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# EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTION OF DELAYED LOGISTIC EQUATION AND ITS ASYMPTOTIC BEHAVIOR*

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**Abstract** In this paper, our main aim is to study the existence and uniqueness of the periodic solution of delayed Logistic equation and its asymptotic behavior. In case the coefficients are periodic, we give some sufficient conditions for the existence and uniqueness of periodic solution. Furthermore, we also study the effect of time-delay on the solution.

**Key Words** Logistic equation; periodic; asymptotic behavior; time delay; uniqueness.

**2000 MR Subject Classification** 35K57.

**Chinese Library Classification** O175.26, O174.29, O175.21.

## 1. Introduction

The delayed Logistic differential equation

$$Lu(t,x) = u(t,x)\left[a(t,x) - b(t,x)u(t,x) - c(t,x)u(t-\tau,x)\right], \quad (t,x) \in R^+ \times \Omega, \tag{1.1}$$

$$B[u](t,x) = 0, \qquad (t,x) \in R^+ \times \partial\Omega, \tag{1.2}$$

$$u(t,x) = \phi(t,x), \qquad (t,x) \in [-\tau,0] \times \overline{\Omega}, \tag{1.3}$$

is given as a model of single-species population growth. In [1] the authors have studied the case that the coefficients vary periodically and the time delay is given as $\tau = mT$, where $m$ is a positive integer and $T$ is the period. In [2] the coefficients only associated with $x$ has been studied, which implies the steady-state solution is globally asymptotic stable for every given $\tau > 0$. In [3-5] the systems of parabolic equations with delays are studied, which imply the quasi-solutions for single equation may be obtained. In this

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paper we study the existence and uniqueness of the periodic solution of the problem (1.1)(1.2) and the asymptotic behavior of the problem (1.1)-(1.3).

We give the hypotheses below:

$(H_1)$ $\Omega$ is a bounded domain in $R^n$ with smooth boundary $\partial\Omega$, and $L$ is an operator defined as $L = \partial/\partial t - \triangle$ , where $\triangle$ denotes the Laplace operator. The boundary condition is given by

$$B[u] = u \qquad \text{or} \qquad B[u] = \frac{\partial u}{\partial n} + \gamma(x)u.$$

$\gamma(x) \in C^{1+\alpha}(\partial\Omega)$ and $\gamma(x) \geq 0$ on $\partial\Omega$, and $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega$, $R^+ = (0, \infty)$.

$(H_2)$ The coefficients $a(t, x)$, $b(t, x)$ and $c(t, x)$ are $T$-periodic in $t$ and Hölder continuous on $[0, T] \times \overline{\Omega}$ with $a(t, x) > 0$; $b(t, x) > 0$; $c(t, x) \geq 0$. We denote $a_1, b_1, c_1$ and $a_2, b_2, c_2$ to be the minimum and maximum values of $a$, $b$, $c$ on $[0, T] \times \overline{\Omega}$ with $c_2 > 0$ respectively.

$(H_3)$ The time delay $\tau$ is a positive constant. $\phi \in C^{0,1}([-\tau, 0] \times \overline{\Omega})$ is a nonnegative bounded function which satisfies the compatibility condition, i.e. $B[\phi(0, x)] = 0$.

Denote $C^{1,\,2}(R^+ \times \Omega)$ to be the set of functions which are once continuously differentiable in $t \in R^+$ and twice continuously differentiable in $x \in \Omega$. Similar notations are used for other function spaces and other domains.

**Lemma 1.1** *If there exists a pair of smooth functions $\overline{u}$, $\underline{u} \in C^{1,\,2}(R^+ \times \Omega) \bigcap C([-\tau, \infty) \times \overline{\Omega})$ (called coupled upper and lower solutions) such that $\overline{u} \geq \underline{u}$ on $[-\tau, \infty) \times \overline{\Omega}$, and they satisfy the following inequalities*

$$\begin{aligned} &L\overline{u}(t, x) \geq \overline{u}(t, x)[a(t, x) - b(t, x)\overline{u}(t, x) - c(t, x)\underline{u}(t - \tau, x)], \\ &L\underline{u}(t, x) \leq \underline{u}(t, x)[a(t, x) - b(t, x)\underline{u}(t, x) - c(t, x)\overline{u}(t - \tau, x)], \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t, x) \in R^+ \times \Omega, \end{aligned} \tag{1.4}$$

$$B[\overline{u}](t, x) \geq 0 \geq B[\underline{u}](t, x), \qquad (t, x) \in R^+ \times \partial\Omega, \tag{1.5}$$

$$\overline{u}(t, x) \geq \phi(t, x) \geq \underline{u}(t, x), \qquad (t, x) \in [-\tau, 0] \times \overline{\Omega}, \tag{1.6}$$

