J. Partial Diff. Eqs. **16**(2003), 335–346
©International Academic Publishers Vol.16 No.4

ON THE GLOBAL ATTRACTOR OF GENERALIZED GINZBURG-LANDAU EQUATION IN ONE SPATIAL DIMENSION

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(Received Nov. 7, 2002)

Abstract The target of this paper is the long time behaviour of solutions for a generalized Ginzburg-Landau equation on $\mathbb{R}$. The authors establish the existence of a global attractor of finite Hausdorff and fractal dimension in a weighted Hilbert space for the equation.
Key Words Global attractor; Hausdorff dimension; Fractal dimension; Generalized Ginzburg-Landau equation.
2000 MR Subject Classification 35Q35, 35Q80.
Chinese Library Classification O175.24.

1. Introduction

The generalized Ginzburg-Landau (GGL) equation in one spatial dimension is the equation of the following form

$$u_t = \alpha_0 u + \alpha_1 u_{xx} + \alpha_2 |u|^2 u + \alpha_3 |u|^2 u_x + \alpha_4 u^2 \bar{u}_x + \alpha_5 |u|^{2\sigma} u,$$

where $\alpha_k = a_k + ib_k (k = 0, 1, 2, 3, 4, 5)$ are complex constants and σ is a positive constant.

The GGL equation arises as a generic amplitude equation close to the onset of instabilities that lead to chaotic dynamics in fluid mechanical systems, as well as in reaction-diffusion processes, nonliear optics, electric foring of liquid crystal and many other fields. For the further background of the equation, the reader may consult Doelman [1], Duan [2-4], Eckhaus and Iooss [5].

For long time behaviours of solutions, Guo and Gao [6] and Guo and Wang [7] have proved the existence of global attractors of the Cauchy problem of GGL equation with periodic boundary condition for the cases of one and two spatial dimensions. They have also discussed Gevrey regularity and approximate inertial manifold for the same problem in two spatial dimensions [8]. Guo and Li [9] have showed the existence of a global attractor in a so-called uniformly weighted space for Cauchy problem of GGL equation in an unbounded two-dimensional-domain, in which $\alpha_0 = a_0 > 0$ and $\operatorname{Re}\alpha_2 = 0$ the so-called super-critical case was considered and the weight function was integrable.

In this paper we are interested in Cauchy problem of the following GGL equation

$$u_t = \alpha_0 u + \alpha_1 u_{xx} + \alpha_2 |u|^2 u + \alpha_3 |u|^2 u_x + \alpha_4 u^2 \bar{u}_x + \alpha_5 |u|^4 u + f, \tag{1}$$

with initial data

$$u|_{t=0} = u^0, \tag{2}$$

where $\alpha_0 = a_0 < 0$, $\operatorname{Re}\alpha_2 > 0$, that is so-called sub-critical case, and $f = f(x)$ is known.

Under the condition $a_0 < 0$, the dynamics of GGL is dissipative, while the compactness of the solution operator to the GGL is absent due to the noncompact embedding $H^1 \hookrightarrow L^2$ in $I\!R$. To obtain certain attracting behavior for Cauchy problem (1) and (2) in some sense, we study the problem in the context of weighted Hilbert spaces $H_{0,r}$ $(r > 0)$ with the norm

$$\|u\|_{0,r} = \int_{-\infty}^{+\infty} |u(x)|^2 (1 + x^2)^r dx.$$

We will also employ weighted Hilbert spaces $H_{l,r}$ whose norm is

$$\|u\|_{l,r}^2 = \sum_{j=0}^{l} \|\partial_x^j u\|_{0,r}^2.$$

Such weighted spaces were used in [10] for Cauchy problem of certain reaction-diffusion equation.

This work is organized as follows: the weighted estimates are presented in the second section, the discussion follows Ref. [11]. The existence and uniqueness of the global solution in the weighted space are shown in the third section by Henry's theorem. In the fourth section the existence of the global attractor is proved in an analogous way as Ref. [10]. Finally the finite Hausdorff and fractal dimensions of the attractor are estimated in the last section.

2. Weighted Estimates of Solutions

In this section a uniform bound in time t for the solutions of (1) and (2) in weighted space $H_{l,r}$ will be figured out, which is crucial for global existence of a solution and the attracting behaviors of the solution.

Before stating the results, we need to prepare some technical tools. Let

$$\varphi(x) = \varphi_\epsilon(x) = (1 + (\epsilon x)^2)^r$$

with $0 < \epsilon \leq 1$. It is not hard to verify that (see [10]),

$$\|u\varphi^{1/2}\| \leq \|u\|_{0,r} \leq \epsilon^{-2r}\|u\varphi^{1/2}\|, \tag{3}$$

$$|\varphi_x(x)| \leq \epsilon r\varphi(x), \tag{4}$$

$$\epsilon^r\|u\|_{l,r} \leq \|u\varphi^{1/2}\|_l \leq \|u\|_{l,r}, \tag{5}$$

where $\|\cdot\|$ and $\|\cdot\|_l$ denote $\|\cdot\|_{0,0}$ and $\|\cdot\|_{l,0}$, respectively. We will write $H^l = H_{l,0}$. It is clear that $H_{0,r} \subseteq H_{0,0} = L^2$ and $H_{l,r} \subseteq H^l$ for any $r \geq 0$.

