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GLOBAL EXISTENCE OF CLASSICAL SOLUTION WITH SMALL INITIAL TOTAL VARIATION FOR QUASILINEAR LINEARLY DEGENERATE HYPERBOLIC SYSTEMS

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Abstract In this paper, the author proves the global existence of classical solution to the Cauchy problem with slowly decaying initial data with small initial total variation for general first order quasilinear linearly degenerate hyperbolic systems. This generalizes the corresponding result of A.Bressan for initial data with compact support.

Key Words Linear degeneracy; small initial total variation; slowly decaying initial data; global classical solution; quasilinear hyperbolic system.
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1. Introduction

Consider the following first order quasilinear strictly hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x) and $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)(i, j = 1, \cdots, n)$.

By strict hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

Let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))(resp.\ r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T)$ be a left (resp. right) eigenvector corresponding to $\lambda_i(u)(i = 1, \cdots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (resp.\ A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.3}$$

We have

$$det|l_{ij}(u)| \neq 0 \quad (equivalently,\ det|r_{ij}(u)| \neq 0). \tag{1.4}$$

All $\lambda_i(u), l_{ij}(u)$ and $r_{ij}(u)(i,j = 1,\cdots,n)$ have the same regularity as $a_{ij}(u)(i,$ $j = 1,\cdots,n)$.

Without loss of generality, we may suppose that

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i,j = 1\cdots,n) \tag{1.5}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1\cdots,n), \tag{1.6}$$

where δ_{ij} stands for the Kronecker's symbol.

Consider the Cauchy problem for the system (1.1) with the following initial data

$$t = 0 : u = \varphi(x), \tag{1.7}$$

where $\varphi(x)$ is a "small" C^1 vector function of x. In the case that $\varphi(x) \in C^2$ possesses compact support and small total variation, suppose that the system (1.1) is strictly hyperbolic and linearly degenerate in the sense of P.D.Lax, A.Bressan proved in [1] the global existence of classical solution to Cauchy problem (1.1) and (1.7). On the other hand, when $\varphi(x) \in C^1$ possesses certain decay properties as $|x| \to +\infty$ and the strictly hyperbolic system (1.1) is only weakly linearly degenerate, Li Ta-tsien, Zhou Yi and Kong De-xing presented in [2–4] a complete result on the global existence of C^1 solution to Cauchy problem (1.1) and (1.7). Moreover, Kong De-xing constructed a counter-example in [5] which shows the necessary decay property of initial data is essential for guaranteeing the global existence of classical solution to Cauchy problem (1.1) and (1.7).

In this paper we will prove that in the case that $\varphi(x) \in C^1$ possesses slowly decaying properties and small total variation, for the strictly hyperbolic and linearly degenerate system (1.1), there exists a unique global classical solution to Cauchy problem (1.1) and (1.7) for all $t \in \mathbb{R}$.

The main result of this paper is the following

Theorem 1.1 *Suppose that in a neighbourhood of* $u = 0$, $A(u) \in C^2$ *and the system (1.1) is strictly hyperbolic and linearly degenerate i.e. on the domain under consideration*

$$\nabla\lambda_i(u)r_i(u) \equiv 0 \quad (i = 1,\cdots,n). \tag{1.8}$$

Suppose furthermore that the initial data satisfy the following properties:

(i) $\varphi(x) \in C^1$;

(ii) $\varphi(x)$ *satisfies the following slowly decaying property*

$$\bar{\theta} \overset{\triangle}{=} \sup_{x\in\mathbb{R}}\{(1+|x|)(|\varphi(x)|+|\varphi'(x)|)\} < +\infty; \tag{1.9}$$

(iii) *The initial total variation is small enough, namely,*

$$\theta \overset{\triangle}{=} \int_{-\infty}^{+\infty} |\varphi'(x)|dx << 1. \tag{1.10}$$

Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, Cauchy problem (1.1) and (1.7) admits a unique global C^1 solution $u = u(t,x)$ for all $t \in \mathbb{R}$.

