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EXPONENTIAL ATTRACTOR FOR A CLASS OF NONCLASSICAL DIFFUSION EQUATION*

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Abstract In this paper, we consider the asymptotic behavior of solutions for a class of nonclassical diffusion equation. We show the squeezing property and the existence of exponential attractor for this equation. We also make the estimates on its fractal dimension and exponential attraction.

Key Words Nonclassical; diffusion equation; squeezing property; exponential attractor

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1. Introduction

The nonclassical diffusion equations

$$u_t - \nu\Delta u_t - \sum_{i=1}^{n}[\sigma(u_{x_i})]_{x_i} + g(u) = f(x,t) \tag{1.1}$$

arise in many different areas of mathematics and physics. They have been used, for instance, to model thermodynamics processes[1], [2], fluid flow in fissured rock [3],consolidation of clay [4], and shear in second order fluids [5-7]. For the physical interpretation of "$\nu\Delta u_t$", we refer to [1-3]. The equations of (1.1) with a one time derivative appearing in the highest order term are called pseudo-parabolic or Sobolev-Galpern equations [8-12]. Aifantis [13] proposed a general frame for establishing the equations. The existence and uniqueness, and regularity of solutions for the nonclassical diffusion equations have been investigated by many authors, such as Showalter [14], Davis [15],

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Quarteroni [16], Karch [17], Shi *et al* [18], Liu and Wang [19], Liu,Wan and Lu [20], Li *et.al* [21] and their references therein.

In this paper,we consider the following initial boundary value problem of the non-classical diffusion equation:

$$u_t - \nu\Delta u_t - \lambda\Delta u + g(u) = f(x), (x,t) \in \Omega \times R^+, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times R^+. \tag{1.4}$$

where λ is a positive constant, $g : R \to R$ is a smooth function, and $f(x) \in L^2(\Omega), \Omega \subset R^n$ is a smooth bounded open set, $\partial\Omega$ is the boundary of Ω. The existence of the compact global attractor in $H_0^1(\Omega)$ for the equation (1.2) has been established by Li *et al* [21]. Our aim of this paper is to show the existence of finite dimensional exponential attractor for this equation.

The outline of this paper is as follows: in Section 2, we state some basic results on the existence of exponential attractors and recall some known results concerning the existence and uniqueness of solutions. Section 3 contains our main results; we first establish the Lipschitz continuity of the dynamical system $S(t)$ associated with Eq.(1.2), then we prove that the semigroup $S(t)$ satisfies the squeezing property and deduce the existence of the exponential attractor.

Throughout this paper, we denote by $\|\cdot\|$ the norm of $H = L^2(\Omega)$ with the usual inner product $(\cdot,\cdot)$. We also use $\|\cdot\|_p$ for the norm of $L^p(\Omega)$ for $1 \le p \le \infty (\|\cdot\|_2 = \|\cdot\|)$. Generally, $\|\cdot\|_X$ denotes the norm of Banach space X.

For convenience, we put $\nu \equiv 1$ in (1.2). In the sequel, we always assume that g satisfies conditions:

$(G_1) g \in C^1(R), \exists \mu \in R$, such that $\varliminf_{|s|\to\infty} \dfrac{g(s)}{s} \ge \mu$;

$(G_2) \exists c, \gamma \ge 0$, such that

$$|g'(s)| \le c(1 + |s|^\gamma),$$

where $0 \le \gamma < \infty$(as $n = 1,2), \gamma \le \dfrac{2n}{n-2}$(as $n \ge 3$).

2. Preliminaries

Let $D(A), V$ be two Hilbert spaces, $D(A)$ be dense in V and compactly imbedded into V.

