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LONG TIME BEHAVIOR OF THE DISSIPATIVE GENERALIZED SYMMETRIC REGULARIZED LONG WAVE EQUATIONS

Shang Yadong, Guo Boling and Fang Shaomei
(Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P.O.Box 8009, Beijing 100088, China)
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Abstract This paper deals with the long time behavior of solutions for the dissipative generalized symmetric regularized long wave equations. We show the existence of global weak attractors for the periodic initial value problem of the equations in $H^1 \times L^2$. The finite dimensionality of the global attractors is also established.

Key Words Symmetric regularized long wave equation, dissipative term, periodic initial value problem, global attractors, Hausdorff dimension, fractal dimension

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1. Introduction

A symmetric version of regularized long wave equation (SRLWE)

$$u_{xxt} - u_t = \rho_x + uu_x, \tag{1.1}$$

$$\rho_t + u_x = 0, \tag{1.2}$$

has been proposed as a model for propagation of weakly nonlinear ion acoustic and space-charge waves[1]. The $sech^2$ solitary wave solutions, the four invariants and some numerical results have been obtained in [1]. Obviously, eliminating ρ from (1.1), we get a class of symmetric regularized long wave equation (SRLWE)

$$u_{tt} - u_{xx} + \frac{1}{2}(u^2)_{xt} - u_{xxtt} = 0. \tag{1.3}$$

The SRLW equation (1.3) is explicitly symmetry in the x and t derivatives and is very similar to the regularized long wave equation which describes shallow water waves and plasma drift waves[2-3]. The SRLW equation (1.1)—(1.2) or (1.3) arises also in many other areas of mathematical physics. Numerical investigation indicated that interactions of solitary waves are inelastic [4], thus the solitary wave of the SRLW equation is not soliton. More recently, Chen Lin ([5]) studied the orbital stability and instability of solitary wave solutions of the generalized SRLW equations. The research on the well-posedness and numerical methods for the equation has aroused more and more interest. In [6] Guo Boling studied the existence, uniqueness and regularity of the periodic initial value problem for a class of the generalized SRLW equations and obtained the error estimates of the spectral approximation. Miao Chenxia [7] considered the initial boundary value problem for symmetric regularized long wave equations with non homogenous boundary value.

In real processes, viscosity, as well as dispersion, plays an important role. Therefore, it is more significant to study the behavior (especially the large time behavior)of the dissipative symmetric regularized long wave equations with damping term

$$u_{xxt} - u_t + \nu u_{xx} = \rho_x + uu_x, \tag{1.4}$$

$$\rho_t + u_x + \gamma\rho = 0. \tag{1.5}$$

where γ, ν are positive constants, which is a reasonable model to render essential phenomena of nonlinear ion acoustic wave motion when take account of dissipation.

In this paper, we consider the following periodic initial value problem for the dissipative generalized symmetric regularized long wave equations with damping term

$$u_t - \nu u_{xx} + \rho_x + f(u)_x - u_{xxt} = g_1(x), (x,t) \in \mathrm{R} \times \mathrm{R}^+, \tag{1.6}$$

$$\rho_t + u_x + \gamma\rho = g_2(x), (x,t) \in \mathrm{R} \times \mathrm{R}^+, \tag{1.7}$$

$$u(x+D,t) = u(x-D,t), \rho(x+D,t) = \rho(x-D,t), x \in \mathrm{R}, t \geq 0, \tag{1.8}$$

$$u(x,0) = u_0(x), \rho(x,0) = \rho_0(x), \quad x \in \mathrm{R}, \tag{1.9}$$

where $D > 0, \gamma, \nu > 0$ are positive constants, $f : \mathrm{R} \to \mathrm{R}$ are C^∞functions, $g_1(x), g_2(x) \in L^2_{per}(\Omega), \Omega = (-D, D)$, we establish the t-independent a priori estimates of the problem (1.6)–(1.9), then we prove the existence of global attractor of the problem (1.6)–(1.9) in $H^1(\Omega) \times L^2(\Omega)$, and establish the finite-dimensionality of Hausdorff and fractal dimension for the global attractor. Since the dynamical system $S(t)$ defined by (1.6) and (1.7) is not compact in $H^1(\Omega) \times L^2(\Omega)$, we cannot construct the global attractor by the method introduced by Temam [8] or Constantin, Foias and Temam[9]. We here employ the techniques developed by Ghidaglia[10] to show the existence of finite dimensional global weak attractor for (1.6)–(1.7) in $H^1(\Omega) \times L^2(\Omega)$. For this purpose, it is necessary that the semigroup $S(t)$ should be weakly continuous in $H^1(\Omega) \times L^2(\Omega)$ for every $t > 0$. We will establish the weak continuity of $S(t)$ in $H^1(\Omega) \times L^2(\Omega)$ by applying a direct method.