For the sake of completeness, in Section 2 we will briefly recall F.John's formulas on the decomposition of waves with some supplements(cf.[3, 6]), which will play an important role in the sequel. In Section 3, we will establish a uniform a priori estimate on the C^0 norm of C^1 solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.7). In Section 4, we will establish a uniform a priori estimate on the C^0 norm of the first order derivative $u_x(t,x)$ of the C^1 solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.7) and then prove Theorem 1.1. In Section 5, an application to the system of planar motion of an elastic string is given.

2. Decomposition of Waves

Suppose that on the domain under consideration, the system (1.1) is strictly hyperbolic and (1.5)-(1.6) hold.

Let

$$v_i = l_i(u)u \quad (i = 1, \cdots, n) \tag{2.1}$$

and

$$w_i = l_i(u)u_x \quad (i = 1, \cdots, n), \tag{2.2}$$

where $l_i(u)$ denotes the i-th left eigenvector.

By (1.5), it is easy to see that

$$u = \sum_{k=1}^{n} v_k r_k(u) \tag{2.3}$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.4}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.5}$$

be the directional derivative with respect to t along the i-th characteristic. We have (cf.[3, 6])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k \quad (i = 1, \cdots, n), \tag{2.6}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u). \tag{2.7}$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \quad \forall\ i, j. \tag{2.8}$$

It follows from (2.6) that

$$\frac{\partial v_i}{\partial t}+\frac{\partial(\lambda_i(u)v_i)}{\partial x}=\sum_{j,k=1}^{n}B_{ijk}(u)v_jw_k \quad (i=1,\cdots,n), \tag{2.9}$$

where

$$B_{ijk}(u)=\beta_{ijk}(u)+\nabla\lambda_i(u)r_k(u)\delta_{ij}. \tag{2.10}$$

We have

$$B_{iii}(u)=\nabla\lambda_i(u)r_i(u),\ \forall i, \tag{2.11}$$

then, in the case that $\lambda_i(u)$ is linearly degenerate in the sense of P.D.Lax, we have

$$B_{iii}(u)\equiv 0, \quad \forall i. \tag{2.12}$$

On the other hand,we have(cf.[3, 6])

$$\frac{dw_i}{d_it}=\sum_{j,k=1}^{n}\gamma_{ijk}(u)w_jw_k \quad (i=1,\cdots,n), \tag{2.13}$$

where

$$\gamma_{ijk}(u)=\frac{1}{2}[(\lambda_j(u)-\lambda_k(u))l_i(u)\nabla r_k(u)r_j(u)-\nabla\lambda_k(u)r_j(u)\delta_{ik}+(j|k)], \tag{2.14}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms. We have

$$\gamma_{ijj}(u)\equiv 0, \quad \forall j\neq i \tag{2.15}$$

and

$$\gamma_{iii}(u)=-\nabla\lambda_i(u)r_i(u), \quad \forall i. \tag{2.16}$$

Then, when the i-th characteristic $\lambda_i(u)$ is linearly degenerate in the sense of P.D.Lax, we have

$$\gamma_{iii}(u)\equiv 0, \quad \forall i. \tag{2.17}$$

It follows from (2.13) that

$$\frac{\partial w_i}{\partial t}+\frac{\partial(\lambda_i(u)w_i)}{\partial x}=\sum_{j,k=1}^{n}\Gamma_{ijk}(u)w_jw_k \quad (i=1,\cdots,n), \tag{2.18}$$

where

$$\Gamma_{ijk}(u)=\frac{1}{2}(\lambda_j(u)-\lambda_k(u))l_i(u)[\nabla r_k(u)r_j(u)-\nabla r_j(u)r_k(u)]. \tag{2.19}$$

We have

$$\Gamma_{ijj}(u)\equiv 0, \quad \forall i,j. \tag{2.20}$$

3. Uniform a Priori Estimate on the C^0 Norm of C^1 Solution

Suppose that $u = u(t, x)$ is the C^1 solution to Cauchy problem (1.1) and (1.7) on the domain $D(T) = \{(t, x)|0 \le t \le T, |x| < \infty\}$. Let

$$L(u(t)) = \sum_{i=1}^{n} \int_{\mathbb{R}} |w_i(t, x)|dx \tag{3.1}$$

and

$$Q(u(t)) = \sum_{i<j} \iint_{x>y} |w_i(t, x)w_j(t, y)|dxdy, \tag{3.2}$$

where $w_i(i = 1, \cdots, n)$ are defined by (2.2).

