J. Partial Diff. Eqs. **16**(2003), 8–17
 Vol.16, No.1

GLOBAL C^1 SOLUTION TO THE INITIAL-BOUNDARY VALUE PROBLEM FOR DIAGONAL HYPERBOLIC SYSTEMS WITH LINEARLY DEGENERATE CHARACTERISTICS

Li Ta-tsien*
(Department of Mathematics, Fudan University, Shanghai 200433, China
E-mail : dqli@fudan.edu.cn)
Peng Yuejun
(Laboratoire de Mathématiques Appliquées, CNRS UMR 6620
Université Blaise Pascal (Clermont-Ferrand 2), F-63177 Aubière cedex, France
E-mail : peng@math.univ-bpclermont.fr)

Dedicated to the 80th birthday of Professor Zhou Yulin

(Received June 4, 2002)

Abstract We prove that the C^0 boundedness of solution implies the global existence and uniqueness of C^1 solution to the initial-boundary value problem for linearly degenerate quasilinear hyperbolic systems of diagonal form with nonlinear boundary conditions. Thus, if the C^1 solution to the initial-boundary value problem blows up in a finite time, then the solution itself must tend to the infinity at the starting point of singularity.

Key Words Initial-boundary value problem, global C^1 solution, quasilinear hyperbolic system of diagonal form, linearly degenerate characteristics.

2000 MR Subject Classification

Chinese Library Classification

1. Introduction

For first order quasilinear hyperbolic systems, generically speaking, the classical solution exists only locally in time and the singularity will appear in a finite time (see [1-3] and the references therein). In some cases, however, the global existence of classical solution can be obtained. For example, for the Cauchy problem for quasilinear hyperbolic systems with weakly linearly degenerate characteristic fields (WLD), the classical solution exists globally in time provided that the initial data are suitably small and decay at the infinity [4]. This tells us that the formation of singularity depends strongly on the character of characteristics of the system. For the initial-boundary value problem, the situation is quite different. Even for quasilinear hyperbolic systems of diagonal form with linearly degenerate characteristic fields (a special case of WLD), the solution itself may blow up in a finite time (see [5]). Then it is natural to ask whether

*Supported by the Special Funds for Major State Basic Research Projects of China.

the classical C^1 solution exists globally in time when the solution itself can be controlled. The answer of this question is positive for homogeneous reducible hyperbolic systems and, more generally, for homogeneous rich hyperbolic systems with linearly degenerate characteristic fields, for which Lax transformation can be used to simplify the equations [5, 6]. In this paper, we want to extend this result to general homogeneous hyperbolic systems of diagonal form with linearly degenerate characteristic fields.

We consider then the following strictly hyperbolic system of diagonal form :

$$\frac{\partial u_i}{\partial t} + \lambda_i(u)\frac{\partial u_i}{\partial x} = 0 \quad (i = 1, ..., n) \tag{1.1}$$

for $t \geq 0$ and $x \in [0, 1]$, where $u = (u_1, ..., u_n)$, the eigenvalues $\lambda_i(u)$ $(i = 1, ..., n)$ are supposed to be smooth and satisfy

$$\lambda_1(u) < ... < \lambda_m(u) < 0 < \lambda_{m+1}(u) < ... < \lambda_n(u) \tag{1.2}$$

for any given u on the domain under consideration. Moreover, suppose $\lambda_i(u)$ $(i = 1, ..., n)$ are all linearly degenerate in the sense of P.D.Lax, i.e.,

$$\frac{\partial \lambda_i(u)}{\partial u_i} \equiv 0 \quad (i = 1, ..., n). \tag{1.3}$$

The system (1.1) is supplemented by the initial conditions

$$t = 0: \quad u_i = \varphi_i(x) \quad (i = 1, ..., n) \tag{1.4}$$

for $x \in [0, 1]$ and the boundary conditions

$$x = 0: \quad u_k = g_k(t, u_1, ..., u_m) \quad (k = m + 1, ..., n), \tag{1.5}$$

$$x = 1: \quad u_j = g_j(t, u_{m+1}, ..., u_n) \quad (j = 1, ..., m) \tag{1.6}$$

for $t \geq 0$, where φ_i and g_i $(i = 1, ..., n)$ are all C^1 functions with respect to their arguments, and the conditions of C^1 compatibility are supposed to be satisfied at points $(t, x) = (0, 0)$ and $(0, 1)$ respectively.

