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GENERALIZED QUASILINEARIZATION METHOD FOR A CLASS OF SEMILINEAR ELLIPTIC SYSTEMS*

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Abstract In this paper, the method of generalized quasilinearization is extended to a class of semilinear elliptic systems, and the sequences which are the solutions of linear differential equations that converge to the unique solution of the given semilinear elliptic system are obtained.
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1. Introduction

The method of quasilinearization was disscussed by Bellman [1], Lakshmikantham and Leela [2] and Lakshmikantham and Vatsala [3,4]. In this paper, it is extended to a class of semilinear elliptic systems.

Consider the following semilinear elliptic system

$$L_i u_i = f_i(x, U), \quad x \in \Omega, \tag{1}$$

$$B_i u_i = \varphi_i(x), \quad x \in \partial\Omega, \tag{2}$$

where $U \equiv (u_1, \cdots, u_n)$, $\Omega \subset R^N$ is a bounded domain with the boundary $\partial\Omega$, and L_i and B_i are elliptic and boundary operators given, respectively, by

$$L_i u_i \equiv -\sum_{j,k=1}^{N} a_{jk}^{(i)}(x)\frac{\partial^2 u_i}{\partial x_j \partial x_k} + \sum_{j=1}^{N} b_j^{(i)}(x)\frac{\partial u_i}{\partial x_j} + c^{(i)} u_i,$$

$$B_i u_i \equiv \alpha_i(x)\frac{\partial u_i}{\partial \nu} + \beta_i(x) u_i,$$

where ν is the unit outer normal vector on $\partial\Omega$, and $\alpha_i(x)$, $\beta_i(x) \in C^{1,\alpha}[\partial\Omega]$, $\beta_i(x) > 0$ and $\partial\Omega$ belongs to the $C^{2,\alpha}$. Moreover, it is assumed that for each $i = 1, \cdots, n$, L_i is uniformly elliptic in Ω and $a_{jk}^{(i)}$, $b_j^{(i)}$, $c^{(i)} \in C^\alpha[\bar\Omega]$, $c^{(i)}(x) \geq 0$, $\varphi_i \in C^{1,\alpha}(\bar\Omega)$, $f_i \in C^\alpha(\bar\Omega \times R^n)$ in Ω.

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2. Comparison Lemmas

We first give the following comparison result.

Lemma 1 *Let $W, V \in C^2(\bar{\Omega})$ be lower and upper solutions of (1)-(2), that is, W, V satisfy*

$$L_i w_i \le f_i(x, Z) \ in\ \Omega, \quad B_i w_i \le \varphi_i(x) \ on\, \partial\Omega,\ for\, Z \in < W, V > \ [5] \ with\, z_i = w_i,$$
$$L_i v_i \ge f_i(x, Z) \ in\, \Omega, \quad B_i v_i \ge \varphi_i(x) \ on\ \partial\Omega,\ for\ Z \in < W, V > \ with\ z_i = v_i.$$

Suppose further that

$$|f_i(x, \tilde{U}) - f_i(x, \tilde{V})| \le K_i |\tilde{U} - \tilde{V}|, \quad \tilde{U}, \tilde{V} \in < W, V >,$$

where $|\tilde{U} - \tilde{V}| = |\tilde{u}_1 - \tilde{v}_1| + \cdots + |\tilde{u}_n - \tilde{v}_n|, \quad c^{(i)}(x) > K_i \ge 0$.

Then $W \le V$ [5], namely, $w_i \le v_i$, $i = 1, \cdots, n$, $x \in \bar{\Omega}$.