The outline of this article is as follows. In Section 2, we show that the solution semigroup $S(t)$ is weakly continuous in $H^1(\Omega) \times L^2(\Omega)$ for every $t > 0$. In Section 3, we derive the uniform a priori estimates in time on the solution of the equations (1.6)–(1.7) in $H^1(\Omega) \times L^2(\Omega)$. Then we show that the existence of global weak attractor for the equations (1.6)–(1.7) in $H^1(\Omega) \times L^2(\Omega)$. The finite dimensionality of the global weak attractor is also deduced.

2. The Nonlinear Solution Semigroup

By Galerkin method, we can easily deduce the following existence results.

Theorem 1 *Assume that* $(u_0,\rho_0)\in H^1_{per}(\Omega)\times L^2_{per}(\Omega), g_1(x)\in L^2_{per}(\Omega), g_2(x)\in L^2_{per}(\Omega)$. *Then the problem (1.6)–(1.9) possesses a unique solution* $(u(t),\rho(t))$ *defined on* R^+ *such that*

$$\begin{aligned} u(t)\in& L^\infty(0,T;H^1(\Omega)),\quad \frac{\partial u}{\partial t}\in L^\infty(0,T;H^1(\Omega)),\\ \rho(t)\in& L^\infty(0,T;L^2(\Omega)),\quad \frac{\partial \rho}{\partial t}\in L^\infty(0,T;L^2(\Omega)),\quad \forall\ T>0. \end{aligned}$$

This shows that the system (1.6)–(1.9) defines a solution semigroup $S(t)$ which maps $H^1(\Omega)\times L^2(\Omega)$ to $H^1(\Omega)\times L^2(\Omega)$ such that $S(t)(u_0,\rho_0)=(u(t),\rho(t))$, the solution of the problem (1.6)–(1.9).

Let $H=L^2(\Omega)$ be Hilbert space endowed with its usual inner product $(\cdot,\cdot)$ and norm $\|\cdot\|, \|\cdot\|_p$ denote the norm of $L^p(\Omega)$ for all $1\le p\le\infty(\|\cdot\|_2=\|\cdot\|), \|\cdot\|_X$ denotes the norm of any Banach space X.

We first establish the following fact about the solution semigroup $S(t)$.

Proposition 1 *Assume that* $g_1(x)\in L^2_{per}(\Omega), g_2(x)\in L^2_{per}(\Omega), (u_0,\rho_0)\in H^1_{per}(\Omega)\times L^2_{per}(\Omega)$. *Then the dynamical system* $S(t): H^1(\Omega)\times L^2(\Omega)\to H^1(\Omega)\times L^2(\Omega)$ *is weakly continuous for every* $t>0$.

Proof $\forall t_1>0$ fixed,we shall show $S(t_1)$ is weakly continuous from $H^1(\Omega)\times L^2(\Omega)$ to $H^1(\Omega)\times L^2(\Omega)$. Assume now that

$$(u_0^k,\rho_0^k)\to(w_0,\sigma_0)\ \text{weakly in}\ H^1(\Omega)\times L^2(\Omega). \tag{2.1}$$

We are going to show that $S(t_1)(u_0^k,\rho_0^k)\to S(t_1)(w_0,\sigma_0)$ weakly in $H^1(\Omega)\times L^2(\Omega)$. Choose $T>t_1$,and denote by $(u_k(t),\rho_k(t))=S(t)(u_0^k,\rho_0^k), (w(t),\sigma(t))=S(t)(w_0,\sigma_0)$. Since the weak convergence implies the boundedness, it follows that

$$\left\|(u_0^k,\rho_0^k)\right\|_{H^1\times L^2}\le R, \tag{2.2}$$

where R is a constant independent of k,$\|(u,\rho)\|^2_{H^1\times L^2}=\|u\|^2_{H^2}+\|\rho\|^2$.