When λ_i $(i = 1, ..., n)$ are constants, it has been proved (see [5]) that the initial-boundary value problem (1.1)-(1.6) always admits a unique global C^1 solution. This shows that without the nonlinearity of system, the nonlinear boundary conditions can not lead to the formation of singularity; otherwise, the C^1 solution may blow up in a finite time. The goal of this paper is to prove that if the C^1 solution to the initial-boundary value problem (1.1)-(1.6) blows up in a finite time, then the solution itself must tend to the infinity at the starting point of singularity. This kind of blow up phenomenon is similar to the breakdown of C^1 solution to the Cauchy problem for inhomogeneous reducible quasilinear hyperbolic systems of diagonal form with linearly degenerate characteristic fields (see [7] or Chapter 2 in [3]).

According to the local existence and uniqueness of C^1 solution (see [8], [9]), there exists $\delta > 0$ depending on the C^1 norm of the given data such that the problem (1.1)-(1.6) has a unique C^1 solution $u = u(t,x)$ on the domain

$$D(\delta) \stackrel{\text{def}}{=} \{(t,x) \,|\, 0 \le t \le \delta,\ 0 \le x \le 1\}. \tag{1.7}$$

Then, in order to get the global existence and uniqueness of C^1 solution $u = u(t,x)$ to the problem (1.1)-(1.6) for all time $t \ge 0$, it suffices to prove that for any given $T_0 > 0$, if this problem admits a unique C^1 solution on $D(T)$ with $0 < T \le T_0$, then the following uniform a priori estimate on the C^1 norm of solution holds :

$$\|u(t,.)\|_{C^1[0,1]} \stackrel{\text{def}}{=} \sum_{i=1}^{n} \Big(\|u_i(t,.)\|_{C^0[0,1]} + \|\partial_x u_i(t,.)\|_{C^0[0,1]}\Big) \le C_1(T_0), \quad \forall\, t \in [0,T], \tag{1.8}$$

where $C_1(T_0)$ is a positive constant independent of T but possibly depending on T_0. Thus, we formulate our theorem as follows.

Theorem 1 *For any given $T_0 > 0$, let $u = u(t,x)$ be the unique C^1 solution to the initial-boundary value problem (1.1)-(1.6) on the domain $D(T)$ with $0 < T \le T_0$. If $u = u(t,x)$ satisfies the following uniform a priori estimate :*

$$\|u(t,.)\|_{C^0[0,1]} \stackrel{\text{def}}{=} \sum_{i=1}^{n} \|u_i(t,.)\|_{C^0[0,1]} \le C_0(T_0), \quad \forall\, t \in [0,T], \tag{1.9}$$

where $C_0(T_0)$ is a positive constant independent of T but possibly depending on T_0, then the initial-boundary value problem (1.1)-(1.6) admits a unique global C^1 solution $u = u(t,x)$ on the domain $\{(t,x) | t \ge 0,\ 0 \le x \le 1\}$.

This paper is organized as follows. In the next section, we give the main steps for proving Theorem 1. Then the proof of Theorem 1 is reduced to get an a priori estimate on an integral form in which we have to control the quantity $\frac{\partial u_l}{\partial x}$ along the i-th characteristic for all $l \ne i$. This is achieved in the last section through an analysis of the relation between the i-th and the l-th characteristics for all $l \ne i$.