Proof Let $Y(x) = W(x) - V(x)$, namely, $y_i(x) = w_i(x) - v_i(x)$, $i = 1, \cdots, n$. If $y_i(x) \le 0$ is not true in Ω, then there exists an $\varepsilon > 0$ and $x_0 \in \bar{\Omega}$ such that

$$w_i(x_0) = v_i(x_0) + \varepsilon, \quad w_i(x) \le v_i(x) + \varepsilon, \quad x \in \bar{\Omega}.$$

If $x_0 \in \partial\Omega$, then $\dfrac{\partial w_i(x_0)}{\partial \nu} \ge \dfrac{\partial v_i(x_0)}{\partial \nu}$ and hence we can get

$$\begin{aligned} B w_i(x_0) &= \alpha_i(x_0) \frac{\partial w_i(x_0)}{\partial \nu} + \beta_i(x_0) w_i(x_0) \\ &\ge \alpha_i(x_0) \frac{\partial v_i(x_0)}{\partial \nu} + \beta_i(x_0)[v_i(x_0) + \varepsilon] > B v_i(x_0), \end{aligned}$$

which is a contradiction.

If $x_0 \in \Omega$, then $\dfrac{\partial w_i(x_0)}{\partial x_j} = \dfrac{\partial v_i(x_0)}{\partial x_j}$, $\sum\limits_{j,k=1}^{N} \left(\dfrac{\partial^2 w_i(x_0)}{\partial x_j \partial x_k} - \dfrac{\partial^2 v_i(x_0)}{\partial x_j \partial x_k} \right) \lambda_j \lambda_k \le 0$, where λ_j, λ_k are positive constants. Then by using the assumptions it follows that

$$\begin{aligned} f_i(x_0, W(x_0)) &\ge L_i w_i(x_0) \ge L_i \left[v_i(x_0) + \varepsilon \right] \ge f_i(x_0, V(x_0)) + c^{(i)}(x_0)\varepsilon \\ &\ge f_i(x_0, W(x_0)) + \left[c^{(i)}(x_0) - K_i \right] \varepsilon, \end{aligned}$$

which contracts with $c^{(i)}(x) > K_i$. Hence the claim is true and the proof is complete.

Evidently one has the following corollary to Lemma 1.

Corollary 2 *For any $P = (p_1, \cdots, p_n)$ with $p_i \in C^2(\Omega)$ satisfying*

$$\begin{aligned} & L_i^{(c_0)} p_i \equiv - \sum_{j,k=1}^{N} a_{jk}^{(i)}(x) \frac{\partial^2 p_i}{\partial x_j \partial x_k} + \sum_{j=1}^{N} b_j^{(i)}(x) \frac{\partial p_i}{\partial x_j} + c_0^{(i)} p_i \le 0, \quad x \in \Omega, \\ & B_i p_i \le 0, \quad x \in \partial\Omega, \end{aligned} \tag{3}$$

where $c_0^{(i)}(x) > 0$. Then one has $p_i(x) \le 0$ in $\bar{\Omega}$.

The following standard existence and uniqueness result for the linear elliptic system is needed.

Lemma 3 *Consider the boundary value problem for the linear elliptic system*

$$L_iu_i = h_i(x), \quad x \in \Omega, \tag{4}$$
$$B_iu_i = \varphi_i(x), \quad x \in \partial\Omega, \tag{5}$$

where $h_i \in C^{\alpha}[\bar{\Omega}]$, $i = 1, \cdots, n$. *Then for each* $i = 1, \cdots, n$ *(4)-(5) has a unique solution* $u_i \in C^{2,\alpha}[\bar{\Omega}]$.

3. The Main Result

Consider the following semilinear elliptic system

$$L_iu_i = f_i(x, U) + g_i(x, U), \quad x \in \Omega, \tag{6}$$
$$B_iu_i = \varphi_i(x), \quad x \in \partial\Omega, \tag{7}$$

where $f_i, g_i, \dfrac{\partial f_i}{\partial u_i}, \dfrac{\partial g_i}{\partial u_i} \in C^{\alpha}(\bar{\Omega} \times R^N)$, $i = 1, 2, \cdots, n$.