We note that $(u_k(t),\rho_k(t))$ satisfies the following equations

$$u_{kt}-\nu u_{kxx}+\rho_{kx}+f(u_k)_x-u_{kxxt}=g_1(x), \tag{2.3}$$
$$\rho_{kt}+u_{kx}+\gamma\rho_k=g_2(x). \tag{2.4}$$

Taking the inner product of(2.3)with u_k in H,we find that

$$\frac{1}{2}\frac{d}{dt}(\|u_k\|^2+\|u_{kx}\|^2)+\nu\|u_{kx}\|^2+\int_\Omega\rho_{kx}u_kdx+\int_\Omega f(u_k)_xu_kdx=(g_1,u_k). \tag{2.5}$$

Let

$$F(s)=\int_0^s f(\tau)d\tau.$$

Then we have

$$\int_\Omega f(u_k)_x u_k dx=-\int_\Omega f(u_k)u_{kx}dx=-\int_\Omega \frac{\partial}{\partial x}F(u_k)dx=0. \tag{2.6}$$

Taking the inner product of (2.4) with ρ_k in H,we get that

$$\frac{1}{2}\frac{d}{dt}\|\rho_k\|^2+\int_\Omega \rho_k u_{kx}dx+\gamma\|\rho_k\|^2=(g_2,\rho_k). \tag{2.7}$$

By using the conditions of Proposition 1, it comes from (2.5)–(2.7) that

$$\begin{aligned}&\frac{d}{dt}(\|u_k\|^2+\|u_{kx}\|^2+\|\rho_k\|^2)+2\nu\|u_{kx}\|^2+2\gamma\|\rho_k\|^2\\ \le&(\|u_k\|^2+\|u_{kx}\|^2+\|\rho_k\|^2)+\|g_1\|^2+\|g_2\|^2.\end{aligned} \tag{2.8}$$

Applying Gronwall lemma to (2.8) we find that

$$\begin{aligned}&\|u_k(t)\|^2+\|u_{kx}(t)\|^2+\|\rho_k(t)\|^2\\ \le\ &e^t(\|u_k(0)\|^2+\|u_{kx}(0)\|^2+\|\rho_k(0)\|^2)+e^t(\|g_1\|^2+\|g_2\|^2)\\ \le\ &R^2e^t+e^t(\|g_1\|^2+\|g_2\|^2).\end{aligned}$$

And hence

$$\|u_k(t)\|_{H^1}^2+\|\rho_k(t)\|^2\le C,\quad \forall\ 0\le t\le T, \tag{2.9}$$

where C is a constant depending on T.

By (2.9) and Agmon inequality

$$\|u\|_\infty\le C\|u\|^{\frac{1}{2}}\|u\|_{H^1},\quad \forall\ u\in H^1(\Omega). \tag{2.10}$$

We obtain that

$$\|u\|_\infty\le C,\quad \forall\ 0\le t\le T. \tag{2.11}$$

Taking the inner product of (2.3) with u_{kt} in H,we find that

$$\begin{aligned}&\|u_{kt}(t)\|^2+\|u_{kxt}(t)\|^2\\ =&-\nu(u_{kx},u_{kxt})+(\rho_k,u_{kxt})+(f(u_k),u_{kxt})+(g_1,u_{kt})\\ \le&\nu\|u_{kx}\|\,|u_{kxt}\|+\|\rho_k\|\,\|u_{kxt}\|+\|f(u_k)\|\,\|u_{kxt}\|+\|g_1\|\,\|u_{kt}(t)\|.\end{aligned} \tag{2.12}$$

By using the smoothness of f and (2.11) we have

$$\|f(u_k)\|_\infty\le C,\quad \forall\ 0\le t\le T. \tag{2.13}$$

And then we get from (2.9) and (2.13) that

$$\|u_{kt}\|^2 + \|u_{kxt}\|^2 \le C\|u_{kt}\| + C\|u_{kxt}\| \le \frac{1}{2}\|u_{kt}\|^2 + \frac{1}{2}\|u_{kxt}\|^2 + C, \tag{2.14}$$

which implies that

$$\|u_{kt}\|_{H^1} \le C, \quad \forall\ 0 \le t \le T. \tag{2.15}$$

We get from(2.4) and(2.9) that

$$\|\rho_{kt}\| \le \|u_{kx}\| + \gamma\|\rho_k\| + \|g_2\| \le C, \quad \forall\ 0 \le t \le T. \tag{2.16}$$