*then the initial-boundary value problem (1.1)-(1.3) has a unique solution $u \in C^{1,\,2}(R^+ \times \Omega) \bigcap C([-\tau,\ \infty) \times \overline{\Omega})$ with $\overline{u} \geq u \geq \underline{u}$ on $[-\tau, \infty) \times \overline{\Omega}$.*

In the case $\overline{u}$, $\underline{u}$ satisfy (1.4)(1.5) with $\overline{u} \geq \underline{u}$ on $R^+ \times \overline{\Omega}$, we also call $\overline{u}$, $\underline{u}$ a pair of upper and lower solutions of the problem (1.1)(1.2). For the proof of *Lemma 1.1* we can refer to [6, 7 ]. As there always exists a positive number $\alpha$ large enough such that $\phi(t, x) \leq \alpha$ on $[-\tau, 0] \times \overline{\Omega}$, it is easy to check that $\alpha$ and 0 is a pair of upper and lower solutions of the problem (1.1)–(1.3), then from *Lemma 1.1* we can get a unique solution $u \in C^{1,\,2}(R^+ \times \Omega) \bigcap C([-\tau,\ \infty) \times \overline{\Omega})$. The following eigenvalue problem

$$L\varphi(t,x) - e(t,x)\varphi(t,x) = \sigma\varphi(t,x), \qquad (t,x) \in R^+ \times \Omega,$$
$$B[u] = 0, \qquad (t,x) \in R^+ \times \partial\Omega, \tag{1.7}$$

where $\varphi$ is $T$-periodic in $t$, has a principal eigenvalue $\sigma(e)$ with positive eigenfunction. If the principal eigenvalue $\sigma(a) < 0$ with $e(t,x) = a(t,x)$, then from Hess [8], we know that the problem below

$$Lu(t,x) = u(t,x)[e(t,x) - b(t,x)u(t,x)], \qquad (t,x) \in R^+ \times \Omega,$$
$$B[u](t,x) = 0, \qquad (t,x) \in R^+ \times \partial\Omega,$$
$$u(t,x) \quad \text{is } T-\text{periodic}, \tag{1.8}$$

has a positive $T$-periodic solution $\theta_0(t,x)$ on $R^+ \times \Omega$. Let $\theta^*(t,x) = c(t,x)\theta_0(t-\tau,x)$, if $\sigma(a-\theta^*) < 0$ with $e(t,x) = a(t,x) - \theta^*(t,x)$, then the problem (1.8) has a positive $T$-periodic solution $\Theta(t,x)$ on $R^+ \times \Omega$. And it is easy to check that $\theta_0$ is an upper $T$-periodic solution of the problem (1.1)(1.2) and $\Theta$ is a lower $T$-periodic solution of (1.1)(1.2) . So through the monotone iteration schemes, and using the method in [3, 4], we can get the following results.

**Theorem 1.1** *Let the hypotheses* $(H_1)-(H_3)$ *hold.*

(1) *If* $\sigma(a) \geq 0$, *then the trivial solution* 0 *is globally asymptotically stable in (1.1)-(1.3) with respect to every nonnegative initial function* $\phi(t,x)$.

(2) *If* $\sigma(a) < 0$, *and* $\sigma(a-\theta^*) < 0$, *then the problem (1.1)(1.2) has a pair of $T$-periodic upper and lower quasi-solutions* $\overline{\theta}$, $\underline{\theta} \in C^{1,\,2}(R^+ \times \Omega)$ *with* $\Theta \leq \underline{\theta} \leq \overline{\theta} \leq \theta_0$. *Moreover, for every nonnegative, nontrivial initial function* $\phi(t,x)$, *the time-dependent solution* $u(t,x)$ *of the problem (1.1)-(1.3) satisfies*

$$\liminf_{t\to\infty}[u(t,x) - \underline{\theta}(t,x)] \geq 0 \geq \limsup_{t\to\infty}[u(t,x) - \overline{\theta}(t,x)], \quad \forall\, x \in \overline{\Omega}. \tag{1.9}$$

**Remark** (1) In case $\overline{\theta} \equiv \underline{\theta}$ , the $T$-periodic solution $\overline{\theta}$ (or $\underline{\theta}$) is a unique periodic solution of the problem (1.1)(1.2). (2) For the operator $L = \partial/\partial t - \triangle$, if substitute $-\triangle$ by uniformly strong elliptic operator, then Theorem 1.1 is also satisfied.