Theorem 4 *Suppose that*

(I) $W^{(0)}(x) \leq V^{(0)}(x)$ *with* $w_i^{(0)}(x), v_i^{(0)}(x) \in C^2(\bar{\Omega})$ *in* Ω *satisfing for each* $i = 1, 2, \cdots, n$,

$$L_iw_i^{(0)} \leq f_i(x, Z) + g_i(x, Z) \text{ in } \Omega, \quad B_iw_i^{(0)} \leq \varphi_i \text{ on } \partial\Omega,$$
$$L_iv_i^{(0)} \geq f_i(x, \bar{Z}) + g_i(x, \bar{Z}) \text{ in } \Omega, \quad B_iv_i^{(0)} \geq \varphi_i \text{ on } \partial\Omega,$$

where $Z, \bar{Z} \in < W^{(0)}, V^{(0)} >$ *with* $z_i = w_i^{(0)}$, $\bar{z}_i = v_i^{(0)}$.

(II) *For each* $i = 1, 2, \cdots, n$, $\dfrac{\partial^2 f_i}{\partial u_j^2}(x, U) \geq 0$, $\dfrac{\partial^2 g_i}{\partial u_j^2}(x, U) \leq 0$, $j = 1, 2, \cdots, n$; $0 < c \leq c^{(i)}(x) - \Big[\dfrac{\partial f_i}{\partial u_i}(x, V^{(0)}) + \dfrac{\partial g_i}{\partial u_i}(x, W^{(0)})\Big]$.

Then in the problems (6)-(7) exist a unique solution $U = (u_1, \cdots, u_n)$ *and monotone sequences* $\{w_i^{(k)}\}, \{v_i^{(k)}\} \in C^{2,\alpha}(\bar{\Omega})$, *such that* $w_i^{(k)} \to u_i$, $v_i^{(k)} \to u_i$ *in* $C^2(\bar{\Omega})$ *with* $w_i^{(0)} \leq u_i \leq v_i^{(0)}$, $i = 1, 2, \cdots, n$.

Proof Consider the following linear elliptic boundary value problems for each $s = 1, 2, \cdots$

$$L_iw_i^{(s+1)} = F_i(x, W^{(s+1)}; W^{(s)}, V^{(s)}) \text{ in } \Omega, \quad B_iw_i^{(s+1)} = \varphi_i \text{ on } \partial\Omega;$$
$$L_iv_i^{(s+1)} = F_i(x, V^{(s+1)}; W^{(s)}, V^{(s)}) \text{ in } \Omega, \quad B_iv_i^{(s+1)} = \varphi_i \text{ on } \partial\Omega,$$

where

$$\begin{aligned}F_i(x,W;W^{(s)},V^{(s)}) &= f_i(x,W^{(s)})+g_i(x,W^{(s)})\\&\quad+\frac{\partial f_i}{\partial u_i}(x,W^{(s)})(w_i-w_i^{(s)})+\frac{\partial g_i}{\partial u_i}(x,V^{(s)})(w_i-w_i^{(s)}),\\G_i(x,W;W^{(s)},V^{(s)}) &= f_i(x,V^{(s)})+g_i(x,V^{(s)})\\&\quad+\frac{\partial f_i}{\partial u_i}(x,V^{(s)})(w_i-v_i^{(s)})+\frac{\partial g_i}{\partial u_i}(x,W^{(s)})(w_i-v_i^{(s)}).\end{aligned}$$

Since for each $i,j=1,2,\cdots,n$, $\frac{\partial f_i}{\partial u_j}(x,U)$ is nondecreasing and $\frac{\partial g_i}{\partial u_j}(x,U)$ is nonincreasing in u_i for each $x\in\bar{\Omega}$, one finds that for any $\eta=(\eta_1,\cdots,\eta_n)$ with $\eta_j\in C^2(\bar{\Omega})$, $w_j^{(0)}\le\eta_j\le v_j^{(0)}$ $(j=1,\cdots,n)$, $c^{(i)}(x)-\frac{\partial f_i}{\partial u_i}(x,\eta)-\frac{\partial g_i}{\partial u_i}(x,\eta)\ge c^{(i)}(x)-\frac{\partial f_i}{\partial u_i}(x,V^{(0)})-\frac{\partial g_i}{\partial u_i}(x,W^{(0)})\ge L>0$ in Ω. In order to prove the existence of unique solutions of the problem (6)-(7), below one needs to show that any $\eta=(\eta_1,\cdots,\eta_n)$ with $\eta_j\in C^2(\bar{\Omega})$, $w_j^{(0)}\le\eta_j\le v_j^{(0)}$ $(j=1,\cdots,n)$, $h_i(x)\in C^{\alpha}(\Omega)$, where

$$h_i(x)=f_i(x,\eta)+g_i(x,\eta)-\frac{\partial f_i}{\partial u_i}(x,\eta)\eta_i-\frac{\partial g_i}{\partial u_i}(x,\eta)\eta_i.$$