By (2.9),(2.15), and(2.16) we find that there exist $\theta \in H^1(\Omega), \vartheta \in L^2(\Omega)$, and $(u(t), \rho(t))$ such that $u(t) \in L^\infty(0, T; H^1(\Omega)), \rho(t) \in L^\infty(0, T; H)$ and a subsequence, which is still denoted by (u_k, ρ_k), such that

$$u_k(t_1) \to \theta \text{ weakly in } H^1(\Omega), \tag{2.17}$$
$$u_k(t) \to u(t) \text{ in } L^\infty(0, T; H^1(\Omega)) \text{ weak star}, \tag{2.18}$$
$$u_{kt}(t) \to u_t(t) \text{ in } L^\infty(0, T; H^1(\Omega)) \text{ weak star}, \tag{2.19}$$
$$\rho_k(t_1) \to \vartheta \text{ weakly in } H, \tag{2.20}$$
$$\rho_k(t) \to \rho(t) \text{ in } L^\infty(0, T; H) \text{ weak star}, \tag{2.21}$$
$$\rho_{kt}(t) \to \rho_t(t) \text{ in } L^\infty(0, T; H) \text{ weak star}. \tag{2.22}$$

By (2.18),(2.19) and a compactness theorem[11] we infer that

$$u_k(t) \to u(t) \text{ in } L^2(0, T; H) \text{ strongly}. \tag{2.23}$$

$\forall v \in H^1_{per}(\Omega), \forall \psi(t) \in C_0^\infty(0, T)$, by (2.3) we claim that

$$\begin{aligned} &\int_0^T (u_{kt}, \psi(t)v)dt + \nu\int_0^T (u_{kx}, \psi(t)v_x)dt + \int_0^T (u_{kxt}, \psi(t)v_x)dt \\ &-\int_0^T (\rho_k, \psi(t)v_x)dt - \int_0^T (f(u_k), \psi(t)v_x)dt \\ =&\int_0^T (g_1, \psi(t)v)dt. \end{aligned} \tag{2.24}$$

Note that

$$\begin{aligned} &\left|\int_0^T (f(u), \psi(t)v_x)dt - \int_0^T (f(u_k), \psi(t)v_x)dt\right| \\ =&\left|\int_0^T (f(u) - f(u_k), \psi(t)v_x)dt\right| \\ \le&\int_0^T \|f(u) - f(u_k)\|\,\|\psi(t)v_x\|\,dt \le \|f'(\xi)\|_\infty \int_0^T \|u - u_k\|\,\|\psi(t)v_x\|\,dt \\ \le& C\|u - u_k\|_{L^2(0,T;H)}\|\psi(t)v_x\|_{L^2(0,T;H)} \to 0. \end{aligned} \tag{2.25}$$

Taking the limit of (2.24) as $k \to \infty$, by (2.18),(2.19),(2.20) and (2.25) we find that

$$\begin{aligned}&\int_0^T (u_t, v)\psi(t)dt + \nu \int_0^T (u_x, v_x)\psi(t)dt + \int_0^T (u_{xt}, v_x)\psi(t)dt \\ &- \int_0^T (\rho, v_x)\psi(t)dt - \int_0^T (f(u), v_x)\psi(t)dt \\ =&\int_0^T (g_1, v)\psi(t)dt. \end{aligned} \tag{2.26}$$

Similarly, $\forall v \in L^2_{per}(\Omega), \forall \psi(t) \in C_0^\infty(0,T)$, by (2.4) we claim that

$$\int_0^T (\rho_{kt}, \psi(t)v)dt + \int_0^T (u_{kx}, \psi(t)v)dt + \gamma \int_0^T (\rho_k, \psi(t)v)dt = \int_0^T (g_2, \psi(t)v)dt. \tag{2.27}$$

Taking the limit of (2.27) as $k \to \infty$, by (2.18),(2.21), and(2.22) we get

$$\int_0^T (\rho_t, v)\psi(t)dt + \int_0^T (u_x, v)\psi(t)dt + \gamma \int_0^T (\rho, v)\psi(t)dt = \int_0^T (g_2, v)\psi(t)dt. \tag{2.28}$$

Hence, the following holds in the sense of distributions

$$u_t - \nu u_{xx} + \rho_x + f(u)_x - u_{xxt} = g_1(x), \tag{2.29}$$

$$\rho_t + u_x + \gamma\rho = g_2(x), \tag{2.30}$$

that is, $(u(t), \rho(t))$ satisfies the equations (1.6)-(1.7).