To make our presentation simpler, we will merely give formal arguments in the proofs and leave out related approximations.

Lemma 1 *Assume that* $a_0 < 0$, $a_5 < 0 < a_1$, $0 \leq a_2 < \sqrt{a_0 a_5}$, $-a_1 a_5 > (b_3 - b_4)^2$, $u^0 \in H_{0,r}$ *and* $f \in H_{0,r}$. *Then, for a proper choice of* $\epsilon \in (0,1)$, *the solution* $u(t)$ *of (1)(2) satisfies the following estimates*

$$\|u(t)\|_{0,r} \leq K_0 \qquad \forall t \geq 0, \tag{6}$$

$$\limsup_{t\to\infty} \|u(t)\varphi^{1/2}\| \leq \rho_0, \tag{7}$$

$$a_1 \int_s^t \|u_x(\tau)\varphi^{1/2}\|^2 d\tau \leq \|u(s)\|_{0,r}^2 + \frac{2\|f\|_{0,r}^2}{|a_0|}(t-s), \quad \forall 0 \leq s < t, \tag{8}$$

where K_0 *depends only on* $\|u^0\|_{0,r}$, $\|f\|_{0,r}$ *and* α_j *for* $j = 0, 1, \cdots, 5$*; while* ρ_0 *depends only on* $\|f\|_{0,r}$ *and* a_0.

Proof Multiplying both sides of (1) by $\overline{u}\varphi$, integrating with respect to x and taking the real part, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\varphi^{1/2}\|^2 = {} & a_0\|u\varphi^{1/2}\|^2 - a_1\|u_x\varphi^{1/2}\|^2 - \mathrm{Re}\,\alpha_1 \int u_x u \varphi_x \\ & + a_2 \int |u|^4\varphi + a_5 \int |u|^6 \varphi + \mathrm{Re}\,(a_3 + a_4) \int |u|^2 u_x \overline{u}\varphi \\ & - (b_3 - b_4)\mathrm{Im} \int |u|^2 u_x \overline{u}\varphi + \mathrm{Re} \int f\overline{u}\varphi,\end{aligned} \tag{9}$$

where $\int$ represents $\int_{-\infty}^{\infty}$. Recalling (4), we have

$$\begin{aligned}\left|\mathrm{Re}\,\alpha_1 \int u_x u \varphi_x\right| & \leq \epsilon r|\alpha_1| \int |u_x|\cdot|u|\varphi \\ & \leq \epsilon C(\|u_x\varphi^{1/2}\|^2 + \|u\varphi^{1/2}\|^2),\end{aligned} \tag{10}$$

$$\begin{aligned}\left|\mathrm{Re}\,(a_3 + a_4) \int |u|^2 u_x \overline{u}\varphi\right| & = \frac{1}{2}|a_3 + a_4| \cdot \left|\int |u|^2(u_x\overline{u} + \bar{u}_x u)\varphi\right| \\ & = \frac{1}{2}|a_3 + a_4| \cdot \left|\int |u|^2(|u|^2)_x\varphi\right| \\ & = \frac{1}{4}|a_3 + a_4| \cdot \left|\int |u|^4\varphi_x\right| \\ & \leq \epsilon C \int |u|^4 \varphi\end{aligned} \tag{11}$$

and

$$\left|(b_3-b_4)\mathrm{Im}\int |u|^2u_x\overline{u}\varphi\right| \le B_1B_2\left(\int |u|^6\varphi\right)^{1/2}\left(\int |u_x|^2\varphi\right)^{1/2} \le \frac{B_1^2}{2}\int |u|^6\varphi+\frac{B_2^2}{2}\int |u_x|^2\varphi, \tag{12}$$

where Young inequality is used with the constants B_1 and B_2 satisfying $B_1B_2=|b_3-b_4|$, $B_2^2-a_1<0$ and $B_1^2+a_5<0$. Noting that $a_2+\epsilon C\le\sqrt{a_0a_5}$ for small ϵ, we have

$$[a_2+\epsilon C]\int |u|^4\varphi \le \sqrt{a_0a_5}\left(\int |u|^6\varphi\right)^{1/2}\left(\int |u|^2\varphi\right)^{1/2} \le \frac{|a_0|}{2}\int |u|^2\varphi+\frac{|a_5|}{2}\int |u|^6\varphi.$$

It is clear that

$$\left|\mathrm{Re}\int f\overline{u}\varphi\right| \le \frac{1}{|a_0|}\|f\varphi^{1/2}\|^2+\frac{|a_0|}{4}\|u\varphi^{1/2}\|^2. \tag{13}$$