As a matter of fact, if $\eta_j\in C^2(\Omega)$, then $\eta_j\in W^{2,q}(\Omega)$ in view of the boundedness of Ω, and $\partial\Omega\in C^{2,\alpha}(\bar{\Omega})$. The imbedding theorem [6] shows that $\eta_j\in C^{1,\alpha}(\bar{\Omega})$.

Therefore one can get

$$\begin{aligned}|f_i(x,\eta(x))-f_i(x,\eta\,(y))| &\le M_0\Big[\,\|x-y\|^{\alpha}+\sum_{j=1}^{n}|\eta_j(x)-\eta_j(y)|^{\alpha}\Big]\\&\le M_0\,\|x-y\|^{\alpha}+\sum_{j=1}^{n}|\eta_j|^{\alpha}_{C^1(\bar{\Omega})}\,\|x-y\|^{\alpha}\\&\le L_0\,\|x-y\|^{\alpha},\end{aligned}$$

where $L_0=M_0\Big[1+\sum\limits_{j=1}^{n}|\eta_j|^{\alpha}_{C^1(\bar{\Omega})}\Big]$, and

$$\begin{aligned}&|\eta_i\,(x)\frac{\partial f_i}{\partial u_i}(x,\eta(x))-\eta_i(y)\frac{\partial f_i}{\partial u_i}(y,\eta\,(y))|\\&\quad\le|\eta_i\,(x)\frac{\partial f_i}{\partial u_i}(x,\eta(x))-\eta_i(y)\frac{\partial f_i}{\partial u_i}(x,\eta\,(x))|\\&\qquad+|\eta_i\,(y)\frac{\partial f_i}{\partial u_i}(x,\eta(x))-\eta_i(y)\frac{\partial f_i}{\partial u_i}(y,\eta\,(y))|\\&\quad\le M_1\,|\eta_i(x)-\eta_i(y)|+M_2\bar{M}_0\Big[\,\|x-y\|^{\alpha}+\sum_{j=1}^{n}|\eta_j(x)-\eta_j(y)|^{\alpha}\Big]\\&\quad\le M_1\,|\eta_i|^{\alpha}_{C^1(\bar{\Omega})}\,\|x-y\|^{\alpha}+M_2\bar{M}_0\Big[\,\|x-y\|^{\alpha}+\sum_{j=1}^{n}|\eta_j|^{\alpha}_{C^1(\bar{\Omega})}\,\|x-y\|^{\alpha}\,\Big]\\&\quad\le\bar{L}_0\,\|x-y\|^{\alpha},\end{aligned}$$

where for $U \in < W^{(0)}, V^{(0)} >$, $i = 1, 2, \cdots, n$,

$$\left|\frac{\partial f_i}{\partial u_i}(x, U)\right| \le M_1, \quad |\eta_i(y)| \le M_2, \quad \bar{L}_0 = M_1 \, |\eta_i|^{\alpha}_{C^1(\bar{\Omega})} + M_2 \bar{M}_0 \Big[1 + \sum_{j=1}^{n} |\eta_j|^{\alpha}_{C^1(\bar{\Omega})}\Big].$$