$\forall v \in H^1_{per}(\Omega), \forall \psi(t) \in C^\infty(0,T)$ with $\psi(T) = 0, \psi(0) = 1$, by (2.3) we obtain

$$\begin{aligned}&- \int_0^T (u_k, v)\psi'(t)dt + \nu \int_0^T (u_{kx}, v_x)\psi(t)dt + \int_0^T (u_{kxt}, v_x)\psi(t)dt \\ &- \int_0^T (\rho_k, v_x)\psi(t)dt - \int_0^T (f(u_k), v_x)\psi(t)dt \\ =&(u_k(0), v) + \int_0^T (g_1, v)\psi(t)dt. \end{aligned} \tag{2.31}$$

Similarly, $\forall v \in L^2_{per}(\Omega), \forall \psi(t) \in C^\infty(0,T)$ with $\psi(T) = 0, \psi(0) = 1$, by (2.4) we get

$$\begin{aligned}&- \int_0^T (\rho_k, v)\psi'(t)dt + \int_0^T (u_{kx}, v)\psi(t)dt + \gamma \int_0^T (\rho_k, v)\psi(t)dt \\ =&(\rho_k(0), v) + \int_0^T (g_2, v)\psi(t)dt. \end{aligned} \tag{2.32}$$

Assumption (2.1) implies that

$$u_k(0) = u_0^k \to w_0 \text{ weakly in } H^1_{per}(\Omega), \tag{2.33}$$

$$\rho_k(0) = \rho_0^k \to \sigma_0 \text{ weakly in } H. \tag{2.34}$$

Then taking the limit of (2.31) and (2.32) as before, by (2.33) and (2.34) we obtain

$$\begin{aligned}
&-\int_0^T (u,v)\psi'(t)dt+\nu\int_0^T (u_x,v_x)\psi(t)dt+\int_0^T (u_{xt},v_x)\psi(t)dt\\
&-\int_0^T (\rho,v_x)\psi(t)dt-\int_0^T (f(u),v_x)\psi(t)dt\\
=&(w_0,v)+\int_0^T (g_1,v)\psi(t)dt, \qquad (2.35)\\
&-\int_0^T (\rho,v)\psi'(t)dt+\int_0^T (u_x,v)\psi(t)dt+\gamma\int_0^T (\rho,v)\psi(t)dt\\
=&(\sigma_0,v)+\int_0^T (g_2,v)\psi(t)dt. \qquad (2.36)
\end{aligned}$$

On the other hand, by (2.29) and (2.30) we infer that

$$\begin{aligned}
&-\int_0^T (u,v)\psi'(t)dt+\nu\int_0^T (u_x,v_x)\psi(t)dt+\int_0^T (u_{xt},v_x)\psi(t)dt\\
&-\int_0^T (\rho,v_x)\psi(t)dt-\int_0^T (f(u),v_x)\psi(t)dt\\
=&(u(0),v)+\int_0^T (g_1,v)\psi(t)dt, \qquad (2.37)\\
&-\int_0^T (\rho,v)\psi'(t)dt+\int_0^T (u_x,v)\psi(t)dt+\gamma\int_0^T (\rho,v)\psi(t)dt\\
=&(\rho(0),v)+\int_0^T (g_2,v)\psi(t)dt. \qquad (2.38)
\end{aligned}$$

Thus it follows from (2.35)–(2.36) and (2.37)–(2.38) that

$$\begin{aligned}
(w_0,v)=&(u(0),v), \quad \forall\ v\in H^1_{per}(\Omega),\\
(\sigma_0,v)=&(\rho(0),v), \quad \forall\ v\in L^2_{per}(\Omega),
\end{aligned}$$

which show that

$$u(0)=w_0, \quad \rho(0)=\sigma_0. \qquad (2.39)$$

And thus by (2.28)–(2.29) and (2.38) we see that

$$(u(t),\rho(t))=S(t)(w_0,\sigma_0)=(w(t),\sigma(t)). \qquad (2.40)$$

$\forall v\in H^1_{per}(\Omega)(L^2_{per}(\Omega)), \forall\psi(t)\in C^\infty[0,t_1]$ with $\psi(t_1)=1,\psi(0)=0$, then repeating the procedure of the proofs of (2.31)–(2.39), by (2.17) and (2.20) we find that

$$(u(t_1),v)=(\theta,v), \quad \forall\ v\in H^1_{per}(\Omega), \qquad (2.41)$$

$$(\rho(t_1),v)=(\vartheta,v), \quad \forall\ v\in L^2_{per}(\Omega). \qquad (2.42)$$