Substituting (10) through (13) into (9), for small ϵ, we get

$$\frac{d}{dt}\|u\varphi^{1/2}\|^2+a_1\|u_x\varphi^{1/2}\|^2\le\frac{a_0}{4}\|u\varphi^{1/2}\|^2+\frac{2}{|a_0|}\|f\varphi^{1/2}\|^2. \tag{14}$$

Applying Gronwall inequality to (14), we have

$$\|u\varphi^{1/2}\|^2\le\|u^0\varphi^{1/2}\|^2e^{a_0t/4}+\frac{8\|f\varphi^{1/2}\|^2}{a_0^2}(1-e^{a_0t/4}), \tag{15}$$

which implies both the estimate (6) by (3) and the estimate (7) immediately. the estimate (8) follows by integrating both sides of the inequality (14) from s to t.

Lemma 2 *Besides the assumptions in Lemma 1, assume that $u^0\in H_{1,r}$. Then, the solution of Cauchy problem (1) and (2) possesses estimates*

$$\|u(t)\|_{1,r}\le K_1,\qquad \forall t\ge 0, \tag{16}$$

$$\|u_x(t)\|_{0,r}\le K_0(t^{-1/2}+1),\qquad \forall t>0, \tag{17}$$

$$\int_0^T\|u_{xx}(\tau)\|_{0,r}^2d\tau\le K_1(1+T),\qquad \forall T>0, \tag{18}$$

$$\limsup_{t\to\infty}\|u(t)\|_{1,r}\le\rho_1, \tag{19}$$

where K_1 and ρ_1 depend only on $\|u^0\|_{1,r}$, $\|f\|_{0,r}$ and α_j's; the dependence of K_0 is the same as that of K_0 in Lemma 1.

Proof Multiplying both sides of (1) by $-\bar{u}_{xx}\varphi$, integrating with respect to x and taking the real part, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_x\varphi^{1/2}\|^2 + a_1\|u_{xx}\varphi^{1/2}\|^2 \\ = a_0\|u_x\varphi^{1/2}\|^2 - \mathrm{Re}\int \bar{u}_x u_t\varphi_x + \mathrm{Re}\,\alpha_0\int u\bar{u}_x\varphi_x + \mathrm{Re}\,\alpha_2\int |u|^2u(-\bar{u}_{xx}\varphi) \\ + \mathrm{Re}\,\alpha_3\int |u|^2u_x(-\bar{u}_{xx}\varphi_x) + \mathrm{Re}\,\alpha_4\int u^2\bar{u}_x(-\bar{u}_{xx}\varphi) \\ + \mathrm{Re}\,\alpha_5\int |u|^4u(-\bar{u}_{xx}\varphi) + \mathrm{Re}\int f(-\bar{u}_{xx}\varphi). \qquad (20)\end{aligned}$$

Write the last seven terms on the right side of (20) by $I_i (i = 1, 2, \cdots, 7)$.

To estimate these terms, we will take advantage of the inequality (4) and another inequality

$$\|u\|_\infty \le \|u_x\varphi^{1/2}\|^{1/2} \cdot \|u\varphi^{1/2}\|^{1/2},$$

which is a corollary of $\|u\|_\infty^2 \le \|u_x\| \cdot \|u\|$ and $\varphi(x) \ge 1$ for $x \in I\!R$.

Let C stand for the constants that may depend on $\|u(t)\|_{0,r}$ in the coming estimates.

$$\begin{aligned}I_1 \le& \left|\mathrm{Re}\int \bar{u}_x u_t\varphi_x\right| \le \epsilon r\int |\bar{u}_x|\cdot|u_t|\varphi \\ \le& \epsilon r\Big(|a_0|\int |u|\cdot|\bar{u}_x|\varphi + |\alpha_1|\int |u_{xx}|\cdot|\bar{u}_x|\varphi + |\alpha_2|\|u\|_\infty^2\int |u|\cdot|\bar{u}_x|\varphi \\ &+ (|\alpha_3| + |\alpha_4|)\|u\|_\infty^2\int |u_x|^2\varphi + |\alpha_5|\cdot\|u\|_\infty^4\int |u|\cdot|\bar{u}_x|\varphi + \int |f|\cdot|\bar{u}_x|\varphi\Big) \\ \le& \epsilon\Big(C + \|u_x\varphi^{1/2}\|^2 + \|u_x\varphi^{1/2}\|^4 + \|u_{xx}\varphi^{1/2}\|^2\Big), \qquad (21)\end{aligned}$$

$$I_2 \le \left|a_0\int u\bar{u}_x\varphi_x\right| \le \epsilon C\Big(\|u\varphi^{1/2}\|^2 + \|u_x\varphi^{1/2}\|^2\Big), \qquad (22)$$