One can obtain similar estimates for $g_i(x, U)$, $i = 1, 2, \cdots, n$. As a result, one finds that

$$\begin{aligned} |h_i(x) - h_i(y)| \le\; & |f_i(x, \eta(x)) - f_i(x, \eta\,(y))| + |g_i(x, \eta(x)) - g_i(x, \eta\,(y))| \\ & + |\eta_i\,(x)\frac{\partial f_i}{\partial u_i}(x, \eta(x)) - \eta_i(y)\frac{\partial f_i}{\partial u_i}(y, \eta\,(y))| \\ & + |\eta_i\,(x)\frac{\partial g_i}{\partial u_i}(x, \eta(x)) - \eta_i(y)\frac{\partial g_i}{\partial u_i}(y, \eta\,(y))| \\ \le\; & C\,\|x - y\|^{\alpha}, \end{aligned}$$

where $C = L_0 + \bar{L}_0 + L_1 + \bar{L}_1$, L_1, $\bar{L}_1$ being corresponding constants relative to the functions g_i. Hence for each $i = 1, 2, \cdots, n$, $h_i(x) \in C^{\alpha}(\Omega)$, and consequently, for each $s \ge 1$, there exists a unique solution $W^{(s)}, V^{(s)}$ with $w_i^s, v_i^s \in C^{2,\,\alpha}(\bar{\Omega})$ of the boundary value problems

$$\begin{aligned} & L_{c_0} w_i^{(s+1)} = f_i(x, W^{(s)}) + g_i(x, W^{(s)}) - \frac{\partial f_i}{\partial u_i}(x, W^{(s)}) w_i^{(s)} - \frac{\partial f_i}{\partial u_i}(x, V^{(s)}) w_i^{(s)}, \quad x \in \Omega, \\ & B_i w_i^{(s+1)} = \varphi_i, \quad x \in \partial\Omega; \\ & L_{c_0} v_i^{(s+1)} = f_i(x, V^{(s)}) + g_i(x, V^{(s)}) - \frac{\partial f_i}{\partial u_i}(x, W^{(s)}) v_i^{(s)} - \frac{\partial f_i}{\partial u_i}(x, V^{(s)}) v_i^{(s)}, \quad x \in \Omega, \\ & B_i v_i^{(s+1)} = \varphi_i, \quad x \in \partial\Omega, \end{aligned}$$

where L_{c_0} is as defined in (3) and $c_0^{(i)}(x) = c^{(i)}(x) - \dfrac{\partial f_i}{\partial u_i}(x, W^{(s)}) - \dfrac{\partial f_i}{\partial u_i}(x, V^{(s)}) > 0$. Below one needs to show that for each $i = 1, 2, \cdots, n$,

$$w_i^{(0)} \le w_i^{(1)} \le \cdots \le w_i^{(s)} \le v_i^{(s)} \le \cdots \le v_i^{(1)} \le v_i^{(0)}, \quad x \in \bar{\Omega}. \tag{8}$$

Because of $W, V \in C^2(\bar{\Omega})$ being the lower and upper solutions of (6)-(7), one can easily obtain for each $i = 1, 2, \cdots, n$ by using Lemma 1

$$w_i^{(0)} \le w_i^{(1)}, \quad v_i^{(1)} \le v_i^{(0)}.$$

To show $w_i^{(1)} \le v_i^{(0)}$, one has

$$\begin{aligned} L_i w_i^{(1)} &= F_i(x, W^{(1)}; W^{(0)}, V^{(0)}) \\ &= f_i(x, W^{(0)}) + g_i(x, W^{(0)}) + \frac{\partial f_i}{\partial u_i}(x, W^{(0)})(w_i^{(1)} - w_i^{(0)}) \\ &\quad + \frac{\partial g_i}{\partial u_i}(x, V^{(0)})(w_i^{(1)} - w_i^{(0)}) \end{aligned}$$

$$
\begin{aligned}
&\leq\ f_i(x,V^{(0)})+g_i(x,V^{(0)})-\frac{\partial f_i}{\partial u_i}(x,W^{(0)})(v_i^{(0)}-w_i^{(0)})\\
&\quad+\frac{\partial f_i}{\partial u_i}(x,W^{(0)})(w_i^{(1)}-w_i^{(0)})-\frac{\partial g_i}{\partial u_i}(x,V^{(0)})(v_i^{(0)}-w_i^{(0)})\\
&\quad+\frac{\partial g_i}{\partial u_i}(x,V^{(0)})(w_i^{(1)}-w_i^{(0)})\\
&=\ G_i(x,W^{(1)};W^{(0)},V^{(0)}) \text{ in } \Omega.
\end{aligned}
$$

But $L_iv_i^{(1)}=G_i(x,V^{(1)};W^{(0)},V^{(0)})$ in Ω and Lemma 1 implies that $w_i^{(1)}\leq v_i^{(0)}$ in $\bar{\Omega}$. Similarly one can prove that $w_i^{(0)}\leq v_i^{(1)}$ in $\bar{\Omega}$ for each $i=1,2,\cdots,n$.