2. Main steps of proving Theorem 1

Since the C^0 norm of solution $u = u(t,x)$ is bounded, it remains to show that

$$\|\partial_x u(t,.)\|_{C^0[0,1]} \stackrel{\text{def}}{=} \sum_{i=1}^{n} \|\partial_x u_i(t,.)\|_{C^0[0,1]} \le C_1(T_0), \quad \forall\, t \in [0,T]. \tag{2.1}$$

To this end, let

$$v_i = \frac{\partial u_i}{\partial x} \quad (i = 1, ..., n) \tag{2.2}$$

and

$$T_1 = \min_{1 \le i \le n} \ \min_{|u| \le C_0(T_0)} \frac{1}{|\lambda_i(u)|} > 0. \tag{2.3}$$

In the sequel, we denote by D_i $(i=1,2,...)$ various constants depending only on T_0. Without loss of generality, we may suppose that $D_1 \geq 1$. Then (2.1) follows if we prove the following a priori uniform estimate for all $t \in [0, T_1]$:

$$\begin{aligned} 1+\|v(t,.)\|_{C^0[0,1]} &\overset{\text{def}}{=} 1+\sum_{i=1}^{n}\|v_i(t,.)\|_{C^0[0,1]} \\ &\leq D_1\left(1+\sum_{i=1}^{n}\|v_i(0,.)\|_{C^0[0,1]}\right) \overset{\text{def}}{=} D_1\left(1+\|v(0,.)\|_{C^0[0,1]}\right). \end{aligned} \tag{2.4}$$

In fact, if (2.4) holds, we may take $u(T_1, x)$ as the new initial data on $t = T_1$ and repeat the same procedure. Since D_1 is independent of T, we have for all $t \in [T_1, 2T_1]$,

$$\begin{aligned} 1+\|v(t,.)\|_{C^0[0,1]} &\leq D_1\left(1+\|v(T_1,.)\|_{C^0[0,1]}\right) \\ &\leq D_1^2\left(1+\|v(0,.)\|_{C^0[0,1]}\right). \end{aligned} \tag{2.5}$$

Hence (2.5) holds for all $t \in [0, 2T_1]$. By induction, repeating this procedure at most $N \leq \left[\frac{T_0}{T_1}\right]+1$ times, we get

$$\begin{aligned} 1+\|v(t,.)\|_{C^0[0,1]} &\leq D_1^N\left(1+\|v(0,.)\|_{C^0[0,1]}\right) \\ &= D_1^N\left(1+\|\varphi'\|_{C^0[0,1]}\right), \end{aligned} \tag{2.6}$$

in which

$$\|\varphi'\|_{C^0[0,1]} \overset{\text{def}}{=} \sum_{i=1}^{n}\|\varphi_i'\|_{C^0[0,1]}.$$

This proves (2.1).

For simplicity, we denote by $v^i = (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)$ for $i = 1, ..., n$. Differentiating the system (1.1) with respect to x and using the condition (1.3), we get

$$\frac{\partial v_i}{\partial t}+\lambda_i(u)\frac{\partial v_i}{\partial x} = -a_i(u, v^i)v_i \quad (i=1,...,n), \tag{2.7}$$

where

$$a_i(u, v^i) = \sum_{l \neq i}\frac{\partial \lambda_i(u)}{\partial u_l}\, v_l. \tag{2.8}$$

Next, differentiating the boundary condition (1.5) with respect to t yields

$$x=0: \quad \frac{\partial u_k}{\partial t} = \frac{\partial g_k}{\partial t}+\sum_{j=1}^{m}\frac{\partial g_k}{\partial u_j}\frac{\partial u_j}{\partial t} \quad (k=m+1,...,n),$$

then, using the system (1.1) we get

$$x=0: \quad -\lambda_k(u)v_k = \frac{\partial g_k}{\partial t}-\sum_{j=1}^{m}\lambda_j(u)\frac{\partial g_k}{\partial u_j}\, v_j \quad (k=m+1,...,n).$$