## 2. Existence and Uniqueness of Periodic Solution

To study the existence of the periodic solution of (1.1)(1.2), it only needs to search for the sufficient conditions for $\overline{\theta} \equiv \underline{\theta}$. From now on, we assume $\sigma(a) < 0$ and $\sigma(a-\theta^*) < 0$. From Theorem 1.1, we know that the coupled upper and lower $T$-periodic quasi-solutions $\overline{\theta}$ and $\underline{\theta}$ of the problem (1.1)(1.2) satisfy the relations below

$$\overline{\theta}_t - \triangle\overline{\theta} = \overline{\theta}(a - b\overline{\theta} - c\underline{\theta}_\tau), \qquad (t,x) \in R^+ \times \Omega,$$
$$\underline{\theta}_t - \triangle\underline{\theta} = \underline{\theta}(a - b\underline{\theta} - c\overline{\theta}_\tau), \qquad (t,x) \in R^+ \times \Omega,$$
$$B[\overline{\theta}] = B[\underline{\theta}] = 0, \qquad (t,x) \in R^+ \times \partial\Omega, \tag{2.1}$$

where $\overline{\theta}_t \equiv \dfrac{\partial\overline{\theta}}{\partial t}$, $\underline{\theta}_t \equiv \dfrac{\partial\underline{\theta}}{\partial t}$, $\overline{\theta}_\tau \equiv \overline{\theta}(t-\tau,x)$ and $\underline{\theta}_\tau \equiv \underline{\theta}(t-\tau,x)$.

From (2.1) we can get the following relations

$$(\overline{\theta}_t - \underline{\theta}_t) - \triangle(\overline{\theta} - \underline{\theta}) = a(\overline{\theta} - \underline{\theta}) - b(\overline{\theta}^2 - \underline{\theta}^2) - c(\overline{\theta}\underline{\theta}_\tau - \underline{\theta}\overline{\theta}_\tau). \tag{2.2}$$

In the following, we search for the conditions for $\overline{\theta} \equiv \underline{\theta}$ associated with different boundary conditions.

## Part A Dirichlet Conditions

For the Dirichlet boundary conditions $\overline{\theta} = \underline{\theta} = 0$ on $\partial\Omega$, consider that $\overline{\theta} - \underline{\theta} \geq 0$, multiply (2.2) by $(\overline{\theta} - \underline{\theta})$, and integrate in $x$ on $\Omega$, then the left-hand $I$ and the right-hand $II$ may be written as the following

$$\begin{aligned} I &= \int_\Omega (\overline{\theta} - \underline{\theta})(\overline{\theta}_t - \underline{\theta}_t)dx - \int_\Omega (\overline{\theta} - \underline{\theta})\triangle(\overline{\theta} - \underline{\theta})dx \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega (\overline{\theta} - \underline{\theta})^2 dx + \int_\Omega |\nabla(\overline{\theta} - \underline{\theta})|^2 dx, \end{aligned} \tag{2.3}$$

$$\begin{aligned} II &= \int_\Omega (\overline{\theta} - \underline{\theta})[a\,(\overline{\theta} - \underline{\theta}) - b\,(\overline{\theta} + \underline{\theta})(\overline{\theta} - \underline{\theta}) - c\,(\overline{\theta}\underline{\theta}_\tau - \underline{\theta}\,\underline{\theta}_\tau + \underline{\theta}\,\underline{\theta}_\tau - \underline{\theta}\overline{\theta}_\tau)]dx \\ &= \int_\Omega [a - b\,(\overline{\theta} + \underline{\theta}) - c\,\underline{\theta}_\tau](\overline{\theta} - \underline{\theta})^2 dx + \int_\Omega c\,\underline{\theta}(\overline{\theta} - \underline{\theta})(\overline{\theta}_\tau - \underline{\theta}_\tau)dx. \end{aligned} \tag{2.4}$$

From Poincar$\acute{e}$ inequality [cf 9],

$$\int_\Omega |\nabla(\overline{\theta} - \underline{\theta})|^2 dx \geq \lambda_1 \int_\Omega (\overline{\theta} - \underline{\theta})^2 dx,$$

where $\lambda_1$ is the principal eigenvalue of $-\triangle$ on $\Omega$ with zero Dirichlet boundary condition. Denote $\|\cdot\|_{L^2(\Omega)}$ (denote $\|\cdot\|$ in simple ) to be the $L^2$ norm on $\Omega$, then

$$I \geq \frac{1}{2}\frac{d}{dt}\|\overline{\theta} - \underline{\theta}\|^2 + \lambda_1\|\overline{\theta} - \underline{\theta}\|^2. \tag{2.5}$$