Analogously,

$$
\begin{aligned}
L_iw_i^{(1)} &=\ F_i(x,W^{(1)};W^{(0)},V^{(0)})\\
&\leq\ f_i(x,W^{(1)})+g_i(x,W^{(1)})+\Big[\frac{\partial g_i}{\partial u_i}(x,V^{(0)})-\frac{\partial g_i}{\partial u_i}(x,W^{(1)})\Big](w_i^{(1)}-w_i^{(0)})\\
&\leq\ F_i(x,W^{(1)};W^{(1)},V^{(1)});\\
L_iv_i^{(1)} &=\ G_i(x,V^{(1)};W^{(0)},V^{(0)})\\
&\geq\ f_i(x,V^{(1)})+g_i(x,V^{(1)})+\Big[\frac{\partial f_i}{\partial u_i}(x,W^{(0)})-\frac{\partial f_i}{\partial u_i}(x,V^{(1)})\Big](v_i^{(1)}-v_i^{(0)})\\
&\geq\ G_i(x,V^{(1)};W^{(1)},V^{(1)}).
\end{aligned}
$$

It then follows by Lemma 1 that $w_i^{(1)}\leq v_i^{(1)}$ in $\bar{\Omega}$. Hence, one has

$$w_i^{(0)}\leq w_i^{(1)}\leq v_i^{(1)}\leq v_i^{(0)} \text{ in } \bar{\Omega}.$$

Next one shall consider that if $w_i^{(s-1)}\leq w_i^{(s)}\leq v_i^{(s)}\leq v_i^{(s-1)}$ in $\bar{\Omega}$, for some $s>1$, then it follows that

$$w_i^{(s)}\leq w_i^{(s+1)}\leq v_i^{(s+1)}\leq v_i^{(s)} \text{ in } \bar{\Omega}.$$

Since $w_i^{(s)},v_i^{(s)}$ satisfy

$$
\begin{aligned}
&L_iw_i^{(s)}=F_i(x,W^{(s)};W^{(s-1)},V^{(s-1)}) \text{ in } \Omega,\quad B_iw_i^{(s)}=\varphi_i \text{ on } \partial\Omega;\\
&L_iv_i^{(s)}=G_i(x,W^{(s)};W^{(s-1)},V^{(s-1)}) \text{ in}\Omega,\quad B_iv_i^{(s)}=\varphi_i \text{ on } \partial\Omega,
\end{aligned}
$$

by the assumptions one can obtain

$$
\begin{aligned}
&L_iw_i^{(s)}\leq F_i(x,W^{(s)};W^{(s)},V^{(s)}) \text{ in } \Omega,\quad B_iw_i^{(s)}=\varphi_i \text{ on } \partial\Omega;\\
&L_iv_i^{(s)}\geq G_i(x,V^{(s)};W^{(s)},V^{(s)}) \text{ in } \Omega,\quad B_iv_i^{(s)}=\varphi_i \text{ on } \partial\Omega.
\end{aligned}
$$

Hence by the above and Lemma 1 one gets $w_i^{(s)}\leq w_i^{(s+1)}$, $v_i^{(s+1)}\leq v_i^{(s)}$ in $\bar{\Omega}$. Similar to the proof of $w_i^{(1)}\leq v_i^{(1)}$, one can easily show $w_i^{(s+1)}\leq v_i^{(s+1)}$. Thus by induction, (8) is valid for all s.