Therefore, the boundary conditions of $v = (v_1, ..., v_n)$ on $x = 0$ can be expressed as

$$x = 0: \quad v_k = \frac{1}{\lambda_k(u)} \left(\sum_{j=1}^{m} \lambda_j(u) \frac{\partial g_k}{\partial u_j} v_j - \frac{\partial g_k}{\partial t} \right) \quad (k = m+1, ..., n). \tag{2.9}$$

Similarly, we have

$$x = 1: \quad v_j = \frac{1}{\lambda_j(u)} \left(\sum_{k=m+1}^{n} \lambda_k(u) \frac{\partial g_j}{\partial u_k} v_k - \frac{\partial g_j}{\partial t} \right) \quad (j = 1, ..., m). \tag{2.10}$$

Noting that for given u, (2.9)-(2.10) are linear boundary conditions for v, (2.9)-(2.10) can be rewritten as

$$x = 0: \quad v_k = \sum_{j=1}^{m} b_{kj}(t) v_j + c_k(t) \quad (k = m+1, ..., n), \tag{2.11}$$

$$x = 1: \quad v_j = \sum_{k=m+1}^{n} b_{jk}(t) v_k + c_j(t) \quad (j = 1, ..., m), \tag{2.12}$$

where b_{jk}, b_{kj}, c_j and c_k are continuous functions of $t \geq 0$. Since $j \neq k$, there is no ambiguity in the above notations. Finally, the initial condition of v is given by

$$t = 0: \quad v_i = \varphi_i'(x) \quad (i = 1, ..., n) \tag{2.13}$$

for $x \in [0, 1]$.

Let $(t, x) \in D(T_1)$. For $i = 1, ..., n$, the i-th characteristic $x = x_i(s)$ passing through (t, x) is defined by

$$x_i'(s) = \lambda_i(u(s, x_i(s))), \quad x_i(t) = x.$$

From the definition (2.3) of T_1, for $j = 1, ..., m$, it is easy to see that there are only two possibilities :

(i) The j-th characteristic $x = x_j(s)$ passing through (t, x) intersects the x-axis at a point $(0, \alpha_j)$. Then it follows from (2.7) and (2.13) that

$$\frac{dv_j(s, x_j(s))}{ds} = -a_j(s, x_j(s)) v_j$$

and

$$t = 0: \quad v_j = \varphi_j'(\alpha_j),$$

in which

$$a_i(s, x_i(s)) = a_i(u(s, x_i(s)), v^i(s, x_i(s))) \quad (i = 1, ..., n). \tag{2.14}$$

Hence, we have

$$v_j(t, x) = v_j(t, x_j(t)) = \varphi_j'(\alpha_j) \exp\left\{ - \int_0^t a_j(s, x_j(s))\, ds \right\}. \tag{2.15}$$

(ii) The j-th characteristic $x = x_j(s)$ passing through (t, x) intersects the line $x = 1$ at a point $(t_j, 1)$ and the k-th $(k > m)$ characteristic $x = x_k(s)$ passing through $(t_j, 1)$ intersects the x-axis at a point $(0, \beta_{jk})$. Similarly to (2.15), we have

$$v_j(t,x) = v_j(t, x_j(t)) = v_j(t_j, 1) \exp\left\{-\int_{t_j}^{t} a_j(s, x_j(s))\, ds\right\}, \tag{2.16}$$

where, noting (2.12),

$$v_j(t_j, 1) = \sum_{k=m+1}^{n} b_{jk}(t_j) v_k(t_j, 1) + c_j(t_j), \tag{2.17}$$

and similarly to (2.15) and noting (2.13), we have

$$v_k(t_j, 1) = \varphi_k'(\beta_{jk}) \exp\left\{-\int_0^{t_j} a_k(s, x_k(s))\, ds\right\} \quad (k = m+1, ..., n). \tag{2.18}$$

Similarly, for $k = m+1, ..., n$, there are only two possibilities :