If set

$$M = \sup_{x\in\Omega,\ 0\leq t\leq T}(a - 2b\,\Theta - c\,\Theta_\tau), \qquad N = \sup_{x\in\Omega,\ 0\leq t\leq T}(c\,\theta_0), \tag{2.6}$$

then from (2.4) and Hölder inequality, also consider that $\Theta \leq \underline{\theta} \leq \overline{\theta} \leq \theta_0$ we can get

$$\begin{aligned} II &\leq \int_\Omega [a - 2b\,\Theta - c\,\Theta_\tau](\overline{\theta} - \underline{\theta})^2 dx + \int_\Omega c\,\theta_0(\overline{\theta} - \underline{\theta})(\overline{\theta}_\tau - \underline{\theta}_\tau)dx \\ &\leq M\int_\Omega (\overline{\theta} - \underline{\theta})^2 dx + N\int_\Omega (\overline{\theta} - \underline{\theta})(\overline{\theta}_\tau - \underline{\theta}_\tau)dx \\ &\leq M\|\overline{\theta} - \underline{\theta}\|^2 + N\|\overline{\theta}_\tau - \underline{\theta}_\tau\| \cdot \|\overline{\theta} - \underline{\theta}\|. \end{aligned} \tag{2.7}$$

From (2.5) and (2.7) we can get

$$\frac{1}{2}\frac{d}{dt}\|\overline{\theta}-\underline{\theta}\|^2 \le (M-\lambda_1)\|\overline{\theta}-\underline{\theta}\|^2 + N\|\overline{\theta}_\tau-\underline{\theta}_\tau\|\cdot\|\overline{\theta}-\underline{\theta}\|$$
$$\le (M+\frac{N}{2}-\lambda_1)\|\overline{\theta}-\underline{\theta}\|^2 + \frac{N}{2}\|\overline{\theta}_\tau-\underline{\theta}_\tau\|^2. \tag{2.8}$$

By integrating (2.8) in $t$ on $[0,T]$, and considering that $\|\overline{\theta}-\underline{\theta}\|^2$ is $T$-periodic in $t$,

$$0 = \int_0^T \left[\frac{1}{2}\frac{d}{dt}\|\overline{\theta}-\underline{\theta}\|^2\right]dt$$
$$\le (M+\frac{N}{2}-\lambda_1)\int_0^T \|\overline{\theta}-\underline{\theta}\|^2 dt + \frac{N}{2}\int_0^T \|\overline{\theta}_\tau-\underline{\theta}_\tau\|^2 dt$$
$$= (M+N-\lambda_1)\int_0^T \|\overline{\theta}-\underline{\theta}\|^2 dt. \tag{2.9}$$

If the inequality $M+N-\lambda_1<0$ is satisfied, then from (2.9) we can get

$$\int_0^T \|\overline{\theta}-\underline{\theta}\|^2 dt \equiv 0,$$

which implies that $\overline{\theta}\equiv\underline{\theta}$ on $R^+\times\overline{\Omega}$.

Now if $\theta^1$ is any other solution with $\Theta\le\theta^1\le\theta_0$, then $\theta^1$ and $\underline{\theta}$ is a pair of upper and lower solutions of the problem (1.1)(1.2) and satisfy the relations in (2.1). The same reason as the previous for $\overline{\theta}$ and $\underline{\theta}$ yields $\theta^1\equiv\underline{\theta}$ on $R^+\times\overline{\Omega}$. Then the problem (1.1)(1.2) has a unique $T$-periodic solution.

To ensure the existence of $\overline{\theta}$ and $\underline{\theta}$ , it needs $\sigma(a-\theta^*)<0$ only if $a-\theta^*>\lambda_1$, i.e. $a-c\,\theta_{0\,\tau}>\lambda_1$. So from the above argument, we get the sufficient conditions for $\overline{\theta}\equiv\underline{\theta}$ as below

$$\text{(i)}\quad a-c\,\theta_{0\,\tau}>\lambda_1,$$
$$\text{(ii)}\quad \sup_{x\in\Omega,\,0\le t\le T}(a-2b\,\Theta-c\,\Theta_\tau)+\sup_{x\in\Omega,\,0\le t\le T}(c\,\theta_0)<\lambda_1. \tag{2.10}$$

**Theorem 2.1** *Let hypotheses $(H_1)-(H_3)$ hold. For the Dirichlet boundary condition, if the conditions in (2.10) are satisfied, then the problem (1.1)(1.2) has a unique smooth $T$-periodic solution $\theta$ on $\overline{\Omega}$. Moreover, for every initial function $\phi(t,x)$ with $\Theta\le\phi\le\theta_0$, the time-dependent solution $u(t,x)$ in (1.1)-(1.3) has the asymptotic behavior*

$$\lim_{t\to\infty}[u(t,\cdot)-\theta(t,\cdot)]=0 \qquad in\ C(\overline{\Omega}).$$

## Part B Neumann Conditions

In the case $\gamma(x)\equiv 0$ on $\partial\Omega$, it is Neumann boundary conditions

$$\frac{\partial\overline{\theta}}{\partial n}=\frac{\partial\underline{\theta}}{\partial n}=0.$$