$$\begin{aligned}I_3 \le& \left|\mathrm{Re}\,\alpha_2\int |u|^2u(-\bar{u}_{xx}\varphi)\right| \le \epsilon\|u_{xx}\varphi^{1/2}\|^2 + C\epsilon^{-1}\|u\|_\infty^4\|u\varphi^{1/2}\|^2 \\ \le& \epsilon\|u_{xx}\varphi^{1/2}\|^2 + C\epsilon^{-1}\|u_x\varphi^{1/2}\|^2, \qquad (23)\end{aligned}$$

$$\begin{aligned}I_4 \le& \left|\mathrm{Re}\,\alpha_3\int |u|^2u_x(-\bar{u}_{xx}\varphi)\right| \le \epsilon\|u_{xx}\varphi^{1/2}\|^2 + C\epsilon^{-1}\|u\|_\infty^4\|u_x\varphi^{1/2}\|^2 \\ \le& \epsilon\|u_{xx}\varphi^{1/2}\|^2 + C\epsilon^{-1}\|u_x\varphi^{1/2}\|^4, \qquad (24)\end{aligned}$$

$$I_5 \le \left|\mathrm{Re}\,\alpha_4\int u^2\bar{u}_x(-\bar{u}_{xx}\varphi)\right| \le \epsilon\|u_{xx}\varphi^{1/2}\|^2 + C\epsilon^{-1}\|u_x\varphi^{1/2}\|^4, \qquad (25)$$

$$\begin{aligned}I_6 \le& \left|\mathrm{Re}\,\alpha_5\int |u|^4u(-\bar{u}_{xx}\varphi)\right| \le \epsilon\|u_{xx}\varphi^{1/2}\|^2 + C\epsilon^{-1}\|u\|_\infty^4\|u\varphi^{1/2}\|^2 \\ \le& \epsilon\|u_{xx}\varphi^{1/2}\|^2 + C\epsilon^{-1}\|u_x\varphi^{1/2}\|^4, \qquad (26)\end{aligned}$$

$$I_7 \le \left|\mathrm{Re}\int f(-\bar{u}_{xx}\varphi)\right| \le \epsilon\|u_{xx}\varphi^{1/2}\|^2 + \epsilon^{-1}\|f\varphi^{1/2}\|^2. \qquad (27)$$

Combining all these estimates and choosing $\epsilon > 0$ small enough, we obtain

$$\frac{d}{dt}\|u_x\varphi^{1/2}\|^2 + a_1\|u_{xx}\varphi^{1/2}\|^2 \le p_1(\|u(t)\|_{0,r})(p_2(\|u(t)\|_{0,r}) + \|u_x\varphi^{1/2}\|^2)^2, \qquad (28)$$

where $p_1(s)$ and $p_2(s)$ are increasing functions in s.

Setting $c_1 := p_1(K_0)$, $c_2 := p_2(K_0)$, where K_0 is the constant in (6) and $y(t) := c_2 + \|u_x(t)\varphi^{1/2}\|^2$, we have (28) in form of

$$\frac{dy}{dt} + a_1\|u_{xx}\varphi^{1/2}\|^2 \le c_1 y^2(t). \tag{29}$$

Therefore,

$$y(t) \le y(s)\exp\left\{c_1\int_s^t y(\tau)d\tau\right\}.$$

Combining this inequality with the estimate (8) from Lemma 1, we can obtain those estimates claimed in this lemma as follows.

When $t \in (0,1]$, we have the following two estimates

$$y(t) \le y(0)\exp\left\{c_1\int_0^t y(\tau)d\tau\right\} \le K_0\left(1 + \|u_x(0)\varphi^{1/2}\|^2\right),$$

if $s = 0$ and

$$y(t) \le y(s)\exp\left\{c_1\int_0^t y(\tau)d\tau\right\} \le K_0\left(1 + \frac{1}{t}\right),$$

if $s \in (0,1]$ is given by

$$y(s) = \frac{1}{t}\int_0^t y(\tau)d\tau,$$

where K_0 always stands for a constant depending only on $\|u^0\|_{0,r}$, $\|f\|_{0,r}$ and α_k.

When $t > 1$, set $s = t-1$ and $s \in [t-1,t]$ satisfying $y(s) = \int_{t-1}^t y(\tau)d\tau$, we have

$$y(t) \le K.$$

The estimates (16), (17) and (19) follow immediately from the above results. The estimate (18) follows by integrating both sides of the inequality (29) and using the above results for $y(t)$ to estimate the integral on the right side.

We conclude this section by a result on the Lipschitz continuity of the solution about its initial value.