It comes from (2.40)–(2.42) that

$$(\theta,\vartheta)=(u(t_1),\rho(t_1))=S(t_1)(w_0,\sigma_0). \qquad (2.43)$$

And then (2.17) and (2.20) imply that

$$S(t_1)(u_0^k,\rho_0^k)\to S(t_1)(w_0,\sigma_0) \text{ weakly in } H^1(\Omega)\times L^2(\Omega),$$

which concludes proposition 1.

3. The Global Attractor

In this section, we construct the global attractor for the dynamical system $S(t)$ generated by the problem (1.6)–(1.9) in $H^1(\Omega)\times L^2(\Omega)$. Hereafter, we always assume that

$$g_1(x)\in L^2_{per}(\Omega),\int_\Omega g_1(x)dx=0 \tag{3.1}$$

and then integrating (1.6) over Ω and applying (1.8) we find that the average of $u(t)$ is conserved, i.e. for all $t>0$:

$$\theta(u(t))=\frac{1}{|\Omega|}\int_\Omega u(x,t)dx=\frac{1}{|\Omega|}\int_\Omega u_0(x)dx=\theta(u_0). \tag{3.2}$$

This shows that the problem (1.6)–(1.9) has not bounded absorbing sets in the whole space $E=H\times H$. To overcome this difficulty, we introduce the subset of E:

$$E_\alpha=\{(u,\rho)\in H\times H, |\theta(u)|\le\alpha\},$$

for some fixed α. The equation (2.2) indicates that E_α is invariant under the semigroup $S(t)$ associated to system(1.6)–(1.9).

In the sequel, we will show that there indeed exist bounded absorbing sets in E_α.

Lemma 1 *Assume that (2.1) holds, $(u_0(x),\rho_0(x))\in H_\alpha, u_0(x)\in H^1_{per}(\Omega)$. Then for the solution $(u(t),\rho(t))$ of the problem (1.6)–(1.9) we have*

$$\|u(t)\|_{H^1}\le K, \|\rho(t)\|\le K, \forall t\ge t_1,$$

where K denotes a constant depending only on the data $(\nu,\gamma,\alpha,f,g_1,g_2,\Omega)$, t_1 depending on the data $(\nu,\gamma,\alpha,f,g_1,g_2,\Omega,R)$ when $\|u_0\|_{H^1}\le R$ and $\|\rho_0\|\le R$.

Proof For convenience, we denote

$$\overline{u}=u-\theta(u), \tag{3.3}$$

where $\theta(u)=\frac{1}{|\Omega|}\int_\Omega u(x)dx$, and then we have

$$\theta(\overline{u})=\frac{1}{|\Omega|}\int_\Omega \overline{u}(x,t)dx=\frac{1}{|\Omega|}\int_\Omega u(x,t)dx-\frac{1}{|\Omega|}\int_\Omega \theta(u)dx=0. \tag{3.4}$$

We note that (1.6),(1.7),(3.2) and (3.3) imply that

$$\overline{u}_t-\nu\overline{u}_{xx}+\rho_x+f(u)_x-\overline{u}_{xxt}=g_1(x), \tag{3.5}$$

$$\rho_t+\overline{u}_x+h(\rho)=g_2(x). \tag{3.6}$$

Taking the inner product of (3.5) with $\overline{u}$ in H we infer that

$$\frac{1}{2}\frac{d}{dt}\left\|\overline{u}\right\|^2+\frac{1}{2}\frac{d}{dt}\left\|\overline{u}_x\right\|^2+\nu\left\|\overline{u}_x\right\|^2+\int_\Omega (f(u))_x\cdot\overline{u}dx+\int_\Omega \rho_x\cdot\overline{u}dx=(g_1,\overline{u}). \tag{3.7}$$