Since $C^{2,\alpha}(\bar{\Omega}) \subset W^{2,q}(\bar{\Omega})$ for $q > 1$, by Theorem A.3.3 in [6] one has

$$\|w_i^{(s)}\|_{W^{2,q}(\bar{\Omega})} \leq C(\|h_i^{(s)}\|_{L^q(\bar{\Omega})} + \|\varphi_i\|_{W^{1,q}(\bar{\Omega})}), \tag{9}$$

where

$$h_i^{(s)}(x) = f_i(x, w_i^{(s)}) + g_i(x, w_i^{(s)}) - \frac{\partial f_i}{\partial u_i}(x, w_i^{(s)})w_i^{(s)} - \frac{\partial g_i}{\partial u_i}(x, v_i^{(s)})w_i^{(s)}.$$

Obviously, $\{h_i^{(s)}(x)\}$ is uniformly bounded in $C(\bar{\Omega})$. Since $C(\bar{\Omega})$ is dense in $L^q(\bar{\Omega})$, $\{h_i^{(s)}(x)\}$ is also uniformly bounded in $L^q(\bar{\Omega})$. This together with (9) shows that $\{w_i^{(s)}(x)\}$ is uniformly bounded in $W^{2,q}(\bar{\Omega})$. For $q = \dfrac{2}{1-\alpha}$, $w_i^{(s)} \in W^{2,q}(\bar{\Omega})$ and hence by imbedding Theorem A.3.5 in [6],

$$\|w_i^{(s)}\|_{C^{1,\alpha}(\bar{\Omega})} \leq C\|w_i^{(s)}\|_{W^{2,q}(\bar{\Omega})}, \quad s = 1, 2, \cdots,$$

for some constant C independent of the elements of $W^{2,q}(\bar{\Omega})$. Thus for each $i = 1, 2, \cdots, n$, $\{w_i^{(s)}(x)\}$is uniformly bounded in $C^{1,\alpha}(\bar{\Omega})$. Therefore $\{h_i^{(s)}(x)\}$ is uniformly bounded in $C^{\alpha}(\bar{\Omega})$. Consequently by Schauder's estimate one finds that

$$\|w_i^{(s)}\|_{C^{2,q}(\bar{\Omega})} \leq C\|h_i^{(s)}\|_{C^{\alpha}(\bar{\Omega})} + \|\varphi_i\|_{C^{1,q}(\bar{\Omega})}, \text{ for all } s,$$

which implies the uniform boundedness of $\{w_i^{(s)}(x)\}$ in $C^{2,\alpha}(\bar{\Omega})$. As a result, one has $\{w_i^{(s)}(x)\}$is relatively compact in $C^2(\bar{\Omega})$, which yields the existence of a subsequence that converges pointwise to $\alpha* \in C^2(\bar{\Omega})$. By the monotone nature of $w_i^{(s)}(x)$in s, $\{w_i^{(s)}(x)\}$ converges pointwise to $\rho_i(x)$ in $\bar{\Omega}$ for each $i = 1, 2, \cdots, n$. Hence the entire sequence $\{w_i^{(s)}(x)\}$converges in $C^2(\bar{\Omega})$ to $\rho_i(x)$, that is, $\lim\limits_{s\to\infty} w_i^{(s)} = \rho_i(x)$ in $C^2(\bar{\Omega})$and $w_i^{(0)} \leq \rho_i \leq v_i^{(0)}$ in $\bar{\Omega}$ for each $i = 1, 2, \cdots, n$. By analogous arguments one can get that $\lim\limits_{s\to\infty} v_i^{(s)} = r_i(x)$ in $C^2(\bar{\Omega})$and $w_i^{(0)} \leq \rho_i \leq r_i \leq v_i^{(0)}$ in $\bar{\Omega}$ for each $i = 1, 2, \cdots, n$. Moreover, it is not difficult to see that ρ_i, r_i, $i = 1, 2, \cdots, n$ are the solutions of (6)-(7). Since $\rho_i \leq r_i$ in $\bar{\Omega}$, by taking $w_i = r_i$, $v_i = \rho_i$, and Lemma 1 shows $\rho_i \geq r_i$, proving $\rho_i = r_i = u_i$ for each $i = 1, \cdots, n$ is the unique solution of (6)-(7). The proof is completed.

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