Set $G(u) := \alpha_2|u|^2u + \alpha_3|u|^2u_x + \alpha_4u^2\bar{u}_x + \alpha_5|u|^4u$ and consider the two solutions $u(t)$ and $v(t)$ of (1) with initial values u^0 and v^0 respectively. Then, repeating those arguments in the proof of Lemma 2, we can get the following

$$\left|\int[G(u)-G(v)](\overline{u}-\overline{v})\varphi\right| \le p(t,\|u_x\|_{0,r}^2,\|v_x\|_{0,r}^2)\|(u-v)\varphi^{1/2}\|^2 + \frac{a_1}{4}\|(u_x-v_x)\varphi^{1/2}\|^2,$$

where $p(t,s_1,s_2)$ is linear about s_1 and s_2. Writing $\omega = u - v$, we have

$$\omega_t = \alpha_0\omega + \alpha_1\omega_{xx} + G(u) - G(v).$$

Multiplying $\overline{\omega}\varphi$, integrating with respect to x and taking the real part, we can obtain

$$\frac{d}{dt}\|\omega\varphi^{1/2}\|^2 \le p(t, \|u_x\|_{0,r}^2, \|v_x\|_{0,r}^2)\|(u-v)\varphi^{1/2}\|^2,$$

combining which with the estimate (8) implies

Lemma 3 *Under the hypotheses of Lemma 1, for the solutions $u_1(t)$ and $u_2(t)$ of the equation (1), we have*

$$\|u_1(t) - u_2(t)\|_{0,r} \le C\|u_1(0) - u_2(0)\|_{0,r}, \quad \forall t \ge 0, \tag{30}$$

where C is only dependent on $\|u^0\|_{0,r}$, $\|v^0\|_{0,r}$, $\|f\|_{0,r}$ and t.

3. Existence and Uniqueness of Global Solutions

In this section we will establish the existence and uniqueness of the global solutions of (1) and (2) in the space $C([0,\infty); H_{1,r}) \cap C^1((0,\infty); H_{1,r})$.

First of all we have the following result for the nonlinear part of (1), the proof of which is a modification of that in [11]. We leave the details to the reader.

Lemma 4 *G is a bounded and locally Lipschitz continuous map from $H_{1,r}$ into $H_{0,r}$, namely*

$$\|G(u)\|_{0,r} \le C(R), \qquad \|G(u) - G(v)\|_{0,r} \le L(R)\|u - v\|_{1,r},$$
$$\forall u, v \in H_{1,r}, \qquad \|u\|_{1,r} \le R, \qquad \|v\|_{1,r} \le R,$$

where $C(\cdot)$ and $L(\cdot)$ are positive increasing functions.

For the linear part of (1) we have

Lemma 5 *$A := \alpha_1\partial_x^2 + a_0 I$ is a sectorial operator in the space $H_{0,r}$ and generates an analytic semigroup $\mathcal{T}$ in the space $H_{0,r}$.*

Proof It has been proved in [12] that A is an infinitesimal generator of a C_0-semigroup $\mathcal{T}$ in $H_{0,r}$. Working on the initial problem

$$\frac{du}{dt} = \alpha_1 u_{xx} + a_0 u, \qquad u|_{t=0} = u^0,$$

we have

$$\begin{cases} \|u(t)\|_{0,r} \le \|u^0\|_{0,r}, & \forall t \ge 0, \\ \|u_{xx}(t)\|_{0,r} \le \dfrac{2\sqrt{2}}{a_1} t^{-1}\|u^0\|_{0,r}, & \forall t > 0, \end{cases}$$

as we did in the proof of Lemmas 1 and 2. Therefore, $\mathcal{T}$ is an analytic semigroup and A is a sectorial operator in the space $H_{0,r}$, see Henry 1.3 [13].

Therefore we have the existence and uniqueness result.

Theorem 1 *Under the condition of Lemma 1, if $u^0 \in H_{1,r}$, then the initial problem (1) and (2) have a unique solution*

$$u \in C([0,\infty); H_{1,r}) \cap C^1((0,\infty); H_{1,r}),$$

$$u \in L^2(0,T;H_{2,r}), \qquad \forall T > 0.$$

Proof Lemma 4 and Lemma 5 imply the local existence and uniqueness of the solution of (1) and (2) by Theorem 3.3.3 in [13] (with $\mathtt{X} = H_{0,r}$, $\alpha = 1/2$). The estimates in the last section guarantee the global existence of the solution of (1) and (2) according to Theorem 3.3.4 in [13].

Therefore, there is a nonlinear operators $\{S(t)\}$ defined by the initial value problems (1) and (2) such that $S(t) : u^0 \to u(t)$ in $H_{1,r}$, $\forall t \geq 0$. Lemma 3 shows that $\{S(t)\}$ can be extended as a semigroup of the nonlinear continuous operators on $H_{0,r}$ and $u(t) = S(t)u^0 \in H_{0,r}$ can be regarded as a generalized solution of the problems (1) and (2) for each $u^0 \in H_{0,r}$. Moreover, the estimate (17) guarantees the boundedness of the operator $S(t)$ from $H_{0,r}$ to $H_{1,r}$ for each $t > 0$.

To conclude this section, we formulate the following result on the smoothness of the above generalized solution.