In this case we can search for the sufficient conditions that only relate to the coefficients and not to $\Theta$, $\theta_0$. In [10] we have given a kind of sufficient conditions. Now we search for another kind in the following. According to (1.4)(1.5), we can get a pair of coupled upper and lower positive constant solutions $k_2$, $k_1$ by the following system

$$\begin{aligned} &k_2(a_2 - b_1k_2 - c_1k_1) = 0, \\ &k_1(a_1 - b_2k_1 - c_2k_2) = 0. \end{aligned} \tag{2.11}$$

If $b_1b_2 > c_1c_2$ and $a_1b_1 > a_2c_2$, we can solve (2.11) and get

$$k_2 = \frac{a_2b_2 - a_1c_1}{b_1b_2 - c_1c_2}, \qquad k_1 = \frac{a_1b_1 - a_2c_2}{b_1b_2 - c_1c_2}, \tag{2.12}$$

and it is easy to check that $0 < k_1 \le \Theta \le \underline{\theta} \le \overline{\theta} \le \theta_0 \le k_2 \le a_2/b_1$.

From (2.6) we can get that

$$\begin{aligned} M &= \sup_{x\in\Omega, 0\le t\le T} (a - 2b\,\Theta - c\,\Theta_\tau) \le [a_2 - (2b_1 + c_1)k_1], \\ N &= \sup_{x\in\Omega, 0\le t\le T} (c\,\theta_0) \le c_2k_2, \end{aligned} \tag{2.13}$$

where $\Theta$ and $\theta_0$ are related to the Neumann boundary condition. The same method as that in Part A reveals that

$$\frac{1}{2}\frac{d}{dt}\|\overline{\theta} - \underline{\theta}\|^2 + \|\nabla(\overline{\theta} - \underline{\theta})\|^2 \le (M + \frac{N}{2})\|\overline{\theta} - \underline{\theta}\|^2 + \frac{N}{2}\|\overline{\theta}_\tau - \underline{\theta}_\tau\|^2. \tag{2.14}$$

Integrating (2.14) in $t$ on $[0, T]$, and considering that $\|\overline{\theta} - \underline{\theta}\|^2$ is $T$-periodic in $t$,

$$\begin{aligned} 0 + \int_0^T \|\nabla(\overline{\theta} - \underline{\theta})\|^2 dt &\le (M + \frac{N}{2}) \int_0^T \|\overline{\theta} - \underline{\theta}\|^2 dt + \frac{N}{2}\int_0^T \|\overline{\theta}_\tau - \underline{\theta}_\tau\|^2 dt \\ &= (M + N)\int_0^T \|\overline{\theta} - \underline{\theta}\|^2 dt \\ &\le [a_2 - (2b_1 + c_1)k_1 + c_2k_2]\int_0^T \|\overline{\theta} - \underline{\theta}\|^2 dt. \end{aligned} \tag{2.15}$$

So if $a_2 - (2b_1 + c_1)k_1 + c_2k_2 < 0$, then the relation (2.15) reveals that

$$\int_0^T \|\overline{\theta} - \underline{\theta}\|^2 dt \equiv 0,$$

which implies that $\overline{\theta} \equiv \underline{\theta}$. And the $T$-periodic solution of (1.1)(1.2) is also unique. As the principal eigenvalue of $-\triangle$ on $\Omega$ with zero Neumann boundary condition is $\lambda_1 = 0$, to ensure the existence of $\overline{\theta}$ and $\underline{\theta}$ , it needs $\sigma(a - \theta^*) < 0$ only if $a(t,x) - c(t,x)\,\theta_0(t - \tau, x) > \lambda_1 = 0$. As $a(t,x) - c(t,x)\,\theta_0(t - \tau, x) \ge a_1 - c_2k_2$, it only needs $a_1 - c_2k_2 > 0$.

It is easy to check that $a_1b_1 > a_2c_2$ implies $b_1b_2 > c_1c_2$ and $a_1 - c_2k_2 > 0$. So from the above argument, we get the sufficient conditions for $\overline{\theta} \equiv \underline{\theta}$ as below

$$\begin{aligned} &\text{(i)} \quad a_1b_1 > a_2c_2, \\ &\text{(ii)} \quad a_2 - (2b_1 + c_1)k_1 + c_2k_2 < 0. \end{aligned} \tag{2.16}$$

where $k_2$, $k_1$ are given by (2.12).