We study

$$\frac{du}{dt} + Au + g(u) = f(x), \quad t > 0, \quad x \in \Omega. \tag{2.1}$$

$$u(0) = u_0, \quad x \in \Omega \tag{2.2}$$

$$u\mid_{\partial\Omega} = 0. \tag{2.3}$$

where Ω is a bounded open set in $R^n, \partial\Omega$ is smooth. A is a positive self adjoint operator with a compact inverse. Let $\{w_n, n = 1, 2, \cdots\}$ denote the complete set of eigenvectors of A, the corresponding eigenvalues are

$$0 < \lambda_1 < \lambda_2 < \cdots \lambda_n \nearrow +\infty \tag{2.4}$$

We assume that the nonlinear semigroup $S(t)$ generated by (2.1)–(2.3) possesses a $(D(A), V)$ type compact attractor, namely, there exists a compact set $\mathcal{A}$ in $V, \mathcal{A}$ attracts all bounded subsets in $D(A)$ in the topology of V and it is invariant under the action of $S(t)$.

Definition 1 *A compact set $M \subset V$ is called an exponential attractor of $(D(A), V)$ type for $(S(t), B)$ if $\mathcal{A} \subseteq M \subseteq B$ and*

1. *$S(t)M \subseteq M, \forall t \geq 0$,*
2. *M has finite fractal dimension, $d_F(M) < +\infty$,*
3. *there exist positive constants c_0, c_1 such that*

$$\mathrm{dist}_V(S(t)B, M) \leq c_0 e^{-c_1 t}, \quad \forall t > 0,$$

where $\mathrm{dist}_V(A, B) = \sup_{x\in A} \inf_{y \in B} \|x - y\|_V$ *, B is a positively invariant set for $S(t)$ in V.*

Definition 2 *If for every $\delta \in (0, \frac{1}{8})$, there exists a time $t^* > 0$, an integer $N_0 \geq 1$, and an orthogonal projection P_{N_0} of rank equal to N_0 such that for every u and v in B, either*

$$\|S(t_*)u - S(t_*)v\|_V \leq \delta \|u - v\|_V \tag{2.5}$$

or

$$\|Q_{N_0}(S(t_*)u - S(t_*)v)\|_V \leq \|P_{N_0}(S(t_*)u - S(t_*)v)\|_V \tag{2.6}$$

then we call $S(t)$ is squeezing in B, where $Q_{N_0} = I - P_{N_0}$.

Theorem 1[22] *Suppose (2.1)–(2.3) satisfies the following conditions:*

1. *there exists a $(D(A), V)$ type compact attractor $\mathcal{A}$,*
2. *there exists a compact set B in V which is positively invariant for $S(t)$,*
3. *$S(t)$ is squeezing and Lipschitz continuous, that is there exists a bounded function $l(t)$ such that $\|S(t)u - S(t)v\|_V \leq l(t) \|u - v\|_V$ for every u, v in B.*

Then (2.1)–(2.3) admits a $(D(A), V)$ type exponential attractor M for $(S(t), B)$ and

$$M = \bigcup_{0 \leq t \leq t_*} S(t) M_* \tag{2.7}$$

where

$$M_* = \mathcal{A} \cup \left(\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} S(t_*)^j (E^{(k)}) \right). \tag{2.8}$$

Moreover,

$$d_F(M) \le 1 + N_0 \log(1+\sqrt{2}l/\delta)/\log\frac{1}{\theta} \tag{2.9}$$

$$\mathrm{dist}_V(S(t)B, M) \le c_0 e^{-c_1 t} \tag{2.10}$$

where $\theta, N_0, E^{(k)}$ *are defined as in [23],* l *is the Lipschitz constant for* $S(t_*)$ *in* B, t_* *is a positive constant.*

We rewrite the equation(1.2) as an abstract differential equation in H:

$$\frac{du}{dt} + \frac{d}{dt}Au + \lambda Au + g(u) = f(x), \tag{2.11}$$

where $A = -\Delta$ is unbounded self-adjoint operator with domain:

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega)$$

Let $V = H_0^1(\Omega)$. The norm of u in $D(A)$ and V is defined by

$$\|u\|_V = (\|u\|^2 + \|\nabla u\|^2)^{1/2},$$
$$\|u\|_{D(A)} = (\|u\|_V^2 + \|\Delta u\|^2)^{1/2}.$$

In [21], the authors have established the following results:

Proposition 1 *Suppose* g *satisfies* $(G_1), (G_2)$ *and* $\mu \ge -\lambda\lambda_1, \lambda_1$ *is the first eigenvalue of the operator* $-\Delta$ *with boundary condition (1.3),* $f \in H, u_0 \in V$*, then the problem (1.2)–(1.4) possesses a unique global solution* $u \in C([0,\infty); V)$*. If* $u_0 \in D(A)$*, then* $u \in L^\infty(0,\infty, D(A))$*.*

This shows that the system (1.2)–(1.4) defines a solution semigroup $S(t)$ which maps V(resp $D(A)$) into V(resp $D(A)$) such that $S(t)u_0 = u(t)$, the solution of the problem (1.2)–(1.4).

Proposition 2 *Under the same conditions of Proposition 1, there exist closed absorbing sets* B_0, B_1 *in* $V, D(A)$ *respectively, and*

$$B_0 = \{u \in V : \|u\|_V \le E_1\}, \tag{2.12}$$

$$B_1 = \{u \in D(A) : \|u\|_{D(A)} \le E_2\}. \tag{2.13}$$

where E_1, E_2 *are constants depending only on the data* $(\lambda, \mu, \gamma, f, \Omega)$ *and the upper bound of* $\|u_0\|_{H^1}$*.*

Denote $\tilde{B}_0 = \{u \in V : \|u\|_V \le E_1, and \, \|u\|_{D(A)} \le E_2\}$. By Proposition 1 of [22], we see that

$$B = \overline{\bigcup_{0 \le t \le t_0(\tilde{B}_0)} S(t)\tilde{B}_0}. \tag{2.14}$$

is a compact positively invariant set in V and is absorbing for all bounded subsets in $D(A)$.

According to Proposition 2 of [22], we also know that the nonlinear semigroup $S(t)$ defined by (1.2)-(1.4) possesses a compact attractor $\mathcal{A}$ of $(D(A), V)-$type, namely, there exists a compact set $\mathcal{A}$ in V, and $\mathcal{A}$ attracts all bounded subsets in $D(A)$ and is invariant under the action of $S(t)$.

In order to establish the existence of the exponential attractor,according to Theorem 1, we need only to show the Lipschitz continuity and the squeezing property of the dynamical system $S(t)$ in B. That is what we proceed to do in the following section.

3. The Exponential Attractor

In this section, we first establish the Lipschitz continuity of the dynamical system $S(t)$ associated with the equation (1.2) in B. Then we show the squeezing property for semigroup $S(t)$ and deduce the existence of the finite fractal dimensional exponential attractor.

To show the Lipschitz continuity of the dynamical system $S(t)$, we now establish Lemma 1.

Lemma 1 *Suppose g satisfies (G_2),then there exists a constant $C_g > 0$ such that*

$$\|g(u) - g(v)\| \leq c_g(1 + \|u\|_V^\gamma + \|v\|_V^\gamma)\,\|u - v\|_V \quad \forall u, v \in V. \tag{3.1}$$

Proof This Lemma can be proved in the same way as that in Lemma 3.1 of Chapter IV in [24]. For the sake of completeness, we give a sketch of the proof of the lemma. By the mean value theorem and the condition (G_2), we have

$$\begin{aligned}\|g(u) - g(v)\|^2 &= \int_\Omega (g(u) - g(v))^2 dx \\ &\leq c^2 \int_\Omega (1 + |\theta u + (1-\theta)v|^\gamma)^2 (u - v)^2 dx \\ &\leq \tilde{c} \int_\Omega (1 + |u|^\gamma + |v|^\gamma)^2 (u - v)^2 dx.\end{aligned} \tag{3.2}$$

By Sobolev embedding,

$$H_0^1(\Omega) \subset L^q(\Omega), \quad \forall q < \infty \quad if \quad n = 2, q = 2n/(n-2), \forall n \geq 3 (= 6 \quad if \quad n = 3). \tag{3.3}$$

