tilde{I}\mathbf{v}=\begin{cases} I_{\delta-1}^0\mathbf{v}_s, & if \quad \mathbf{x}\in\Omega_s,\\ I_{\delta-1}\mathbf{v}_d, & if \quad \mathbf{x}\in\Omega_d,\end{cases}\qquad \tilde{I}q=\begin{cases} I_{\delta-2}q_s, & if \quad \mathbf{x}\in\Omega_s,\\ I_{\delta-1}q_d, & if \quad \mathbf{x}\in\Omega_d,\end{cases}$$

where $I_{\delta-1}q:C^0(\Omega)\to P_{N-1,K}(\Omega)$, then $\tilde{I}\mathbf{v}\in X_\delta$, $\tilde{I}q\in M_\delta$.

Now we turn to estimate the terms on the right-hand side of (5.3).

For the first two terms, we have, for all $\mathbf{v}_\delta\in X_\delta$,

$$\begin{aligned}a(e_{\mathbf{u}},\mathbf{v})+b(\mathbf{v},e_p)=&a(e_{\mathbf{u}},\mathbf{v}-\mathbf{v}_\delta)+b(\mathbf{v}-\mathbf{v}_\delta,e_p)+(a_\delta-a)(\mathbf{u}_\delta,\mathbf{v}_\delta)\\ &+(b_\delta-b)(\mathbf{v}_\delta,p_\delta)+\sum_{k=1}^{K_s}((\mathbf{f},\mathbf{v}_\delta)_{\Omega_s^k}-(\mathbf{f},\mathbf{v}_\delta)_{\Omega_s^k}^{GL}).\end{aligned} \tag{5.4}$$

Taking $\mathbf{v}_\delta=\tilde{I}\mathbf{v}$, then $\mathbf{v}_\delta\in X_\delta$, we obtain

$$(a_\delta-a)(\mathbf{u}_\delta,\mathbf{v}_\delta)=0,\qquad (b_\delta-b)(\mathbf{v}_\delta,p_\delta)=0. \tag{5.5}$$

Moreover [3],

$$\sum_{T\in\mathcal{T}_s}((\mathbf{f},\mathbf{v}_\delta)_{\Omega_s^k}-(\mathbf{f},\mathbf{v}_\delta)_{\Omega_s^k}^{GL})\lesssim\sum_{T\in\mathcal{T}_s}h_T^{\min\{r,N+1\}}N^{-r}\|\mathbf{f}\|_{r,\Omega_s}\|\mathbf{v}_\delta\|_{1,\Omega_s},\quad r\geq 2, \tag{5.6}$$

and

$$\begin{aligned}&a(e_u,\mathbf{v}-\tilde{I}\mathbf{v})+b(\mathbf{v}-\tilde{I}\mathbf{v},e_p)\\ =&\sum_{k=1}^{K_d}\int_{\Omega_d^k}\boldsymbol{\kappa}^{-1}(\mathbf{u}_d-\mathbf{u}_{d,\delta})\cdot(\mathbf{v}_d-I_\delta\mathbf{v}_d)d\mathbf{x}+\sum_{k=1}^{K_s}\int_{\Omega_s^k}2\nu D(\mathbf{u}_s-\mathbf{u}_{s,\delta}):D(\mathbf{v}_s-I_\delta^0\mathbf{v}_s)d\mathbf{x}\\ &+\sum_{k=1}^{K_\Gamma}\int_{\Gamma^k}(\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_s-\mathbf{u}_{s,\delta})\cdot\boldsymbol{\tau})((\mathbf{v}_s-I_\delta^0\mathbf{v}_s)\cdot\boldsymbol{\tau})d\sigma+\sum_{k=1}^{K_\Gamma}\int_{\Gamma^k}(p_d-p_{d,\delta})((\mathbf{v}_s-I_\delta^0\mathbf{v}_s)\cdot\mathbf{n}_s)d\sigma\\ &-\sum_{k=1}^{K_s}\int_{\Omega_s^k}(p_s-p_{s,\delta})\nabla\cdot(\mathbf{v}_s-I_\delta^0\mathbf{v}_s)d\mathbf{x}+\sum_{k=1}^{K_d}\int_{\Omega_d^k}\nabla(p_d-p_{d,\delta})\cdot(\mathbf{v}_d-I_\delta\mathbf{v}_d)d\mathbf{x}.\end{aligned}$$