By Lemma 3.1 in [4], we have

Lemma 3.1 *Suppose that in a neighbourhood of $u = 0$, the system (1.1) is hyperbolic, (1.5)-(1.6) hold and*

$$\lambda_1(u) \le \lambda_2(u) \le \cdots \le \lambda_n(u). \tag{3.3}$$

Suppose furthermore that $u = u(t, x)$ is the C^1 solution to Cauchy problem (1.1) and (1.7) on the domain $D(T)$ such that the C^0 norm of $u(t, x)$ is bounded and the integrals appearing in (3.4)-(3.5) make sense, then we have

$$\frac{dL(u(t))}{dt} \le C_1 \int_{\mathbb{R}} \Lambda(t, x)dx, \ \forall t \in [0, T] \tag{3.4}$$

and

$$\frac{dQ(u(t))}{dt} \le (C_2 L(u(t)) - 1) \int_{\mathbb{R}} \Lambda(t, x)dx, \ \forall t \in [0, T], \tag{3.5}$$

where C_1 and C_2 are two positive constants depending only on the C^0 norm of $u(t, x)$ but independent of T, and

$$\Lambda(t, x) = \sum_{i>j} (\lambda_i(u) - \lambda_j(u))|w_i(t, x)w_j(t, x)|. \tag{3.6}$$

Henceforth, $C_i(i = 1, 2, \cdots)$ denote positive constants dependent only on the C^0 norm of u(t,x) but independent of T.

From Lemma 3.1 we get immediately the following

Lemma 3.2 *Under the assumptions of Lemma 3.1, if the total variation of $u(t, x)$ is small enough (namely, $L(u(t))$ is small enough), then there exists a positive constant M dependent only on the C^0 norm of $u(t, x)$ but independent of T, such that*

$$F(u(t)) \triangleq L(u(t)) + MQ(u(t)) \tag{3.7}$$

is a non-increasing function of t, i.e.

$$\frac{dF(u(t))}{dt} \le 0, \ \forall t \in [0, T]. \tag{3.8}$$

Now we derive a uniform a priori estimate on the C^0 norm of C^1 solution $u = u(t, x)$.

Lemma 3.3 *Suppose that in a neighbourhood of $u = 0$ the system (1.1) is hyperbolic and (3.3) holds. Suppose furthermore that $u = u(t, x)$ is the C^1 solution to Cauchy problem (1.1) and (1.7) on the domain $D(T) = \{(t, x)|0 \le t \le T, |x| < \infty\}$ and the initial data satisfy the assumptions given in Theorem 1.1. Let*

$$\gamma \overset{\triangle}{=} L(\varphi). \tag{3.9}$$

Then there exists $\gamma_0 > 0$ so small that for any given $\gamma \in [0, \gamma_0]$, there exist two positive constants κ_1 and κ_2 independent of γ and T, such that the following uniform a priori estimates hold:

$$\|u(t, \cdot)\|_{C^0} = \sup_{x \in \mathbb{R}} |u(t, x)| \le \kappa_1 \gamma, \quad \forall t \in [0, T] \tag{3.10}$$

and

$$L(u(t)) \le \gamma + \kappa_2 \gamma^2, \quad \forall t \in [0, T]. \tag{3.11}$$

Remark 3.1 Noting (3.1) , (1.9)-(1.10) and (2.2), it follows from (3.9) that there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, we have

$$\gamma \le C_0 \theta, \tag{3.12}$$

where θ is defined by (1.10) and C_0 is a positive constant independent of φ.

Proof of Lemma 3.3 For the time being we assume that on the domain $D(T)$ we have

$$|u(t, x)| \le \delta, \tag{3.13}$$

where $\delta > 0$ is a small constant. At the end of this proof we shall explain that this hypothesis is reasonable.