For $n \leq 3$, we use the Hölder inequality and majorize the last expression (3.2) by

$$\begin{aligned}&\tilde{c}\Big(\int_\Omega (1 + |u|^\gamma + |v|^\gamma)^3 dx\Big)^{2/3}\Big(\int_\Omega (u - v)^6 dx\Big)^{1/3} \\ &\qquad \leq \hat{c}(1 + \|u\|_V + \|v\|_V)^{2\gamma}\,\|u - v\|_V^2 \quad (\text{by}(G_2)\text{and}(3.3)).\end{aligned} \tag{3.4}$$

For $n \geq 4$, we use the Hölder inequality and majorize (3.2) by

$$\begin{aligned}&\tilde{c}\Big(\int_\Omega (1 + |u|^\gamma + |v|^\gamma)^n dx\Big)^{2/n}\Big(\int_\Omega (u - v)^{2n/(n-2)} dx\Big)^{(n-2)/n} \\ &\qquad \leq \hat{c}(1 + \|u\|_V + \|v\|_V)^{2\gamma}\,\|u - v\|_V^2 \quad (\text{by}(G_2)\text{and}(3.3))\end{aligned} \tag{3.5}$$

and (3.1) follows in all cases.

Lemma 2 *Assume that g satisfies $(G_1),(G_2)$ and $\mu \geq -\lambda\lambda_1, \lambda_1$ is the first eigenvalue of the operator $-\Delta$ with boundary condition (1.3), $f \in H, u_1(t), u_2(t)$ are two solutions of the problem (1.2)-(1.4) with initial values $u_{01}, u_{02} \in B$. Then we have*

$$\|u_1(t) - u_2(t)\|_V \leq e^{C_1 t} \|u_{01} - u_{02}\|_V .$$

where C_1 is a constant depending only on the data $(\lambda, \mu, \gamma, f, \Omega)$.

Proof Let $w(t) = u_1(t) - u_2(t)$. Then by (1.2) we find that

$$w_t - \Delta w_t - \lambda\Delta w + g(u_1) - g(u_2) = 0, \tag{3.6}$$

$$w(0) = u_{01} - u_{02}. \tag{3.7}$$

Taking the inner product of (3.6) with w in H, we infer that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_V^2 + \lambda\|\nabla w\|^2 &\leq |(g(u_1) - g(u_2), w)| \leq \|g(u_1) - g(u_2)\|\,\|w\| \\ &\leq c_g(1 + \|u_1\|_V^\gamma + \|u_2\|_V^\gamma)\|w\|_V\|w\| \quad (\text{by}(3.1)) \\ &\leq C_1\|w\|_V^2 \quad (u_1, u_2 \in B). \end{aligned}\tag{3.8}$$

From the Gronwall lemma we see that

$$\|w(t)\|^2 + \|\nabla w(t)\|^2 \leq e^{2C_1 t}(\|w(0)\|^2 + \|\nabla w(0)\|^2),$$

this concluds the proof.

It is well known that the operator $A = -\Delta$ from $D(A)$ to H with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ is an unbounded self-adjoint positive operator and the inverse A^{-1} is compact. And thus there exists an orthonormal basis $\{w_i\}_{i=1}^\infty$ of H consisting of eigenvectors of A such that

$$Aw_i = \lambda_i w_i, \quad 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i \to +\infty \quad \text{as} \quad i \to \infty.$$

For all N, denote by $P = P_N : H \to \text{span}\,\{w_1, w_2, \cdots, w_N\}$ the orthogonal projection, and write $Q = Q_N = I - P_N$.