Theorem 2 *Besides the assumptions in Lemma 1, if $f \in H_{l,r}$, $\forall l \geq 0$, the generalized solution of (1) and (2) $u(t) = S(t)u^0 \in C^\infty$ for $t > 0$. In particular, it is a classical solution and $u(t) \in H_{l,r}, \forall l \geq 0$, $t > 0$.*

Proof By successively differentiating $G(u)$ and using the bounds from the previous step at each stage one can obtain the local Lipschitz continuity of G from $H_{l+1,r}$ to $H_{l,r}$, which implies that $u \in H_{l+1}$ for $l = 0, 1, 2, \ldots$ as $u^0 \in H_{1,r}$ by Henry 3.5 [13].

When $u^0 \in H_{0,r}$, for any $\tau > 0$, noting $u(\tau) = S(\tau)u^0 \in H_{1,r}$ and applying the above arguments to $S(t)u(\tau)$, we see that $u(t+\tau) \in H_{l+1}$ for $t > 0$. Therefore, $u(t) \in H_{l+1}$, for any $t > 0$. Hence $u(t) \in C^\infty$ by embedding $H_{l+1,r} \hookrightarrow H^{l+1} \hookrightarrow C^l$, for any $l > 0$. On the other hand, $u_t(t) = Au + G(u) + f \in H_{l-1,r}$ for $l = 1, 2, \ldots$ implies $u_t(t) \in C^\infty$ for $t > 0$. Differentiating the equation, we have

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial t} &= \alpha_0 u_x + \alpha_1 u_{xxx} + \alpha_2 \left[2|u|^2 u_x + u^2 \overline{u}_x\right] \\ &\quad + \alpha_3 \left[u_x^2 \overline{u} + u|u_x|^2 + |u|^2 u_{xx}\right] + \alpha_4 \left[u^2 \overline{u}_{xx} + 2u|u_x|^2\right] \\ &\quad + \alpha_5 \left[3|u|^4 u_x + 2u^3 \overline{u}\overline{u}_x\right] + f_x \in H_{l-2}, \qquad \forall l > 2,\end{aligned}$$

and thus $\dfrac{\partial^2 u}{\partial x \partial t} \in C^\infty$. Repeating this process, we will have that u is C^∞ in both x and t.

4. Existence of Attractors

In this section we will demonstrate the existence of the attractors of (1) and (2). Firstly, we formulate two definitions and a proposition following [10].

Definition 1 *A set Q is an absorbing set of a semigroup $S(t)$ acting on a Hilbert space H, provided that there is a $t_0 > 0$ for any bounded set $B \subset H$ so that $S(t)B \subset Q$ as $t \geq t_0$.*

Definition 2 *A set $\mathcal{A} \subset H$ is a global attractor of a semigroup $S(t)$ acting in a Hilbert space H, if the following conditions are fulfilled:*

1. *$\mathcal{A}$ is compact in H,*
2. *$\mathcal{A}$ is an invariant set of $S(t)$, namely, $S(t)\mathcal{A} = \mathcal{A}$, for all $t > 0$,*
3. *$\mathcal{A}$ is an attracting set of $S(t)$ in H, namely, every neighborhood Q of the set $\mathcal{A}$ in H is an absorbing set of $S(t)$.*

Proposition 1 *A semigroup $S(t)$ in a Hilbert space H with a compact attracting set possesses a global attractor.*

Now we are ready to build up a global attractor of the semigroup $S(t)$ in $H_{0,r}$.

It follows from Lemma 1 that $Q_0 = \{u \in H_{0,r}; \|u\|_{0,r} \le 2\rho_1\}$ is an absorbing set of $S(t)$ in $H_{0,r}$. Lemma 2 implies that $Q_1 = S(1)Q_0$ is bounded in $H_{1,r}$. Therefore, $Q_2 = \bigcup_{t\ge 0} S(t)Q_1$ is an invariant absorbing set of $S(t)$ in $H_{0,r}$, which is bounded in $H_{1,r}$.

In order to construct a compact attracting set of $S(t)$, we begin with the following problem,

$$\omega_t = a_0\omega + \alpha_1\omega_{xx} + G(S(t)u^0), \qquad \omega|_{t=0} = 0. \tag{31}$$

Recalling that for any $u^0 \in Q_2$ and $f \in H_{0,r}$, $G(S(t)u^0) \in H_{1,r}$, we have that the unique solution $\omega(t, u^0) \in H_{1,r}$ and $\|\omega(t, u^0)\|_{1,r} \le C$ with C dependent on Q_2.

Set $Q_3 = \{\omega(t, u^0) : t \ge 0, u^0 \in Q_2\}$. Obviously, Q_3 is bounded in $H_{1,r}$. In fact, we have

Lemma 6 *Q_3 is bounded in $H_{0,r(1+\delta)}$ $(0 < \delta < 1)$.*

Proof For $0 < \delta < 1$, we have

$$\left|2\text{Re}\int G(u)\overline{\omega}\varphi^{1+\delta}dx\right| \le |a_0|\|\omega\varphi^{\frac{1+\delta}{2}}\|^2 + \frac{1}{|a_0|}\|G(u)\varphi\|^2 \le |a_0|\|\omega\varphi^{\frac{1+\delta}{2}}\|^2 + C,$$

where C is determined by Q_2.