Noting (1.9), we have

$$\varphi(x) = \int_{-\infty}^{x} \varphi'(\xi) d\xi,$$

then

$$|\varphi(x)| \le \int_{-\infty}^{+\infty} |\varphi'(\xi)| d\xi \le C_3 \gamma. \tag{3.14}$$

By continuity, there exists $\tau_0 > 0$ so small that

$$\|u(t, \cdot)\|_{C^0} \le 2C_3 \gamma, \ \forall t \in [0, \tau_0] \tag{3.15}$$

and

$$\|w(t, \cdot)\|_{C^0} \le c_4, \ \forall t \in [0, \tau_0]. \tag{3.16}$$

We now prove that, when $\tau_0 > 0$ is suitably small, we have

$$\sup_{0\le t\le\tau_0}\sup_{x\in\mathbb{R}}\{(1+|x|)(|v(t,x)|+|w(t,x)|\}\le C\bar{\theta}<+\infty, \tag{3.17}$$

where C is a positive constant dependent on τ_0.

In fact, by (3.13) and (1.9), for every $i = 1,\cdots,n$, we have

$$\begin{cases} |v_i(0,x)| = |l_i(\varphi(x))\varphi(x)| \le C_5\bar{\theta}(1+x)^{-1}, \ \forall x\in\mathbb{R} \\ |w_i(0,x)| = |l_i(\varphi(x))\varphi'(x)| \le C_5\bar{\theta}(1+x)^{-1}, \ \forall x\in\mathbb{R}. \end{cases} \tag{3.18}$$

According to the local existence and uniqueness (cf.[7]), there exists $\tau_0 > 0$ such that Cauchy problem (1.1) and (1.7) admits a unique C^1 solution $u = u(t,x)$ on $0 \le t \le \tau_0$.

Let

$$V(0,\tau) = \max_{i=1,\cdots,n}\|(1+x)v_i(t,x)\|_{C^0(\{0\le t\le\tau\}\times\mathbb{R})} \tag{3.19}$$

and

$$W(0,\tau) = \max_{i=1,\cdots,n}\|(1+x)w_i(t,x)\|_{C^0(\{0\le t\le\tau\}\times\mathbb{R})}, \tag{3.20}$$

where $\tau\in[0,\tau_0]$.

For every $i = 1,\cdots,n$, on the domain $0\le t\le\tau_0$, passing through any given point (t,x) we draw the i-th characteristic $\xi = x_i(s;y)$ which intersects the x-axis at point $(0,y)$. We have

$$\begin{cases} \dfrac{dx_i(s;y)}{ds} = \lambda_i(u(s,x_i(s;y))), 0\le s\le t\le\tau_0, \\ x_i(t;y) = x. \end{cases} \tag{3.21}$$

Noting (3.13), it is easy to see that when τ_0 is small, we have

$$C_6(1+|y|)\le 1+|x_i(s;y)|\le C_7(1+|y|), \ \forall s\in[0,\tau_0]. \tag{3.22}$$

Integrating (2.6) and (2.13) along this characteristic, we get

$$v_i(t,x) = v_i(0,y) + \int_0^t\sum_{j,k=1}^n\beta_{ijk}(u)v_jw_k(s,x_i(s;y))ds, \ \forall t\in[0,\tau_0] \tag{3.23}$$

and

$$w_i(t,x) = w_i(0,y) + \int_0^t\sum_{j,k=1}^n\gamma_{ijk}(u)w_jw_k(s,x_i(s;y))ds, \ \forall t\in[0,\tau_0]. \tag{3.24}$$

Multiplying both sides of (3.23)-(3.24) by $(1+|x|)$ respectively and summing up them, noting (3.22),(3.18) and (3.13), we get

$$Z(t)\le C_8(\bar{\theta}+\int_0^t Z^2(s)ds), \ \forall t\in[0,\tau_0], \tag{3.25}$$

where

$$Z(t) = V(0,t) + W(0,t).$$

Then, we have

$$Z(t) \le \frac{C_8\bar{\theta}}{1 - C_8^2\bar{\theta}t}, \quad \forall t \in [0, \tau_0]. \tag{3.26}$$

Hence, we can choose τ_0 so small that (3.17) holds on $[0, \tau_0]$, then the integrals appearing in (3.4)-(3.5) make sense on the domain $D(\tau_0) = \{(t,x)|0 \le t \le \tau_0, |x| < +\infty\}$.