**Theorem 2.2** *Let hypotheses $(H_1)-(H_3)$ hold. For Neumann boundary condition, if the conditions in (2.16) are satisfied, then the problem (1.1)(1.2) has a unique smooth $T$-periodic solution $\theta$ on $\overline{\Omega}$. Moreover, for every initial function $\phi(t,x)$ with value in $[k_1, k_2]$, the time-dependent solution $u(t,x)$ in (1.1)-(1.3) has the asymptotic behavior*

$$\lim_{t\to\infty}[u(t,\cdot) - \theta(t,\cdot)] = 0 \qquad in\ C(\overline{\Omega}).$$

## Part C Robin Conditions

For the Robin boundary condition

$$\frac{\partial\overline{\theta}}{\partial n} + \gamma\,\overline{\theta} = 0, \qquad \frac{\partial\underline{\theta}}{\partial n} + \gamma\,\underline{\theta} = 0 \quad \text{on} \quad R^+ \times \partial\Omega,$$

by use of the method in *Part A* and consider that $\gamma(x) \geq 0$ on $\partial\Omega$, we can get

$$\begin{aligned} I &= \int_\Omega (\overline{\theta} - \underline{\theta})(\overline{\theta}_t - \underline{\theta}_t)dx - \int_\Omega (\overline{\theta} - \underline{\theta})\triangle(\overline{\theta} - \underline{\theta})dx \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega (\overline{\theta} - \underline{\theta})^2 dx + \int_{\partial\Omega} \gamma(s)(\overline{\theta} - \underline{\theta})^2 ds + \int_\Omega |\nabla(\overline{\theta} - \underline{\theta})|^2 dx \\ &\geq \frac{1}{2}\frac{d}{dt}\|\overline{\theta} - \underline{\theta}\|^2 + \|\nabla(\overline{\theta} - \underline{\theta})\|^2. \end{aligned} \tag{2.17}$$

Further calculation as in *Part B* reveals that (2.14) also holds for Robin boundary condition. So the sufficient condition for $\overline{\theta} \equiv \underline{\theta}$ should be $M + N < 0$, i.e.

$$\sup_{x\in\Omega,\ 0\leq t\leq T}(a - 2b\,\Theta - c\,\Theta_\tau) + \sup_{x\in\Omega,\ 0\leq t\leq T}(c\,\theta_0) < 0. \tag{2.18}$$

where $\Theta$, $\theta_0$ are related to Robin boundary condition.

**Theorem 2.3** *Let hypotheses $(H_1) - (H_3)$ hold. For Robin boundary condition, if $\sigma(a - \theta^*) < 0$ and the inequality in (2.18) are satisfied, then the problem (1.1)(1.2) has a unique smooth $T$-periodic solution $\theta$ on $\overline{\Omega}$. Moreover, for every nonnegative nontrivial initial function $\phi(t,x)$ with value in $[0, P]$, the time-dependent solution $u(t,x)$ in (1.1)-(1.3) has the asymptotic behavior*

$$\lim_{t\to\infty}[u(t,\cdot) - \theta(t,\cdot)] = 0 \qquad in\ C(\overline{\Omega}).$$

## 3. Effect of Time-Delay

In this section, we assume $\sigma(a) < 0$, and the boundary condition is Dirichlet type. We'll study the effect of time–delay on the solution $u(t,x)$ of (1.1)-(1.3). Let function $\theta(t,x)$ be the $T$-periodic solution of the following problem with no time–delay

$$\frac{\partial\theta}{\partial t} - \Delta\theta = \theta\,[a-(b+c)\,\theta], \qquad (t,x)\in R^+\times\Omega, \tag{3.1}$$

$$\theta = 0, \qquad (t,x)\in R^+\times\partial\Omega. \tag{3.2}$$

From the previous argument we know that $u(t,x)$ and $\theta(t,x)$ are nonnegative nontrivial on $R^+\times\overline{\Omega}$. From the equations (1.1) and (3.1),

$$\begin{aligned}
&\frac{\partial}{\partial t}(u-\theta) - \Delta(u-\theta)\\
&\quad= a(u-\theta) - b(u^2-\theta^2) - c(uu_\tau - \theta^2)\\
&\quad= a(u-\theta) - b(u+\theta)(u-\theta) - c\left[(uu_\tau - \theta u_\tau) + (\theta u_\tau - \theta^2)\right]\\
&\quad= [a - b(u+\theta) - cu_\tau](u-\theta) - c\theta(u_\tau-\theta).
\end{aligned} \tag{3.3}$$