By using integration by parts, Lemma 2.2, and the first and fourth equations in (2.1), we obtain

$$
\begin{aligned}
&a(e_{\mathbf{u}},\mathbf{v}-\tilde{I}\mathbf{v})+b(\mathbf{v}-\tilde{I}\mathbf{v},e_p) \qquad (5.7)\\
=&-\sum_{k=1}^{K_d}\int_{\Omega_d^k}(\boldsymbol{\kappa}^{-1}\mathbf{u}_{d,\delta}+\nabla p_{d,\delta})\cdot(\mathbf{v}_d-I_\delta\mathbf{v}_d)d\mathbf{x}\\
&+\sum_{k=1}^{K_s}\int_{\Omega_s^k}(\mathbf{f}+\nabla\cdot(2\nu D(\mathbf{u}_{s,\delta}))-\nabla p_{s,\delta})\cdot(\mathbf{v}_s-I_\delta^0\mathbf{v}_s)d\mathbf{x}\\
&+\sum_{e\in\mathcal{E}(\Omega_s)}\int_e[-2\nu D(\mathbf{u}_{s,\delta})\mathbf{n}_s+p_{s,\delta}\mathbf{n}_s](\mathbf{v}_s-I_\delta^0\mathbf{v}_s)d\sigma\\
&+\sum_{e\in\mathcal{E}(\Gamma)}\int_e\Big\{-2\nu D(\mathbf{u}_{s,\delta})\mathbf{n}_s+p_{s,\delta}\mathbf{n}_s-p_{d,\delta}\mathbf{n}_s-\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_{s,\delta}\cdot\boldsymbol{\tau})\boldsymbol{\tau}\Big\}(\mathbf{v}_s-I_\delta^0\mathbf{v}_s)d\sigma.
\end{aligned}
$$

Now combining (5.4) $\sim$ (5.7), using the Cauchy-Schwarz inequality and Lemma 5.1 gives

$$
\begin{aligned}
&a(e_u,\mathbf{v})+b(\mathbf{v},e_p) \qquad (5.8)\\
\le&\sum_{k=1}^{K_s}\|\mathbf{f}+\nabla\cdot(2\nu D(\mathbf{u}_{s,\delta}))-\nabla p_{s,\delta}\|_{0,\Omega_s^k}\|\mathbf{v}_s-I_\delta^0\mathbf{v}_s\|_{0,\Omega_s^k}\\
&+\sum_{e\in\mathcal{E}(\Omega_s)}\|[-2\nu D(\mathbf{u}_{s,\delta})\mathbf{n}_s+p_{s,\delta}\mathbf{n}_s]\|_{0,e}\|\mathbf{v}_s-I_\delta^0\mathbf{v}_s\|_{0,e}\\
&+\sum_{k=1}^{K_d}\|\boldsymbol{\kappa}^{-1}\mathbf{u}_{d,\delta}+\nabla p_{d,\delta}\|_{0,\Omega_d^k}\|\mathbf{v}_d-I_\delta\mathbf{v}_d\|_{0,\Omega_d^k}+\sum_{T\in\mathcal{T}_s}h_T^{\min\{r,N+1\}}N^{-r}\|\mathbf{f}\|_{r,\Omega_s}\|I_\delta^0\mathbf{v}_s\|_{1,\Omega_s}\\
&+\sum_{e\in\mathcal{E}(\Gamma)}\Big\|-2\nu D(\mathbf{u}_{s,\delta})\mathbf{n}_s+p_{s,\delta}\mathbf{n}_s-p_{d,\delta}\mathbf{n}_s-\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_{s,\delta}\cdot\boldsymbol{\tau})\boldsymbol{\tau}\Big\|_{0,e}\|\mathbf{v}_s-I_\delta^0\mathbf{v}_s\|_{0,e}\\
\lesssim&\sum_{T\in\mathcal{T}_s}\frac{h_T}{N}\|\mathbf{f}+\nu\nabla\cdot(2D(\mathbf{u}_{s,\delta}))-\nabla p_{s,\delta}\|_{0,T}\|\nabla\mathbf{v}_s\|_{0,\Omega_T}+\sum_{T\in\mathcal{T}_s}h_T^{\min\{r,N+1\}}N^{-r}\|\mathbf{f}\|_{r,\Omega_s}\|\mathbf{v}_s\|_{1,\Omega_s}\\
&+\sum_{T\in\mathcal{T}_d}\|\boldsymbol{\kappa}^{-1}\mathbf{u}_{d,\delta}+\nabla p_{d,\delta}\|_{0,T}\|\mathbf{v}_d\|_{0,\Omega_T}+\sum_{e\in\mathcal{E}(\Omega_s)}\sqrt{\frac{h_e}{N}}\|[2\nu D(\mathbf{u}_{s,\delta})\mathbf{n}_s-p_{s,\delta}\mathbf{n}_s]\|_{0,e}\|\nabla\mathbf{v}_s\|_{0,\Omega_T}\\
&+\sum_{e\in\mathcal{E}(\Gamma)}\sqrt{\frac{h_e}{N}}\Big\|2\nu D(\mathbf{u}_{s,\delta}\mathbf{n}_s)-p_{s,\delta}\mathbf{n}_s+p_{d,\delta}\mathbf{n}_s+\frac{\nu\alpha}{\sqrt{\tilde{\kappa}}}(\mathbf{u}_{s,\delta}\cdot\boldsymbol{\tau})\boldsymbol{\tau}\Big\|_{0,e}\|\nabla\mathbf{v}_s\|_{0,\Omega_e}.
\end{aligned}
$$

For the last term in (5.3) we have $b(e_{\mathbf{u}},q)=-b(\mathbf{u}_\delta,q)$, and $b_\delta(\mathbf{u}_\delta,\tilde{I}q)=0$.