In order to apply Lemma 3.2, it is necessary to explain that $L(u(t))$, the total variation of C^1 solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.7) is small enough on the domain $D(\tau_0)$.

Noting (3.17), multiplying both sides of (2.18) by $sgnw_i$ and integrating it from $-\infty$ to $+\infty$ with respect to x yield

$$\frac{d}{dt}\int_{-\infty}^{+\infty} |w_i(t,x)|dx \le \int_{-\infty}^{+\infty} \sum_{j,k=1}^{n} |\Gamma_{ijk}(u)w_j w_k(t,x)|dx, \forall t \in [0,\tau_0]. \tag{3.27}$$

Noting (3.13) and (3.16), summing up the above formula with respect to i, we get

$$\frac{dL(u(t))}{dt} \le C_9 L(u(t)), \quad \forall t \in [0, \tau_0]. \tag{3.28}$$

Then, noting (3.9), we obtain

$$L(u(t)) \le \gamma e^{C_9\tau_0}, \quad \forall t \in [0, \tau_0]. \tag{3.29}$$

Hence, if $\tau_0 > 0$ is suitably small, we have

$$L(u(t)) \le 2\gamma, \quad \forall t \in [0, \tau_0]. \tag{3.30}$$

Thus, when $\gamma_0 > 0$ is small enough, for any given $\gamma \in [0, \gamma_0]$, we can apply Lemma 3.2 on the domain $D(\tau_0)$ to get

$$L(u(t)) \le L(u(t)) + MQ(u(t)) \le \gamma + M\gamma^2, \quad \forall t \in [0, \tau_0], \tag{3.31}$$

where M is given by Lemma 3.2.

Noting (3.15) and (3.31), it is sufficient to choose $\kappa_1 \ge 2C_3$ and $\kappa_2 \ge M$ such that Lemma 3 .3 holds on $D(\tau_0)$.

We next prove that there exist the positive constants κ_1 and κ_2 independent of γ and T, such that Lemma 3.3 holds on the domain $D(T)$, provided that $\gamma_0 > 0$ is small enough.

For this purpose, it suffices to show that if $\gamma_0 > 0$ is small enough, for any given $\gamma \in [0, \gamma_0]$, we can choose $\kappa_1, \kappa_2 > 0$ such that for any given $T_0 (0 \le T_0 \le T)$ when

$$||u(t,\cdot)||_{C^0} \le 2\kappa_1\gamma, \quad \forall t \in [0, T_0], \tag{3.32}$$

and

$$L(u(t)) \leq \gamma + 2\kappa_2\gamma^2, \quad \forall t \in [0, T_0], \tag{3.33}$$

we have

$$||u(t, \cdot)||_{C^0} \leq \kappa_1\gamma, \quad \forall t \in [0, T_0] \tag{3.34}$$

and

$$L(u(t)) \leq \gamma + \kappa_2\gamma^2, \quad \forall t \in [0, T_0]. \tag{3.35}$$

First of all, by the method of getting (3.17), we can prove

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} \{(1 + |x|)(|v(t, x)| + |w(t, x)|)\} \leq C_T\bar{\theta} < +\infty, \tag{3.36}$$

where C_T is a positive constant dependent on T.

Noting (3.32)-(3.33), if $\gamma_0 > 0$ is small enough, for any given $\gamma \in [0, \gamma_0]$, by (3.36) we can apply Lemma 3.2 on the domain $D(T_0)$ to get

$$L(u(t)) \leq \gamma + M\gamma^2, \quad \forall t \in [0, T_0]. \tag{3.37}$$

So we can choose $\kappa_2 \geq M$ for getting (3.35).

On the other hand, when $\gamma_0 > 0$ is small enough, for any given $\gamma \in [0, \gamma_0]$, by (3.36) and (3.33) and noting (3.13), we have

$$||u(t, \cdot)||_{C^0} = \sup_{x \in \mathbb{R}} \left| \int_{-\infty}^{x} u_\xi(t, \xi) d\xi \right| \leq C_{10}L(u(t)) \leq 2C_{10}\gamma, \quad \forall t \in [0, T_0]. \tag{3.38}$$

Then, taking $\kappa_1 \geq 2C_{10}$, we get (3.34).

