J. Partial Diff. Eqs. **15**(2002), 7–12
©International Academic Publishers Vol.15 No.1

ON THE $W^{1,q}$ ESTIMATE FOR WEAK SOLUTIONS TO A CLASS OF DIVERGENCE ELLIPTIC EQUATIONS*

Zhou Shuqing
(Wuhan Inst. of Physics and Math., Chinese Acad. Sci., Wuhan 430071;
Dept. of Math of Hunan Norm. Univ., Changsha 410081, China)
(E-mail: zhoushuqing97@263.net)
Deng Songhai
(Dept. of Math of Xiangya Med. Inst. in Mid-east Univ., Changsha 410078, China)
Li Xiaoyong
(Dept. of Math of Hunan Norm. Univ., Changsha 410081, China)
(Received Mar. 9, 2001)

Abstract Local $W^{1,q}$ estimates for weak solutions to a class of equations in divergence form

$$D_i(a_{ij}(x)|Du|^{p-2}D_ju) = 0$$

are obtained, where $q > p$ is given. These estimates are very important in obtaining higher regularity for the weak solutions to elliptic equations.

Key Words Divergence elliptic equation; local $W^{1,q}$ estimate; reverse Hölder inequality.
2000 MR Subject Classification 35J65.
Chinese Library Classification 0175.25.

1. Introduction

Using compactness method, Avellanda and Lin Fanghua in [1] obtained L^p theory for elliptic systems of periodic structure

$$L^\varepsilon = -\frac{\partial}{\partial x^\alpha}\left[A^{\alpha\beta}_{ij}(\frac{x}{\varepsilon})\frac{\partial}{\partial x^\beta}\right] = f.$$

Using the results in [1], they in [2] also obtained $C^{0,\alpha}, C^{1,\alpha}$ and $C^{0,1}$ regularity for homogenization problem:

$$\begin{cases} \sum_{i,j=1}^{n} a^{ij}(\frac{x}{\varepsilon})\frac{\partial^2 u_\varepsilon}{\partial x^i x^j} = f(x), & x \in D, \\ u_\varepsilon(x) = g(x), & x \in \partial D, \end{cases}$$

* This project is supported by the National Natural Science Foundation of China.

under certain conditions, where $\varepsilon > 0, D$ is smooth domain in R^n. Using Calderón-Zygmund decompositions theorem [3] and measure theory [4], Caffarelli and Petal in [5] established a determinant theorem for the weak solutions which have higher integrability to a class of homogenization problems, and using this theorem, the authors obtained higher integrability for weak solutions to equations

$$\mathrm{div}(a(x, Du)) = 0, \tag{1}$$

then using this result, the authors obtained corresponding results for homogenization problem with periodic structure in [1] and [2]. By the method different from that in [1-2] and [5], Kilpeläinen and Koskela [6] obtained global integrability for the weak solutions to the equation (1). Li Gongbao and Martio [7] obtained local and global integrability for the gradient of the weak solutions to the equation (1). They also in [8] obtained that the weak solution to the equation (1) with very weak boundary value is exclusive. The L^p estimates established in [1] played crucial role in obtaining the results in [2]. But Caffarelli and Petal in [5] didn't obtain corresponding L^p estimates.

In this paper, we discuss the weak solutions in $W^{1,p}$ to the following equation

$$D_i(a_{ij}(x)|Du|^{p-2}D_j u) = 0. \tag{2}$$

Using the method in [5], we obtain L^q integrability for the gradient of the weak solutions to the equation (2),where q is given to be bigger than p, then establish the reverse Hölder inequality for the equation (2) by the method in [9] and [10], and obtain local $W^{1,q}$ estimate for weak solutions to the equation (2).

2. $W^{1,q}$ Estimate

In this section, we discuss the weak solution in $W^{1,p}$ to the elliptic equation of divergence structure

$$D_i(a_{ij}(x)|Du|^{p-2}D_j u) = 0, \tag{3}$$

where, a_{ij} satisfies:

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \tag{4}$$

where, $\lambda, \Lambda > 0$ are constants.