(iii) The k-th characteristic $x = x_k(s)$ passing through (t, x) intersects the x-axis at a point $(0, \alpha_k)$. In this case

$$v_k(t,x) = v_k(t, x_k(t)) = \varphi_k'(\alpha_k) \exp\left\{-\int_0^{t} a_k(s, x_k(s))\, ds\right\}. \tag{2.19}$$

(iv) The k-th characteristic $x = x_k(s)$ passing through (t, x) intersects the t-axis at a point $(t_k, 0)$ and the j-th $(j \le m)$ characteristic $x = x_j(s)$ passing through $(t_k, 0)$ intersects the x-axis at a point $(0, \beta_{kj})$. Similarly, we have

$$v_k(t,x) = v_k(t, x_k(t)) = v_k(t_k, 0) \exp\left\{-\int_{t_k}^{t} a_k(s, x_k(s))\, ds\right\}, \tag{2.20}$$

where

$$v_k(t_k, 0) = \sum_{j=1}^{m} b_{kj}(t_k) v_j(t_k, 0) + c_k(t_k) \tag{2.21}$$

and

$$v_j(t_k, 0) = \varphi_j'(\beta_{kj}) \exp\left\{-\int_0^{t_k} a_j(s, x_j(s))\, ds\right\} \quad (j = 1, ..., m). \tag{2.22}$$

Since both the C^0 norm of $u = u(t,x)$ and the C^1 norm of φ_i $(i = 1, ..., n)$ are bounded, it is easy to see from (2.8) and (2.15)-(2.22) that in order to prove (2.4), it suffices to estimate the term

$$\int_{t_1}^{t} |v_l(s, x_i(s))|\, ds \quad (l \ne i;\ i, l = 1, ..., n), \tag{2.23}$$

for any given $t_1, t \in [0, T_1]$ such that $(s, x_i(s)) \in D(T_1)$ as $s \in [t_1, t]$, where $a_i(s, x_i(s))$ is defined by (2.8) and (2.14). This is the task of the next section.

3. Estimate of (2.23)

Lemma 1 *Under the assumption of Theorem 1, for all $i = 1, ..., n$ and for any given $t_1, t \in [0, T_1]$ such that $(s, x_i(s)) \in D(T_1)$ as $s \in [t_1, t]$, we have*

$$\int_{t_1}^{t} |v_l(s, x_i(s))| \, ds \le D_2 \quad (l \neq i). \tag{3.1}$$

Proof For any given $i = 1, ..., n$, let $x = x_i(s)$ be the i-th characteristic passing through any given point $(t, x) \in D(T_1)$. For any given $l \neq i$, by the definition (2.3) of T_1, there are only three possibilities for the l-th characteristic passing through the same point (t, x).

Case 1 : This l-th characteristic intersects the x-axis at a point $(0, y_{il}(t))$. We denote by $x = x_l(s, y_{il}(t))$ this l-th characteristic. Since these two characteristics coincide at the point $(t, x) = (t, x_i(t))$, we have

$$x_i(t) = x_l(t, y_{il}(t)) \quad (l \neq i). \tag{3.2}$$

Differentiating (3.2) with respect to t and using the definition of characteristics, we obtain

$$\lambda_i(u(t, x_i(t))) = \lambda_l(u(t, x_i(t))) + \frac{\partial x_l(t, y_{il}(t))}{\partial y} \, y'_{il}(t).$$

Then it follows from the strict hyperbolicity (1.2) that $y'_{il}(t)$ $(l \neq i)$ never vanishes, hence, $t \to y_{il}(t)$ is a strictly monotone function. From the l-th equation in (1.1), u_l is constant along any l-th characteristic, then

$$u_l(t, x_i(t)) = u_l(t, x_l(t, y_{il}(t))) = \varphi_l(y_{il}(t)),$$

which yields

$$\frac{du_l(t, x_i(t))}{dt} = \varphi'_l(y_{il}(t)) \, y'_{il}(t). \tag{3.3}$$

On the other hand, note the system (1.1), a direct computation gives

$$\frac{du_l(t, x_i(t))}{dt} = \left(\frac{\partial u_l}{\partial t} + \lambda_i(u) \frac{\partial u_l}{\partial x} \right)(t, x_i(t)) = \left((\lambda_i(u) - \lambda_l(u)) \frac{\partial u_l}{\partial x} \right)(t, x_i(t)). \tag{3.4}$$