By (3.10), when $\gamma_0 > 0$ is small enough, for any given $\gamma \in [0, \gamma_0]$, we always have

$$||u(t, \cdot)||_{C^0} \leq \kappa_1\gamma < \frac{\delta}{2}, \quad \forall t \in [0, T]. \tag{3.39}$$

This shows the validity of the hypothesis (3.13). The proof of Lemma 3.3 is complete.

Remark 3.2 By (3.10) and (3.12), when $\theta_0 > 0$ is small enough, for any given $\theta \in [0, \theta_0]$, on any existence domain $D(T)$ of C^1 solution $u = u(t, x)$ to Cauchy problem (1.1) and (1.7), (3.12) always holds. This is the uniform a priori estimate on the C^0 norm of C^1 solution $u = u(t, x)$.

4. The Proof of Theorem 1.1

It suffices to prove the global existence of C^1 solution $u = u(t, x)$ to Cauchy Problem (1.1) and (1.7) on $t \geq 0$.

Introduce first the following functional (cf.[1])

$$R_i(u) = \sum_{i>j} \int_{x}^{+\infty} |w_j(t, y)| dy + \sum_{i<j} \int_{-\infty}^{x} |w_j(t, y)| dy$$

$$+\alpha\sum_{j<k}\iint_{y<z}|w_j(t,z)w_k(t,y)|dydz, \tag{4.1}$$

where $\alpha>0$ is a constant to be determined later.

Now define

$$M_1=\max\left\{\sup_{|u|\le\delta}\sum_{i,j,k}|\gamma_{ijk}(u)|,\ \sup_{|u|\le\delta}\sum_{i,j,k}|\gamma_{ijk}(u)|\right\}>0 \tag{4.2}$$

and

$$\sigma=\min_{j<k}\inf_{|u|\le\delta}\{\lambda_k(u)-\lambda_j(u)\}>0. \tag{4.3}$$

By (2.5) and noting (3.13), we have (cf.[1])

$$\begin{aligned}\frac{dR_i(u)}{d_it}=&\frac{\partial R_i(u)}{\partial t}+\lambda_i(u)\frac{\partial R_i(u)}{\partial x}\\=&-\sigma\sum_{j\ne i}|w_j(t,x)|-[\alpha\sigma-nM_1-\alpha nM_1L(u(t))]\\&\cdot\sum_{j<k}\int_{-\infty}^{+\infty}|w_j(t,x)w_k(t,x)|dx.\end{aligned} \tag{4.4}$$

Set $\alpha=\dfrac{2nM_1}{\sigma}>0$ and choose a constant $V_0\in(0,1)$ such that $\alpha nM_1V_0\le\dfrac{\alpha\sigma}{2}$. When $L(u(t))\le V_0$, we have

$$\alpha\sigma-nM_1-\alpha nM_1L(u(t))\ge 0. \tag{4.5}$$

Therefore, it follows from (4.4) that

$$\frac{dR_i(u)}{d_it}\le-\sigma\sum_{j\ne i}|w_j(t,x)|. \tag{4.6}$$

Set $\beta=\dfrac{5nM_1}{\sigma}>0$ and $V_1=\min\{V_0,\dfrac{1}{2nM_1}\}>0$ such that if $V_0>0$ is small enough, we have

$$\beta[V_1+\alpha V_1^2]\le\ln 2. \tag{4.7}$$

By (4.1), we have

$$R_i(u)\le L(u(t))+\alpha(L(u(t)))^2, \tag{4.8}$$

hence

$$\exp(\beta R_i(u))\le 2, \tag{4.9}$$

provided that $L(u(t))\le V_1$.

We now prove that the C^0 norm of the first order derivative $u_x(t,x)$ of C^1 solution $u=u(t,x)$ to Cauchy problem (1.1) and (1.7) always remains uniformly bounded.