We have the following theorem and corollary:

Theorem 2.1 *Suppose q is bigger than p; if there exists $\epsilon > 0$,*

$$||a(x) - I|| \le \epsilon, \tag{5}$$

where $a(x) = (a_{ij}), I$ is identical matrix and if $u \in W^{1,p}$ is a weak solution to the equation (3), then $W^{1,q}_{loc}(\Omega)$, and for $\forall R, B_R \subset \Omega$,

$$\left[\oint_{B_{\frac{R}{2}}} (|Du|^q + |u|^q)dx\right]^{\frac{1}{q}} \le \left[\oint_{B_R} (|Du|^p + |u|^p)dx\right]^{\frac{1}{p}}, \tag{6}$$

where B_R is a ball centered in x, with radius R, here, $\fint_{B_R} udx = \frac{1}{|B_R|}\int_{B_R} udx.$

Corollary 2.2 *If a_{ij} is continuous and $u \in W^{1,p}$ is a weak solution to the equation (3), then $\forall q > 0, u \in W^{1,q}_{loc}$.*

3. Some Preliminary Lemmas and Proof of Theorem 2.1

To prove Theorem 2.1, we first discuss the weak solution to p-harmonic function, i.e. p-Laplacian

$$-\Delta_p u \equiv -\mathrm{div}(|Du|^{p-2}Du) = 0. \tag{7}$$

Lemma 3.1 [5] *Suppose u is a p-harmonic function, $Q, 2Q$ are cubes with same center, while the length is different in Factor two. Then*

$$||Du||^p_{L^\infty(Q)} \le C(n,p)\frac{1}{|2Q|}\int_{2Q}|Du|^p dx. \tag{8}$$

We give a proof different from that in [5] and [11].

Proof Denote the length of Q by l. Let $R = \frac{5\sqrt{2}}{2}l$. Let B_R denote the ball with the same center as the cube Q, and with radius R.

We consider the following Dirichlet problem:

$$\begin{cases} \int_{2Q}|Du|^{p-2}Du\cdot D\varphi dx = 0, & x\in 2Q, \forall\varphi\in W_0^{1,p}(B_R),\\ u=0, & x\in B_R\setminus 2Q. \end{cases}$$

By [10], $\forall 0<\rho<R$, we have

$$\int_{B_\rho}|Du|^p dx \le C(\frac{\rho}{R})^n\int_{B_R}|Du|^p dx. \tag{9}$$

Then by Theorem 1.1 in Chapter 3 in [12], for $\forall 0<\rho<R, u\in C^{0,1}(B_\rho)$, furthermore, for all $x, y\in Q, x\neq y$,

$$\frac{|u(x)-u(y)|}{|x-y|} \le C(n,p)(\fint_{B_R}|Du|^p dx)^{\frac{1}{p}}. \tag{10}$$

Let $y\to x$ in (10), we obtain

$$||Du||^p_{L^\infty(\Omega)} \le C(n,p)\fint_{B_R}|Du|^p dx = C(n,p)\frac{1}{|2Q|}\int_{2Q}|Du|^p dx. \tag{11}$$

Lemma 3.2 *Suppose $u\in W^{1,p}$ is a weak solution to the equation (3), and for some Q,*

$$\frac{1}{|Q|}\int_Q |Du|^p dx \le \lambda. \tag{12}$$

Let u_h be a solution to Dirichlet Problem

$$\begin{cases} \Delta_p u_h \equiv -\mathrm{div}(|Du_h|^{p-2}Du_h) = 0, & x\in Q,\\ u_h = u, & x\in\partial Q \end{cases} \tag{13}$$

and suppose (5) holds. Then

$$\frac{1}{|Q|}\int_Q |Du_h|^p dx \le \frac{1}{|Q|}\int_Q |Du|^p dx, \tag{14}$$

$$\frac{1}{|Q|}\int_Q |D(u-u_h)|^p dx \le C\epsilon^\alpha \frac{1}{|Q|}\int_Q |Du|^p dx, \tag{15}$$

where $\alpha = \dfrac{p}{p-1}$ *when* $2 \le p \le N; \alpha = p$ *when* $1 < p < 2$.

Proof Using $\varphi = u - u_h$ as a testing function in the definition of weak solution, we immediately obtain (14).

We now prove (15).