To multiply (3.3) by $(u-\theta)$ and integrate in $x$ on $\Omega$, also by use of the Poincaré inequality, the left-hand may be written as

$$\begin{aligned}
I &= \int_\Omega [\frac{\partial}{\partial t}(u-\theta) - \Delta(u-\theta)](u-\theta)dx\\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega (u-\theta)^2dx + \int_\Omega |\nabla(u-\theta)|^2dx\\
&\geq \frac{1}{2}\frac{d}{dt}\|u-\theta\|^2 + \lambda_1\|u-\theta\|^2,
\end{aligned} \tag{3.4}$$

where $\lambda_1$ is the principal eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. To estimate the right-hand we give some denotations as below

$$\begin{aligned}
&M = \sup_{x\in\Omega,\ 0\leq t\leq T}(a - b\,\theta), \quad N = \sup_{x\in\Omega,\ 0\leq t\leq T}(c\,\theta),\\
&K(\tau) = \sup_{0\leq t\leq T}\|c\theta(\theta_\tau-\theta)\|^2.
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
II &= \int_\Omega [a - b(u+\theta) - cu_\tau](u-\theta)^2dx - \int_\Omega c\theta(u_\tau-\theta)(u-\theta)dx\\
&\leq \int_\Omega (a-b\theta)(u-\theta)^2dx + \int_\Omega |c\theta(u_\tau-\theta)|\cdot|u-\theta|dx\\
&\leq M\|u-\theta\|^2 + \|c\theta(u_\tau-\theta)\|\cdot\|u-\theta\|\\
&\leq M\|u-\theta\|^2 + (\|c\theta(u_\tau-\theta_\tau)\| + \|c\theta(\theta_\tau-\theta)\|)\cdot\|u-\theta\|\\
&\leq M\|u-\theta\|^2 + N\|u-\theta\|\cdot\|u_\tau-\theta_\tau\| + \|c\theta(\theta_\tau-\theta)\|\cdot\|u-\theta\|\\
&\leq (M + \frac{N}{2} + \frac{1}{2})\|u-\theta\|^2 + \frac{N}{2}\|u_\tau-\theta_\tau\|^2 + \frac{1}{2}K(\tau).
\end{aligned} \tag{3.6}$$

So from (3.4) and (3.6) we have

$$\frac{1}{2}\frac{d}{dt}\|u-\theta\|^2 \le (M+\frac{N}{2}+\frac{1}{2}-\lambda_1)\|u-\theta\|^2+\frac{N}{2}\|u_\tau-\theta_\tau\|^2+\frac{1}{2}K(\tau), \tag{3.7}$$

i.e.

$$\frac{d}{dt}\|u-\theta\|^2 \le (2M+N+1-2\lambda_1)\|u-\theta\|^2+N\|u_\tau-\theta_\tau\|^2+K(\tau). \tag{3.8}$$

Denote $y(t)=\|u(t,\cdot)-\theta(t,\cdot)\|^2$, then the relation (3.8) may be rewritten as

$$\frac{d}{dt}y(t) \le (2M+N+1-2\lambda_1)y(t)+Ny(t-\tau)+K(\tau). \tag{3.9}$$

If $2M+2N+1-2\lambda_1 \neq 0$, let $\varepsilon(\tau)=K(\tau)/(2\lambda_1-2M-2N-1)$, then $\xi(t)=y(t)-\varepsilon(\tau)$ solves

$$\frac{d}{dt}\xi(t) \le (2M+N+1-2\lambda_1)\xi(t)+N\xi(t-\tau). \tag{3.10}$$

The corresponding ordinary equation is

$$\frac{d}{dt}\xi_1(t) = (2M+N+1-2\lambda_1)\xi_1(t)+N\xi_1(t-\tau). \tag{3.11}$$

And the characteristic equation is

$$\mu=(2M+N+1-2\lambda_1)+Ne^{-\tau\mu}. \tag{3.12}$$

Consider that $N>0$, refer to the *Hayes Theorem* in [11] and the results in [12], if $2M+2N+1-2\lambda_1<0$, then the equation (3.12) only has roots with negative real-part i.e. $\mathrm{Re}\mu<0$, and 0 of the equation (3.11) is asymptotic stable, i.e.

$$\lim_{t\to\infty}\xi_1(t)=0. \tag{3.13}$$

From the comparison of (3.10) and (3.11), we can have $\xi(t)\le\xi_1(t)$, and consider that $y(t)=\xi(t)+\varepsilon(\tau)$, we can get the relation below

$$y(t)\le\xi_1(t)+\varepsilon(\tau). \tag{3.14}$$

Moreover, to ensure the existence of $T$-periodic solution $\theta(t,x)$ in the problem (3.1)(3.2) it needs $\sigma(a)<0$ only if $a(t,x)>\lambda_1$. So we can get the following result.