Let

$$L = \max_{i=1,\cdots,n} \sup_{x\in\mathbb{R}} |w_i(0,x)|, \tag{4.10}$$

$$T = \sup\left\{t \ge 0 | \max_{i=1,\cdots,n} \sup_{x\in\mathbb{R}} |w_i(t',x)| \le 3L,\ \forall t' \in [0,t]\right\} > 0 \tag{4.11}$$

and

$$\tilde{w}_i(t,x) = |w_i(t,x)| \exp(\beta R_i(u)) \quad (i = 1,\cdots,n). \tag{4.12}$$

By (4.6), we have

$$|w_i(t,x)| \le \tilde{w}_i(t,x) \le 2|w_i(t,x)| \quad (i = 1,\cdots,n). \tag{4.13}$$

Suppose $T < +\infty$.

By the definition of T (see (4.11)), noting (3.36), there exist a point $(\bar{t},\bar{x})$ and an index $i_0 \in \{1,\cdots,n\}$ such that

$$0 < \bar{t} < T, \quad \tilde{w}_{i_0}(\bar{t},\bar{x}) > 2L \tag{4.14}$$

and

$$\tilde{w}_j(t,x) < \tilde{w}_{i_0}(\bar{t},\bar{x}), \quad \forall x \in \mathbb{R},\ \forall t < \bar{t},\ \forall j. \tag{4.15}$$

Let $x = x(t)$ be the i_0-th characteristic passing through point $(\bar{t},\bar{x})$. By continuity, there exists $\tau \in (0,\bar{t})$ on this characteristic such that

$$\tilde{w}_{i_0}(t,x(t)) \ge \frac{1}{2}\tilde{w}_{i_0}(\bar{t},\bar{x}) > L, \quad \forall t \in [\tau,\bar{t}\,]. \tag{4.16}$$

Noting (2.13),(4.6) and the hypothesis (2.17) that the system (1.1) is linearly degenerate, by (4.12), for any given $t \in [\tau,\bar{t}\,]$, the directional derivative of $\tilde{w}_{i_0}(t,x)$ along the characteristic $x = x(t)$ can be written as

$$\begin{aligned}
&\frac{d\tilde{w}_{i_0}(t,x(t))}{d_{i_0}t}\\
=&\left[\frac{d|w_{i_0}(t,x(t))|}{d_{i_0}t} + \beta\frac{dR_{i_0}(u)}{d_{i_0}t}|w_{i_0}(t,x)|\right]\exp(\beta R_{i_0}(u))\\
\le&\left[\sum_{j,k=1}^{n}|\gamma_{ijk}(u)w_jw_k(t,x(t))| - \beta\sigma\sum_{j\ne i_0}|w_{i_0}(t,x(t))w_j(t,x(t))|\right]\exp(\beta R_{i_0}(u))\\
\le&\left[M_1\left(\sum_{\substack{j,k\ne i_0\\ j<k}}|w_jw_k(t,x(t))| + \sum_{j\ne i_0}|w_{i_0}w_j(t,x(t))|\right)\right.\\
&\left.-\beta\sigma\sum_{j\ne i_0}|w_{i_0}w_j(t,x(t))|\right]\exp(\beta R_{i_0}(u)).
\end{aligned} \tag{4.17}$$

Noting (4.13) and (4.15)-(4.16), we have

$$\begin{aligned}&|w_j(t,x(t))| \le \tilde{w}_j(t,x(t)) \le \tilde{w}_{i_0}(\bar{t},\bar{x}) \le 2\tilde{w}_{i_0}(t,x(t)) \le 4|w_{i_0}(t,x(t))|,\\ &\forall\ x\in\mathbb{R},\quad \forall\ t\in[\tau,\bar{t}\ ],\quad \forall\ j.\end{aligned} \tag{4.18}$$

Hence, by (4.17) we get

$$\frac{d\tilde{w}_{i_0}(t,x(t))}{d_{i_0}t} \le -[\beta\sigma - 4nM_1 - M_1]\sum_{j\neq i_0}|w_{i_0}w_j(t,x(t))|\exp(\beta R_{i_0}(u)),\ \forall t\in[\tau,\bar{t}\ ]. \tag{4.19}$$

By the choice of β, we have

$$\beta\sigma - 4nM_1 - M_1 \ge 0, \tag{4.20}$$

then

$$\frac{d\tilde{w}_{i_0}(t,x(t))}{d_{i_0}t} \le 0,\quad \forall t\in[\tau,\bar{t}\ ]. \tag{4.21}$$

Hence

$$\tilde{w}_{i_0}(t,x(t)) \ge \tilde{w}_{i_0}(\bar{t},\bar{x}),\quad \forall t\in[\tau,\bar{t}\ ]. \tag{4.22}$$

This contradicts (4.15).