Multiplying (31) by $\overline{\omega}\varphi^{1+\delta}$ with $\delta \in (0,1)$, integrating with respect to x and taking the real part, we get

$$\frac{d}{dt}\left\|\omega\varphi^{\frac{1+\delta}{2}}\right\|^2 + 2a_1\left\|(\omega\varphi^{\frac{1+\delta}{2}})_x\right\|^2 - 2a_0\left\|\omega\varphi^{\frac{1+\delta}{2}}\right\|^2 \le 2\text{Re}\int G(u)\overline{\omega}\varphi^{1+\delta}dx.$$

The rest part of the proof is by Gronwall inequality.

Therefore we get that Q_3 is compact in $H_{0,r}$ by the following lemma.

Lemma 7 *If a set Q is bounded in $H_{0,r+\delta}(\delta > 0)$ and in $H_{1,r}$, then Q is compact in $H_{0,r}$. (See [10]).*

Now we can show the existence of the global attractor. We first prove that $Q = Q_3 + f_0$ is a compact attracting set of $S(t)$ in $H_{0,r}$, where $f_0 \in H_{1,r}$ is the unique solution of

$$\alpha_0 f_0 + \alpha_1 f_{0xx} = f,$$

with $f \in H_{0,r}$.

For bounded $B \subset H_{0,r}$, there exists $t_1 > 0$ depending on B only such that $S(t)B \subset Q_2$, $\forall t > t_1$. For $u^0 \in B$, let $u_1(t) = S(t+t_1)u^0 \in Q_2$ and write $u_1(t) = v(t) - f_0 + \omega(t, u^0) + f_0$, where $\omega(t, u^0) + f_0 \in Q$ and $v(t) \in H_{1,r}$ is the unique solution of the problem

$$v_t = a_0 v + \alpha_1 v_{xx} + f, \qquad v(0) = u_1(0).$$

Noting $(v - f_0)_t = a_0(v - f_0) + \alpha_1 (v - f_0)_{xx}$, we have

$$\|(v - f_0)\|_{0,r}^2 \leq \|(v(0) - f_0)\|_{0,r}^2 e^{2a_0 t},$$

which implies $\mathrm{dist}\,(u_1(t), Q) \leq \|v(t) - f_0\|_{0,r} \leq \|u_1(0) - f_0\|_{0,r} e^{a_0 t} \leq Ce^{a_0 t}$, where C depends on Q_2. Therefore, $\mathrm{dist}\,(S(t)B, Q) \leq C' e^{a_0 t}$ for $t > t_1$ with C' dependent on Q, namely, Q is also an attracting set of $S(t)$ in $H_{0,r}$. When $f \in H_{l,r}$, $\forall l > 0$, Theorem 2 shows that Q is bounded in $H_{l,r}$ for any $l > 0$ which implies $Q \subset C^\infty$. Then applying Proposition 1, we have the result

Theorem 3 *Under the condition of Theorem 1, the semigroup determined by (1) and (2) possesses a global attractor $\mathcal{A}$ in $H_{0,r}$, which is bounded in $H_{1,r}$, $\forall r > 0$. Moreover, if $f \in H_{l,r}$, $\forall l > 0$, then $\mathcal{A} \subset C^\infty \cap H_{l,r}$, $\forall l > 0$.*

5. The Dimension of the Attractor

In this section we will prove that the attractor $\mathcal{A}$ obtained last section is finite dimensional.

We first show the differentiability of the operator $S(t)$ with respect to u^0 on $\mathcal{A}$.

Lemma 8 *Under the conditions of Theorem 1, the operator $S(t)$ for any $t \geq 0$ is uniformly Frechlet differentiable on $\mathcal{A}$ in the metric of $H_{0,0} = L^2$.*

Proof The differential $DS(t, u^0)v^0$ of the operator $S(t)$ at $u^0 \in \mathcal{A}$ with $v^0 \in L^2$ should be the solution of the equation (1) in variations

$$v_t = \alpha_0 v + \alpha_1 v_{xx} + \Gamma_0(u)v, \qquad v(0) = v^0, \tag{32}$$

where $u(t) = S(t)u^0$,

$$\begin{aligned}\Gamma_0(u)v :=\ & \alpha_2 \left(2|u|^2 v + u^2 \overline{v}\right) + \alpha_3 \left(|u|^2 v_x + u u_x \overline{v} + \overline{u} u_x v\right) \\ & + \alpha_4 \left(u^2 \overline{v_x} + 2u\bar{u}_x v\right) + \alpha_5 \left(3|u|^4 v + 2|u|^2 u^2 \overline{v}\right),\end{aligned}$$