Therefore,

$$v_l(t, x_i(t)) = \frac{\varphi'_l(y_{il}(t)) \, y'_{il}(t)}{(\lambda_i(u) - \lambda_l(u))(t, x_i(t))} \, . \tag{3.5}$$

This allows to obtain the following estimate for any given $t_1 \in [0, T_1]$ such that $(s, x_i(s)) \in D(T_1)$ as $s \in [t_1, t]$ and the l-th characteristics passing through $(s, x_i(s))$ as $s \in [t_1, t]$ intersect the x-axis :

$$\int_{t_1}^{t} | \, v_l(s, x_i(s)) \, | \, ds \quad = \quad \int_{t_1}^{t} \left| \frac{1}{(\lambda_i(u) - \lambda_l(u))(s, x_i(s))} \, \varphi'_l(y_{il}(s)) \, y'_{il}(s) \right| ds$$

$$
\begin{aligned}
&\le D_3 \left| \int_{t_1}^{t} y'_{il}(s)\, ds \right| \\
&= D_3 \,|\, y_{il}(t) - y_{il}(t_1) \,| \\
&\le D_3.
\end{aligned}
\tag{3.6}
$$

Here, we have used the fact that $0 \le y_{il}(t) \le 1$ and $t \to y_{il}(t)$ is a strictly monotone function.

Case 2 : This l-th characteristic intersects the line $x = 1$ at a point $(\tau_{il}(t), 1)$. We denote by $x = x_l(s, \tau_{il}(t))$ this l-th characteristic. Obviously, we have $l \le m$ and

$$
x_i(t) = x_l(t, \tau_{il}(t)) \quad (l \ne i,\ l \le m). \tag{3.7}
$$

Differentiating (3.7) with respect to t and using the definition of characteristics, we obtain

$$
\lambda_i(u(t, x_i(t))) = \lambda_l(u(t, x_i(t))) + \frac{\partial x_l(t, \tau_{il}(t))}{\partial y} \tau'_{il}(t).
$$

It follows again from the strict hyperbolicity (1.2) that $t \to \tau_{il}(t)$ is a strictly monotone function.

Let us look at the k-th $(k > m)$ characteristic passing through the point $(\tau_{il}(t), 1)$. By the definition of T_1, this k-th characteristic must intersect the x-axis at a point $(0, y_{ilk}(t))$ and can be denoted by $x = x_k(s, y_{ilk}(t))$. Now, we show that $t \to y_{ilk}(t)$ is still a strictly monotone function. Indeed, from the definition of $y_{ilk}(t)$, we may write

$$
y_{ilk}(t) = y^{(k)}(\xi)
$$

with $\xi = \tau_{il}(t)$. Since $t \to \tau_{il}(t)$ is a strictly monotone function, it remains to show that $\xi \to y^{(k)}(\xi)$ is also a strictly monotone function. Noting that the k-th characteristic $x = x_k(s, y_{ilk}(t))$ passes through the point $(\tau_{il}(t), 1)$, we have

$$
1 = x_k(\tau_{il}(t), y_{ilk}(t)) = x_k(\xi, y^{(k)}(\xi)).
$$

Differentiating the above relation with respect to ξ gives

$$
\begin{aligned}
0 &= \frac{\partial x_k(\xi, y^{(k)}(\xi))}{\partial \xi} + \frac{\partial x_k(\xi, y^{(k)}(\xi))}{\partial y} (y^{(k)}(\xi))' \\
&= \lambda_k(u(\xi, x_k(\xi, y^{(k)}(\xi)))) + \frac{\partial x_k(\xi, y^{(k)}(\xi))}{\partial y} (y^{(k)}(\xi))'.
\end{aligned}
$$

Since $\lambda_k(u) > 0$, we obtain $(y^{(k)}(\xi))' \ne 0$, which yields the strict monotonicity of $\xi \to y^{(k)}(\xi)$. This shows that $t \to y_{ilk}(t)$ is a strictly monotone function.