Similarly, taking the inner product of (3.6) with ρ in H we get

$$\frac{1}{2}\frac{d}{dt}\left\|\rho\right\|^2+\gamma\left\|\rho\right\|^2+\int_\Omega \overline{u}_x\cdot\rho dx\le(g_{2,}\rho). \tag{3.8}$$

Adding (3.7) to (3.8) and noting that

$$\begin{aligned}\int_\Omega (f(u))_x\cdot\overline{u}dx&=-\int_\Omega f(u)\cdot\overline{u}_xdx-\int_\Omega f(u)\cdot u_xdx\\&=-\int_\Omega (F(u))_xdx=-F(u(D))+F(u(-D))=0\end{aligned} \tag{3.9}$$

where $F(s)=\int_0^s f(\tau)d\tau$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\left\|\overline{u}\right\|^2+\left\|\overline{u}_x\right\|^2+\left\|\rho\right\|^2)+\nu\left\|\overline{u}_x\right\|^2+\gamma\left\|\rho\right\|^2\\\le&(g_1,\overline{u})+(g_{2,}\rho).\end{aligned} \tag{3.10}$$

We recall the Poincare inequality

$$\left\|v\right\|\le C_1\left\|v_x\right\|,\quad \text{if }\int_\Omega v(x)dx=0, \tag{3.11}$$

it follows from (3.4) that

$$\left\|\overline{u}\right\|\le C_1\left\|\overline{u}_x\right\|,\qquad \forall\ t\ge 0. \tag{3.12}$$

Thus, Applying Hölder and Young inequalities we obtain

$$|(g_1,\overline{u})|\le\left\|g_1\right\|\left\|\overline{u}\right\|\le C_1\left\|g_1\right\|\left\|\overline{u}_x\right\|\le\frac{1}{4}\nu\left\|\overline{u}_x\right\|^2+\frac{C_1^2}{\nu}\left\|g_1\right\|^2, \tag{3.13}$$

$$|(g_2,\rho)|\le\left\|g_2\right\|\left\|\rho\right\|\le\frac{1}{4}\gamma\left\|\rho\right\|^2+\frac{1}{\gamma}\left\|g_2\right\|^2. \tag{3.14}$$

In the sequel, we denote any constants depending only on the data $(\nu,\gamma,\alpha,f,g_1,g_2,\Omega)$ by C and $C_i(i=1,2,\cdots)$.

By (3.10),(3.13) and (3.14)we have

$$\frac{d}{dt}(\left\|\overline{u}\right\|^2+\left\|\overline{u}_x\right\|^2+\left\|\rho\right\|^2)+\frac{3\nu}{2}\left\|\overline{u}_x\right\|^2+\frac{3\gamma}{2}\left\|\rho\right\|^2\le 2C_2, \tag{3.15}$$

where $C_2=\frac{C_1^2}{\nu}\left\|g_1\right\|^2+\frac{1}{\gamma}\left\|g_2\right\|^2$.

Due to

$$\begin{aligned}\frac{3\nu}{2}\left\|\overline{u}_x\right\|^2=&\frac{\nu}{2}\left\|\overline{u}_x\right\|^2+\nu\left\|\overline{u}_x\right\|^2\ge\frac{\nu}{2}\left\|\overline{u}_x\right\|^2+\nu C_1^{-2}\left\|\overline{u}\right\|^2\\\ge&C_3(\left\|\overline{u}\right\|^2+\left\|\overline{u}_x\right\|^2),\end{aligned} \tag{3.16}$$

where $C_3 = min\{\frac{1}{2}\nu, \nu C_1^{-2}\}$, by (3.15) we claim that

$$\frac{d}{dt}(\|\overline{u}\|^2 + \|\overline{u}_x\|^2 + \|\rho\|^2) + C_4(\|\overline{u}\|^2 + \|\overline{u}_x\|^2 + \|\rho\|^2) \le 2C_2, \quad \forall\, t \ge 0, \tag{3.17}$$

where $C_4 = \min\{C_3, \dfrac{3}{2}\gamma\}$.