Thus

$$
b(\mathbf{u}_\delta,q)=b(\mathbf{u}_\delta,q-\tilde{I}q)+(b-b_\delta)(\mathbf{u}_\delta,\tilde{I}q). \qquad (5.9)
$$

Using (5.9) and the exactitude of the Gauss type quadratures, we obtain

$$
\begin{aligned}
&b(e_{\mathbf{u}},q)=b(\mathbf{u}_\delta,q-\tilde{I}q)\\
=&\sum_{k=1}^{K_\Gamma}\int_{\Gamma^k}(q_d-I_\delta q_d)(\mathbf{u}_{s,\delta}\cdot\mathbf{n}_s)d\sigma-\sum_{k=1}^{K_s}\int_{\Omega_s^k}(q_s-I_\delta q_s)\nabla\cdot\mathbf{u}_{s,\delta}d\mathbf{x}+\sum_{k=1}^{K_d}\int_{\Omega_d^k}\nabla(q_d-I_\delta q_d)\cdot\mathbf{u}_{d,\delta}d\mathbf{x}\\
=&-\sum_{k=1}^{K_s}\int_{\Omega_s^k}(q_s-I_\delta q_s)\nabla\cdot\mathbf{u}_{s,\delta}d\mathbf{x}-\sum_{k=1}^{K_d}\int_{\Omega_d^k}(q_d-I_\delta q_d)\nabla\cdot\mathbf{u}_{d,\delta}d\mathbf{x}\\
&+\sum_{e\in\mathcal{E}(\Omega_d)}\int_e[\mathbf{u}_{d,\delta}\cdot\mathbf{n}_d](q_d-I_\delta q_d)d\sigma+\sum_{k=1}^{K_\Gamma}\int_{\Gamma^k}(q_d-I_\delta q_d)(\mathbf{u}_{s,\delta}\cdot\mathbf{n}_s+\mathbf{u}_{d,\delta}\cdot\mathbf{n}_d)d\sigma\\
\le&\sum_{k=1}^{K_s}\|q_s-I_\delta q_s\|_{0,\Omega_s^k}\|\nabla\cdot\mathbf{u}_{s,\delta}\|_{0,\Omega_s^k}+\sum_{k=1}^{K_d}\|q_d-I_\delta q_d\|_{0,\Omega_d^k}\|\nabla\cdot\mathbf{u}_{d,\delta}\|_{0,\Omega_d^k}\\
&+\sum_{e\in\mathcal{E}(\Omega_d)}\|[\mathbf{u}_{d,\delta}\cdot\mathbf{n}_d]\|_{0,e}\|q_d-I_\delta q_d\|_{0,e}+\sum_{e\in\mathcal{E}(\Gamma)}\|\mathbf{u}_{s,\delta}\cdot\mathbf{n}_s+\mathbf{u}_{d,\delta}\cdot\mathbf{n}_d\|_{0,e}\|q_d-I_\delta q_d\|_{0,e}.
\end{aligned}
$$

Then using lemma 5.1, we obtain

$$
\begin{aligned}
b(e_{\mathbf{u}},q)\lesssim&\sum_{T\in\mathcal{T}_s}\|\nabla\cdot\mathbf{u}_{s,\delta}\|_{0,T}\|q_s\|_{0,\Omega_T}+\sum_{T\in\mathcal{T}_d}\frac{h_T}{N}\|\nabla\cdot\mathbf{u}_{d,\delta}\|_{0,T}\|\nabla q_d\|_{0,\Omega_T}\\
&+\sum_{e\in\mathcal{E}(\Omega_d)}\sqrt{\frac{h_e}{N}}\|[\mathbf{u}_{d,\delta}\cdot\mathbf{n}_d]\|_{0,e}\|\nabla q_d\|_{0,\Omega_e}+\sum_{e\in\mathcal{E}(\Gamma)}\sqrt{\frac{h_e}{N}}\|\mathbf{u}_{s,\delta}\cdot\mathbf{n}_s+\mathbf{u}_{d,\delta}\cdot\mathbf{n}_d\|_{0,e}\|\nabla q_d\|_{0,\Omega_e}.
\end{aligned}
\tag{5.10}
$$

Finally a direct combination of (5.3), (5.8), and (5.10) leads to the desired result (5.2). □

Remark 5.1. Based on a similar technique as for the single domain spectral method, a lower bound for the spectral element case can be equally obtained. We omit the details of the proof to avoid a too technical discussion.

6 Concluding remarks

In this paper, we have considered a spectral (element) approximation to the Stokes/Darcy coupled equations. The purpose was to derive some *a posteriori* error estimates for the solutions of the discrete problems. The main results of this work include: 1) We obtained the *a posteriori* error indicators for the single domain spectral method, and established their lower bounds and upper bounds; 2) Some numerical tests are carried out to show sharpness of these estimates; 3) A generalization to the spectral element case has been discussed, and *a posteriori* error estimate of the spectral element solution has been obtained.

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