When $2 \le p \le N$, by Proposition 5.1 in [13] and (5), we obtain

$$\begin{aligned}
&\int_Q |D(u-u_h)|^p dx\\
\le\ & \int_Q \langle |Du|^{p-2}Du - |Du_h|^{p-2}Du_h, Du - Du_h\rangle dx\\
=\ & \int_Q \langle |Du|^{p-2}Du, Du - Du_h\rangle dx\\
=\ & C\int_Q \langle (I-a(x))|Du|^{p-2}Du, Du - Du_h\rangle dx\\
& + C\int_Q \langle a(x)|Du|^{p-2}Du, Du - Du_h\rangle dx\\
=\ & C\int_Q \langle (I-a(x))|Du|^{p-2}Du, Du - Du_h\rangle dx\\
\le\ & C\epsilon \left(\int_Q |Du|^p dx\right)^{\frac{p-1}{p}} \left(\int_Q |D(u-u_h)|^p dx\right)^{\frac{1}{p}},
\end{aligned} \tag{16}$$

from which we get (15).

When $1 < p < 2$, by Proposition 5.2 in [13] and (5) and (14), calculating as before, we obtain

$$\begin{aligned}
&\int_Q |D(u-u_h)|^p dx\\
\le\ & C\int_Q (|Du|^p + |Du_h|^p)^{\frac{2-p}{2}} \left(\langle |Du|^{p-2}Du - |Du_h|^{p-2}Du_h, Du - Du_h\rangle\right)^{\frac{p}{2}} dx\\
\le\ & C\left(\int_Q (|Du|^p + |Du_h|^p)dx\right)^{\frac{2-p}{2}} \left(\int_Q \langle |Du|^{p-2}Du - |Du_h|^{p-2}Du_h, Du - Du_h\rangle dx\right)^{\frac{p}{2}}\\
\le\ & C\left(\int_Q |Du|^p dx\right)^{\frac{2-p}{2}} \left[\epsilon \left(\int_Q |Du|^p dx\right)^{\frac{p-1}{p}} \left(\int_Q |D(u-u_h)|^p dx\right)^{\frac{1}{p}}\right]^{\frac{p}{2}},
\end{aligned} \tag{17}$$

from which we obtain

$$\int_Q |D(u-u_h)|^p dx \le C\epsilon^{\frac{p}{2}} \left(\int_Q |Du|^p dx\right)^{\frac{1}{2}} \left(\int_Q |D(u-u_h)|^p dx\right)^{\frac{1}{2}}, \tag{18}$$

therefore, (15) also holds when $1 < p < 2$.

We now prove Theorem 2.1:

By Lemma 3.1, Lemma 3.2 and Theorem A in [5], we obtain that $u \in W^{1,q}_{loc}$.

We now prove the estimate (6) holds.

Choose a ball $B_R \subset \Omega, \eta$ a standard cut-off function , choose $\varphi = \eta^p(u-u_R)$, where $u_R = \frac{1}{|B_R|}\int_{B_R} udx$, as a testing function in (3). Using (4) and (5), we obtain

$$\begin{aligned} & \lambda \int_{B_R} \eta^p |Du|^p dx \\ \le\ & C\int_{B_R} \eta^p a_{ij}|Du|^{p-2}D_iuD_judx \\ =\ & -p\int_{B_R} \eta^{p-1}a_{ij}(u-u_R)|Du|^{p-2}D_iuD_judx \\ \le\ & (1+\epsilon)\int_{B_R}\eta^p|Du|^pdx + (1+\epsilon)\theta^{-(p-1)}\int_{B_R}|D\eta|^p|u-u_h|^pdx. \end{aligned} \tag{19}$$