This shows that $T=+\infty$, then Cauchy Problem (1.1) and (1.7) admits a unique C^1 solution $u=u(t,x)$ for all $t\ge 0$. The proof of Theorem 1.1 is finished.

5. System of Planar Motion of an Elastic String

Consider the following Cauchy problem for the system of planar motion of an elastic string:

$$\begin{cases} u_t - v_x = 0,\\ v_t - (\dfrac{T(r)}{r}u)_x = 0,\end{cases} \tag{5.1}$$

$$t=0: u=\tilde{u}_0 + u_0(x)\quad v=v_0(x), \tag{5.2}$$

where $u=(u_1,u_2)^T, v=(v_1,v_2)^T,\ r=|u|=\sqrt{u_1^2+u_2^2}$, $T(r)$ is a suitably smooth function of $r>1$ such that

$$T'(\tilde{r}_0) > \frac{T(\tilde{r}_0)}{\tilde{r}_0} > 0, \tag{5.3}$$

where $\tilde{r}_0=|\tilde{u}_0|=\sqrt{|\tilde{u}_0^1|^2+|\tilde{u}_0^2|^2}>1$, $\tilde{u}_0=(\tilde{u}_0^1,\tilde{u}_0^2)^T$is a constant vector and $(u_0(x),v_0(x))$ is C^1 vector function.

Let

$$U=(u_1,u_2,v_1,v_2)^T=(u,v)^T. \tag{5.4}$$

By (5.3), the system (5.1) is strictly hyperbolic with the following real eigenvalues:

$$\lambda_1=-\sqrt{T'(r)}<\lambda_2=-\sqrt{\frac{T(r)}{r}}<\lambda_3=\sqrt{\frac{T(r)}{r}}<\lambda_4=\sqrt{T'(r)}. \tag{5.5}$$

The corresponding left and right eigenvectors are

$$\begin{cases} l_1(U)//\left(\sqrt{T'(r)}u_1, \sqrt{T'(r)}u_2, u_1, u_2\right), \\ l_2(U)//\left(-\sqrt{\dfrac{T(r)}{r}}u_2, \sqrt{\dfrac{T(r)}{r}}u_1, -u_2, u_1\right), \\ l_3(U)//\left(-\sqrt{\dfrac{T(r)}{r}}u_2, \sqrt{\dfrac{T(r)}{r}}u_1, u_2, -u_1\right), \\ l_4(U)//\left(\sqrt{T'(r)}u_1, \sqrt{T'(r)}u_2, -u_1, -u_2\right) \end{cases} \tag{5.6}$$

and

$$\begin{cases} r_1(U)//\left(u_1, u_2, \sqrt{T'(r)}u_1, \sqrt{T'(r)}u_2\right)^T, \\ r_2(U)//\left(-u_2, u_1, -\sqrt{\dfrac{T(r)}{r}}u_2, \sqrt{\dfrac{T(r)}{r}}u_1\right)^T, \\ r_3(U)//\left(u_2, -u_1, -\sqrt{\dfrac{T(r)}{r}}u_2, \sqrt{\dfrac{T(r)}{r}}u_1\right)^T, \\ r_4(U)//\left(-u_1, -u_2, \sqrt{T'(r)}u_1, \sqrt{T'(r)}u_2\right)^T. \end{cases} \tag{5.7}$$

It is easy to know that λ_2 and λ_3 are always linearly degenerate. If

$$T''(r) \equiv 0, \quad \forall\, r > 1, \tag{5.8}$$

λ_1 and λ_4 are also linearly degenerate, then all eigenvalues are linearly degenerate. By Theorem 1.1 we have

Theorem 5.1 *Suppose that the initial data $(u_0(x), v_0(x))$ satisfy the assumptions given in Theorem 1.1 and (5.8) holds. Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, Cauchy problem (5.1)-(5.2) admits a unique C^1 solution $u = u(t, x)$ for all $t \in \mathbb{R}$.*

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