In the following, we will use:

$$\left\|A^{\frac{1}{2}}u\right\| \geq \lambda_{N+1}^{\frac{1}{2}}\|u\|, \quad u \in Q_N H, \tag{3.9}$$

$$\left\|A^{\frac{1}{2}}u\right\| = \|\nabla u\|, \qquad u \in D(A^{1/2}), \tag{3.10}$$

$$\|AQ_N u\| = \|Q_N A u\| \leq \|Au\|, \qquad u \in D(A). \tag{3.11}$$

Before proving the squeezing property, we state

Lemma 3 *Assume that $u_1(t)$ and $u_2(t)$ are two solutions of the problem (1.2)-(1.4) with initial values $u_{01}, u_{02} \in B$. Then $q = Q_N(u_1 - u_2)$ satisfies that*

$$\|q(t)\|^2 + \|\nabla q(t)\|^2 \leq (e^{-C_3 t} + \frac{2C_2\lambda_{N+1}^{-1}}{C_3 + 2C_1}e^{2C_1 t})\|u_{01} - u_{02}\|_V^2 .$$

where C_1 is the constant in Lemma 2, C_2, and C_3 depend only on the data $(\lambda, \mu, \gamma, f, \Omega)$.

Proof Let $w(t) = u_1(t) - u_2(t), q(t) = Q_N w(t)$. Applying Q_N to (3.6) we obtain

$$\frac{\partial q(t)}{\partial t} - \Delta \frac{\partial q(t)}{\partial t} - \lambda \Delta q(t) + Q_N(g(u_1) - g(u_2)) = 0. \tag{3.12}$$

Taking the inner product of (3.12) with q in H, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|q\|^2 + \|\nabla q\|^2) + \lambda \|\nabla q\|^2 &= \int_\Omega Q_N(g(u_1) - g(u_2))q dx \\ &= \int_\Omega (g(u_1) - g(u_2))q dx.\end{aligned} \tag{3.13}$$

According to Lemma 1, Hölder inequality and Young's inequality, we infer that

$$\begin{aligned}\left|\int_\Omega (g(u_1) - g(u_2))q dx\right| &\le \|g(u_1) - g(u_2)\| \, \|q\| \\ &\le c_g(1 + \|u_1\|_V^\gamma + \|u_2\|_V^\gamma) \|w\|_V \, \lambda_{N+1}^{-1/2} \|\nabla q\| \quad (\text{by}(3.9), (3.10)) \\ &\le C_2 \lambda_{N+1}^{-1} \|w\|_V^2 + \frac{\lambda}{2} \|\nabla q\|^2 \quad (\text{by} u_1, u_2 \in B).\end{aligned} \tag{3.14}$$

where C_2 depends only on the data $(\lambda, \mu, \gamma, f, \Omega)$, Combining with (3.13) and (3.14) we get

$$\frac{d}{dt}(\|q\|^2 + \|\nabla q\|^2) + \lambda \|\nabla q\|^2 \le 2C_2 \lambda_{N+1}^{-1} \|w\|_V^2. \tag{3.15}$$

Due to

$$\begin{aligned}\lambda \|\nabla q\|^2 &= \frac{1}{2}\lambda \|\nabla q\|^2 + \frac{1}{2}\lambda \|\nabla q\|^2 \\ &\ge \frac{1}{2}\lambda \|\nabla q\|^2 + \frac{1}{2}\lambda\lambda_{N+1} \|q\|^2 \quad (\text{by}(3.9), (3.10)) \\ &\ge C_3(\|q\|^2 + \|\nabla q\|^2),\end{aligned} \tag{3.16}$$

where $C_3 = \min\{\frac{1}{2}\lambda, \frac{1}{2}\lambda\lambda_{N+1}\}$, and by Theorem 1, (3.15) and (3.16), we find that

$$\begin{aligned}&\frac{d}{dt}(\|q\|^2 + \|\nabla q\|^2) + C_3(\|q\|^2 + \|\nabla q\|^2) \\ &\quad \le 2C_2 \lambda_{N+1}^{-1} \|w\|_V^2 \le 2C_2 \lambda_{N+1}^{-1} e^{2C_1 t} \|w(0)\|_V^2.\end{aligned}$$

Then it follows from Gronwall Lemma that

$$\begin{aligned}\|q(t)\|^2 + \|\nabla q(t)\|^2 &\le (\|q(0)\|^2 + \|\nabla q(0)\|^2) e^{-C_3 t} + \frac{2C_2 \lambda_{N+1}^{-1}}{2C_1 + C_3} e^{2C_1 t} \|w(0)\|_V^2 \\ &\le (\|w(0)\|^2 + \|\nabla w(0)\|^2) e^{-C_3 t} + \frac{2C_2 \lambda_{N+1}^{-1}}{2C_1 + C_3} e^{2C_1 t} \|w(0)\|_V^2 \\ &\le (e^{-C_3 t} + \frac{2C_2 \lambda_{N+1}^{-1}}{2C_1 + C_3} e^{2C_1 t})(\|w(0)\|^2 + \|\nabla w(0)\|^2).\end{aligned} \tag{3.17}$$

which concludes the proof of Lemma 3.