which has a unique solution for $v^0 \in H^1$. The boundedness of $\mathcal{A}$ in $H_{1,r}$ and $H_{1,r} \hookrightarrow H^1$ implies that $\mathcal{A}$ is bounded in H^1. For u^0 and $u^0 + v^0 \in \mathcal{A}$, write $\psi = S(t)(u^0 + v^0) - S(t)u^0 - D(S(t)u^0)v^0$. It is clear that ψ satisfies

$$\psi_t = \alpha_0 \psi + \alpha_1 \psi_{xx} + \Gamma_0(u_1)\psi + \Gamma_1(u_2, u_1), \qquad \psi(0) = 0, \tag{33}$$

where $u_1 = S(t)u^0, u_2 = S(t)(u^0 + v^0)$ and $\Gamma_1(u_1, u_2) = G(u_2) - G(u_1) - \Gamma_0(u_1)(u_2 - u_1)$. It can be proved that $\|\psi(t)\| \leq C\|v^0\|^{3/2}$ for $0 < t \leq T$ with C dependent on $\mathcal{A}$ so that $S(t)$ is uniformly differentiable on $\mathcal{A}$.

So far we have the following facts: (1) $\mathcal{A}$ is compact in L^2 for $\mathcal{A}$ is compact in $H_{0,r}$, and $\mathcal{A}$ is bounded in H^1 for $\mathcal{A}$ is bounded in $H_{1,r}$, (2) $S(t)\mathcal{A} = \mathcal{A}$, and (3) the differential $DS(t, u^0)$ of $S(t)$ is bounded on $\mathcal{A}$.

Now we can estimate the Hausdorff and fractal dimensions of $\mathcal{A}$ by Theorem 3.3 in Chapter 5 of [14]. The strategy is that $\dim_H(\mathcal{A}) \le m$ and $\dim_F(\mathcal{A}) \le 2m$ if one can establish the following inequality for an integer m,

$$\limsup_{t\to\infty}\left\{\sup_{u^0\in\mathcal{A}}\sup_{\|v_j^0\|\le 1, j=1,\dots m}\frac{1}{t}\int_0^t \mathrm{Re}\,(\mathrm{Tr}\,[F^{'}(u(\tau)Q_m(\tau)])d\tau\right\}\le 0, \tag{34}$$

where, in our case, $F^{'}(u(\tau)) = \alpha_0 I + \alpha_1\partial_x^2 + \Gamma_0(u(\tau))$, $Q_m(\tau) = Q(\tau, u^0, v_1^0, ..., v_m^0)$ is the projection from L^2 onto span $\{v_1(\tau), ..., v_m(\tau)\}$ with $v_j(\tau) = D(S(\tau)u^0)v_j^0$, $v_j^0 \in L^2, j = 1, ..., m$.

For any fixed $\tau > 0$, let $\varphi_1, \varphi_2, \cdots, \varphi_m \in H^1$ be the standard orthogonal basis of span $\{v_1(\tau), ..., v_m(\tau)\} \subset L^2$. We have

$$\begin{aligned}\mathrm{Re}\,\mathrm{Tr}\,[F^{'}(u(\tau)Q_m(\tau)] &= \sum_{j=1}^{m}\mathrm{Re}\,(F^{'}(u(\tau))\varphi_j, \varphi_j)\\ &= \sum_{j=1}^{m}\left\{a_0\|\varphi_j\|^2 - a_1\|\varphi_{jx}\|^2 + \mathrm{Re}\,(\Gamma_0(u(\tau))\varphi_j, \varphi_j)\right\}.\end{aligned}$$

By the facts $u(\tau) \in \mathcal{A}$ and $\mathcal{A}$ is bounded in H^1, and the generalized Sobolev-Lieb-Thirring inequality(see [14])

$$\int \rho(x)^3 dx \le K\sum_{j=1}^{m}\|\varphi_{jx}\|^2,$$

where $\rho(x) = \sum_{j=1}^{m}|\varphi_j(x)|^2$, the constant K does not depend on the family φ_j and m, we get that

$$\begin{aligned}\mathrm{Re}\,(\Gamma_0(u(\tau))\varphi_j, \varphi_j)\} &\le C\int[|u(\tau)|^2 + |u(\tau)|^4 + |u(\tau)|\,|u_x(\tau)|\,]\rho(x)dx\\ &\le C_0 + a_1\sum_{j=1}^{m}\|\varphi_{jx}\|^2,\end{aligned}$$

where the constant $C_0 > 0$ is determined by $\mathcal{A}$ and the coefficients of the equation (1). Hence

$$\mathrm{Re}\,\mathrm{Tr}\,[F^{'}(u(\tau)Q_m(\tau)] \le a_0 m + C_0$$

and consequently the inequality (34) holds so long as $m > -C_0/a_0$.

Theorem 4 *Under the conditions of Theorem 1, the global attractor $\mathcal{A}$ obtained in Theorem 3 has finite Hausdorff and fractal dimensions.*

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