From $x_i(t) = x_l(t, \tau_{il}(t))$ and the fact that u_p is constant along any p-th characteristic $(p = 1, ..., n)$, we have

$$
u_l(t, x_i(t)) = u_l(t, x_l(t, \tau_{il}(t))) = u_l(\tau_{il}(t), 1).
$$

Then, it follows from the boundary condition (1.6) that

$$\begin{aligned} u_l(t, x_i(t)) &= g_l(\tau_{il}(t), u_{m+1}(\tau_{il}(t), 1), ..., u_n(\tau_{il}(t), 1)) \\ &= g_l(\tau_{il}(t), u_{m+1}(0, y_{il,m+1}(t)), ..., u_n(0, y_{iln}(t))) \\ &= g_l(\tau_{il}(t), \varphi_{m+1}(y_{il,m+1}(t)), ..., \varphi_n(y_{iln}(t))). \end{aligned}$$

Therefore,

$$\frac{du_l(t, x_i(t))}{dt} = \frac{\partial g_l}{\partial \tau} \tau'_{il}(t) + \sum_{m+1}^{n} \frac{\partial g_l}{\partial u_k} \varphi'_k(y_{ilk}(t))\, y'_{ilk}(t).$$

This together with (3.4) gives

$$v_l(t, x_i(t)) = \frac{1}{(\lambda_i(u) - \lambda_l(u))(t, x_i(t))} \left(\frac{\partial g_l}{\partial \tau} \tau'_{il}(t) + \sum_{m+1}^{n} \frac{\partial g_l}{\partial u_k} \varphi'_k(y_{ilk}(t))\, y'_{ilk}(t) \right).$$

Thus, for any given $t_1 \in [0, T_1]$ such that $(s, t_i(s)) \in D(T_1)$ as $s \in [t_1, t]$ and the l-th characteristics passing through $(s, x_i(s))$ as $s \in [t_1, t]$ intersect the line $x = 1$, we have

$$\begin{aligned} & \int_{t_1}^{t} |\, v_l(s, x_i(s))\,|\, ds \\ = \ & \int_{t_1}^{t} \Big| \frac{1}{(\lambda_i(u) - \lambda_l(u))(s, x_i(s))} \left(\frac{\partial g_l}{\partial \tau} \tau'_{il}(s) + \sum_{m+1}^{n} \frac{\partial g_l}{\partial u_k} \varphi'_k(y_{ilk}(s))\, y'_{ilk}(s) \right) \Big|\, ds \\ \leq \ & D_4 \left(\Big| \int_{t_1}^{t} \tau'_{il}(s)\, ds \Big| + \sum_{k=m+1}^{n} \Big| \int_{t_1}^{t} y'_{ilk}(s)\, ds \Big| \right) \\ = \ & D_4 \left(|\, \tau_{il}(t) - \tau_{il}(t_1)\,| + \sum_{k=m+1}^{n} |\, y_{ilk}(t) - y_{ilk}(t_1)\,| \right) \\ \leq \ & D_4(T_1 + n - m). \end{aligned}$$

Here, we have used the monotonicity of $\tau_{il}(t)$ and $y_{ilk}(t)$ and the properties $0 \leq \tau_{il}(t) \leq t \leq T_1$ and $0 \leq y_{ilk}(t) \leq 1$.

Case 3 : The l-th characteristic intersects the line $x = 0$ at a point. Obviously, we have $l > m$. Similar results can be obtained as in Case 2.

Thus, the proof of Lemma 1 is complete.

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