Applying Gronwall lemma we get

$$\begin{aligned} &\|\overline{u}(t)\|^2 + \|\overline{u}_x(t)\|^2 + \|\rho(t)\|^2 \\ \le&(\|\overline{u}(0)\|^2 + \|\overline{u}_x(0)\|^2 + \|\rho(0)\|^2)e^{-C_4 t} + \frac{2C_2}{C_4} \\ \le&(1 + C_1^2)\,\|\overline{u}_x(0)\|^2\, e^{-C_4 t} + \|\rho(0)\|^2\, e^{-C_4 t} + \frac{2C_2}{C_4} \\ \le&(1 + C_1^2)\,\|u_x(0)\|^2\, e^{-C_4 t} + \|\rho(0)\|^2\, e^{-C_4 t} + \frac{2C_2}{C_4} \\ \le&(2 + C_1^2)R^2 e^{-C_4 t} + \frac{2C_2}{C_4}, \qquad \forall t \ge 0 \\ \le&\frac{4C_2}{C_4}, \qquad \forall\, t \ge t_*. \end{aligned} \tag{3.18}$$

where $t_* = \frac{1}{C_4}\ln\frac{C_4(2+C_1^2)R^2}{2C_2}$.

Since

$$\int_\Omega \overline{u}(x,t)\cdot\theta(u(x,t))dx = \theta(u)\int_\Omega \overline{u}(x,t)dx = 0, \tag{3.19}$$

we see that $\overline{u}(t)$ and $\theta(u(t))$ are orthogonal in H. Thus,we get that

$$\begin{aligned} \|u(t)\|^2 &= \|\overline{u}(t)\|^2 + \|\theta(u(t))\|^2 = \|\overline{u}(t)\|^2 + |\theta(u)|^2\,|\Omega| \\ &= \|\overline{u}(t)\|^2 + |\theta(u_0)|^2\,|\Omega| \le \|\overline{u}(t)\|^2 + \alpha^2\,|\Omega|\,. \end{aligned} \tag{3.20}$$

And so, we claim that

$$\begin{aligned} \|u(t)\|_{H^1}^2 + \|\rho(t)\|^2 &= \|u(t)\|^2 + \|u_x(t)\|^2 + \|\rho(t)\|^2 \\ &\le \|\overline{u}(t)\|^2 + \alpha^2\,|\Omega| + \|\overline{u}_x(t)\|^2 + \|\rho(t)\|^2 \\ &\le \frac{4C_2}{C_4} + \alpha^2\,|\Omega|\,. \end{aligned}$$

which concludes Lemma 1.

We observe that Lemma 2 shows that there exists constant K such that the ball

$$B_1 = \Big\{(u,\rho) \in H^1(\Omega)\times L^2(\Omega) : \|u\|_{H^1} \le K, \|\rho\| \le K.\Big\} \tag{3.21}$$

is an absorbing set in $H^1(\Omega)\times H\cap H_\alpha$.

Let

$$\mathcal{A}_1 = \bigcap_{s\ge 0}\overline{\bigcup_{t\ge s} S(t)B_1}\ , \tag{3.22}$$

where the closure is taken with respect to the weak topology of $H^1(\Omega) \times L^2(\Omega)$. And then by Proposition 1 we know that $\mathcal{A}_1$ is a global weak attractor for $S(t)$. More precisely, we have the following

Theorem 2 *Assume that the conditions of Lemma 1 hold. Then the set* $\mathcal{A}_1$ *defined by (3.22) satisfies that*

(i) $\mathcal{A}_1$ *is bounded and weakly closed in* $H^1(\Omega) \times L^2(\Omega) \cap H_\alpha$;

(ii) $S(t)\mathcal{A}_1 = \mathcal{A}_1$ *,*$\forall t \geq 0$;

(iii) For every bounded set X *in* $H^2(\Omega) \times H^1(\Omega)$*,*$S(t)X$ *converges to* $\mathcal{A}_1$ *with respect to the* $H^1(\Omega) \times L^2(\Omega)-$*weak topology as* $t \to \infty$.

Proof The proof of this theorem is similar to that of [10], and so omitted here.

Quite analogous to [12] where we dealt with the finite dimensionality of the global attractor in $H^2(\Omega) \times H^1(\Omega)$, we also deduce the finite dimensionality of the global attractor $\mathcal{A}_1$ in $H^1(\Omega) \times L^2(\Omega)$ here, that is, we have

Theorem 3 *The attractor* $\mathcal{A}_1$ *of Theorem 2 has finite fractal and Hausdorff dimensions in* $H^1(\Omega) \times L^2(\Omega)$.

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