By Choosing θ sufficiently small, (19) implies

$$\int_{B_{\frac{R}{2}}} |Du|^pdx \le CR^{-p}\int_{B_R}|u-u_R|^pdx. \tag{20}$$

Choosing p' such that $\max\{1, \dfrac{np}{n+p}\} < p' < p$, from (20) and Hölder inequality and interpolation theorem, we obtain

$$\begin{aligned} & \oint_{B_{\frac{R}{2}}} |Du|^pdx \\ \le\ & CR^{-p}\left[\left(\oint_{B_R}|u-u_R|^pdx\right)^{\frac{1}{p}}\right]^p \\ \le\ & C\left(\oint_{B_R}|Du|^{p'}dx\right)^{\frac{p}{p'}}, \end{aligned} \tag{21}$$

while

$$\oint_{B_{\frac{R}{2}}}|u|^pdx \le CR\oint_{B_R}|Du|^pdx + C\left(\oint_{B_R}|u|^{p'}dx\right)^{\frac{p}{p'}}. \tag{22}$$

Adding (22) to (21), we obtain

$$\oint_{B_{\frac{R}{2}}}(|Du|^p+|u|^p)dx \le CR\oint_{B_R}(|Du|^p+|u|^p)dx + C\left(\oint_{B_R}(|Du|^{p'}+|u|^{p'})dx\right)^{\frac{p}{p'}}. \tag{23}$$

Letting $g = |Du|^{p'} + |u|^{p'}$, and choosing R_0 sufficiently small such that $\theta = CR < CR_0 < 1$, we get

$$M_{\frac{1}{2}d(x)}(g)^{\frac{p}{p'}}(x) \le C\left(M_{d(x)}(g)(x)\right)^{\frac{p}{p'}} + \theta M_{d(x)}(g)^{\frac{p}{p'}}(x), \tag{24}$$

where $M_{d(x)}(f)(x)$ is local maximum function of $f(x), d(x) \leq R_0$, thus by Proposition 1.1 in Chapter 5 in [12], there exists q' such that, for $\forall t \in [p, q'), u \in W^{1,t}_{loc}$ and

$$\left[\oint_{B_{\frac{R}{2}}} (|Du|^t + |u|^t) dx \right]^{\frac{1}{t}} \leq C \left[\oint_{B_R} (|Du|^p + |u|^p) dx \right]^{\frac{1}{p}}. \tag{25}$$

The first part of the theorem shows that $q' > p$, thus the estimate (6) holds.

Acknowledgement: The authors are grateful for their tutor Fang Ainong for his careful tuition.

References

[1] Avellanda M. and Lin Fanghua, Compactness method in the theory of homogenization, *Comm. Pure Appl. Math.*, **40**(1987), 803-847.

[2] Avellanda M. and Lin Fanghua, Compactness method in the theory of homogenization II, *Comm. Pure Appl. Math.*, **42**(1989), 139-172.

[3] Stein E. M., Singular Integrals and Differentiability Properties of Functions(in Chinese, translated by Minde C., etc.), Press of Beijing Univ., 1986.

[4] Halmos P. R., Measure Theory, Springer-Verlarg, World Books Publishing House, 1998.

[5] Caffarelli L. A. and Peral I., On $W^{1,p}$ estimates for elliptic equatins in divergence form, *Comm. Pure Appl. Math.*, **11**(1)(1998), 1-21.

[6] Kilpeläinen T. and Koskela P., Global integrability of the gradients of solutions to partial differential equations, *Nonlinear Analysis, TMA*, **23**(7)(1994), 899-909.

[7] Li Gongbao and Martio O., Local and global integrability of gradients in obstacle problem, *Acad. Sci. Fenn. Ser. A. I. Math.*, **19**(1994), 25-34.

[8] Li Gongbao and Martio O., Uniqueness of solutions with very weak boundary values, *Nonlinear Analysis, TMA*,(to appear in).

[9] Gehring F. W., The L^p-integrability of the partial derivatives of quasi-conformal mappings, *Acta Math.*, **130**(1973), 265-277.

[10] Choe H. J., Regularity theory for a general class of quasi-linear elliptic differential equations and obstacle problems, *Arch. Rat. Mech. Anal.*, **114**(1991), 383-394.

[11] Dibenetto E., $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear analysis, TMA*, **7**(8)(1983), 827-850.

[12] Giaquinta, M., Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Princeton Univ. Press, Princeton, New Jersey, 1983.

[13] Glowinski R. and Marroco A., Sur l'approximation, par elements finis d'ordre un, et la resolution, par penalisation-dualite, d'une classe de problems Dirichlet non linearies, RAIRO, 1975, R-2; 41-76.