We now show the squeezing property.

Theorem 2 *Assume that g satisfies $(G_1), (G_2)$ and $\mu \geq -\lambda\lambda_1, \lambda_1$ is the first eigenvalue of the operator $-\Delta$ with boundary condition (1.3), $f \in H$, then the semigroup $S(t)$ associated with problem (1.2)–(1.4) is squeezing in B.*

Proof Let $t_* > 0$ fixed, and $w(t) = u_1(t) - u_2(t)$. we now assume that

$$\|P_N w(t_*)\|_V \leq \|Q_N w(t_*)\|_V . \tag{3.18}$$

Then we can deduce

$$\begin{aligned}
\|w(t_*)\|_V^2 &= \|w(t_*)\|^2 + \|\nabla w(t_*)\|^2 \\
&= \|P_N w(t_*)\|^2 + \|Q_N w(t_*)\|^2 + \|\nabla P_N w(t_*)\|^2 + \|\nabla Q_N w(t_*)\|^2 \\
&= \|P_N w(t_*)\|_V^2 + \|Q_N w(t_*)\|_V^2 \\
&\leq 2 \|Q_N w(t_*)\|_V^2 \\
&\leq 2(e^{-C_3 t_*} + \frac{2C_2 \lambda_{N+1}^{-1}}{2C_1 + C_3} e^{2C_1 t_*}) \|w(0)\|_V^2 .
\end{aligned} \tag{3.19}$$

Let t_* be large enough so that

$$e^{-C_3 t_*} < \frac{1}{256}. \tag{3.20}$$

Next we choose N_0 large enough so that

$$\frac{2C_2 \lambda_{N_0+1}^{-1}}{2C_1 + C_3} e^{2C_1 t_*} \leq \frac{1}{256}. \tag{3.21}$$

From (3.19)–(3.21) we obtain that

$$\|w(t_*)\|_V^2 \leq \frac{1}{64} \|w(0)\|_V^2 \tag{3.22}$$

(3.22) completes the proof of Theorem 2.

Now we conclude this paper by giving our main result:

Theorem 3 *Assume that g satisfies $(G_1), (G_2)$ and $\mu \geq -\lambda\lambda_1, \lambda_1$ is the first eigenvalue of the operator $-\Delta$ with boundary condition (1.3), $f \in H$, there exists N_0 large enough such that*

$$\lambda_{N_0+1} \geq 512 C_2 (2C_1 + C_3)^{-1} 2^{16 C_1 / C_3}.$$

Then for the nonlinear semigroup $S(t)$ defined in (1.2), $(S(t)_{t \geq 0}, B)$ admits an exponential attractor M of $(D(A), V)$ type and

$$d_F(M) \leq 1 + 16 c_0 \sqrt{2C_2 (2C_1 + C_3)^{-1}} 2^{8\sqrt{C_1/C_3}}, \tag{3.23}$$

$$\mathrm{dist}_V(S(t)B, M) \leq c_1 e^{-c_2 t}. \tag{3.24}$$

where c_0, c_1, c_2, C_1, C_2, *and* C_3 *are constants independent of the solution of the equation.*

Proof By Theorem 1 and Theorem 2, we know $(S(t)_{t\geq 0}, B)$ admits an exponential attractor M of $(D(A), V)$ type. We note that $\lambda_N \sim N^2$ and let δ, θ be fixed, thus obtain the estimate (3.23) of fractal dimension. Theorem 3 therefore is proved.

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