**Theorem 3.1** *Let the hypotheses* $(H_1)-(H_3)$ *hold. For each fixed* $\tau>0$, *if* $a(t,x)>\lambda_1$ *and* $M+N+1/2<\lambda_1$ *are satisfied, then the solution* $u(t,x)$ *of the delayed problem (1.1)–(1.3) and* $T$*-periodic solution* $\theta(t,x)$ *of the no-delay problem (3.1)(3.2) have the relation*

$$\lim_{t\to\infty}\|u(t,\cdot)-\theta(t,\cdot)\|^2_{L^2(\Omega)}\le\frac{K(\tau)}{2(\lambda_1-M-N)-1}, \tag{3.15}$$

*where* $M$, $N$, $K(\tau)$ *are given by (3.5).*

We have the prior estimate for $K(\tau)$ as follows:

$$\begin{aligned} K(\tau) &= \sup_{0\le t\le T} \|c\theta(\theta_\tau-\theta)\|^2 \\ &\le \sup_{0\le t\le T,\, x\in\Omega} (c\theta)^2 \cdot \sup_{0\le t\le T} \|\theta_\tau-\theta\|^2 \\ &= N^2 \sup_{0\le t\le T} \|\theta_\tau-\theta\|^2 \le N^2 G, \quad \forall\, \tau>0, \end{aligned} \tag{3.16}$$

where $G=\|\sup_{0\le t\le T}\theta(t,\cdot)-\inf_{0\le t\le T}\theta(t,\cdot)\|^2$.

In the following we consider the case $0<\tau\ll 1$. Set $f(t,\tau)=\|\theta(t-\tau,\cdot)-\theta(t,\cdot)\|^2$. As the solution $\theta(t,x)$ of the no-delay problem (3.1)(3.2) is continuous in $t$, it is easy to see that $f(t,\tau)$ is also continuous in $t$ and $\tau$. So we can get

$$\lim_{\tau\to 0} K(\tau)=\lim_{\tau\to 0}\sup_{0\le t\le T} f(t,\tau)=0. \tag{3.17}$$

From (3.15)(3.17) and the Faton Lemma,

$$\begin{aligned} \int_\Omega \lim_{t\to\infty}|u(t,x)-\theta(t,x)|dx &\le \lim_{t\to\infty}\int_\Omega |u(t,x)-\theta(t,x)|dx \\ &\le \lim_{t\to\infty}|\Omega|^{1/2}\Big(\int_\Omega |u(t,x)-\theta(t,x)|^2dx\Big)^{1/2} \\ &= |\Omega|^{1/2}\lim_{t\to\infty}\|u(t,x)-\theta(t,x)\|=0, \end{aligned} \tag{3.18}$$

where $|\Omega|$ is the measure of $\Omega$. So for every $x\in\Omega$,

$$\lim_{t\to\infty}|u(t,x)-\theta(t,x)|=0.$$

Hence we can get a corollary as follows.

**Corollary 3.1** *Let hypotheses $(H_1)-(H_3)$ hold. If $a(t,x)>\lambda_1$ and $M+N+1/2<\lambda_1$ are satisfied, then for every $\tau>0$,*

$$\lim_{t\to\infty}\|u(t,\cdot)-\theta(t,\cdot)\|^2_{L^2(\Omega)}\le \frac{N^2G}{2(\lambda_1-M-N)-1}, \tag{3.19}$$

*where $G=\|\sup_{0\le t\le T}\theta(t,\cdot)-\inf_{0\le t\le T}\theta(t,\cdot)\|^2$, and $M$, $N$ are given by (3.5). Moreover,*

$$\lim_{t\to\infty}[u(t,x)-\theta(t,x)]=0 \quad (as\ \ \tau\to 0) \quad \forall\, x\in\overline{\Omega}. \tag{3.20}$$

**Remark** If the Dirichlet boundary condition is substituted by Neumann or Robin type, we can get similar results.

## 4. Applications

In this section, we give some numerical results for the asymptotic behavior in the delayed diffusive Logistic equation (1.1)-(1.3) on one-dimension spatial domain $\Omega = (0,1)$.

**Example 1** We consider the Dirichlet boundary problem below

$$\partial u(t,x)/\partial t - \triangle u(t,x) = u(t,x)[a(t,x) - b(t,x)u(t,x) - c(t,x)u(t-\frac{1}{2},x)],$$
$$(t,x) \in (0,+\infty)\times(0,1), \qquad (4.1)$$

$$u(t,0) = u(t,1) = 0, \qquad t \in [0,+\infty), \qquad (4.2)$$

$$u(t,x) = \phi(t,x), \qquad (t,x) \in [-\frac{1}{2},0]\times[0,1]. \qquad (4.3)$$

We know the principal eigenvalue of $-\triangle$ in $\Omega = (0,1)$ with zero Dirichlet boundary conditions is $\lambda_1 = \pi^2$. Consider $\theta_0 \le a_2/b_1$ to satisfy the condition (i) in (2.10) only if

$$a - c\,\theta_{0\,\tau} \ge a_1 - c_2 \times \frac{a_2}{b_1} > \lambda_1 = \pi^2$$
$$\text{i.e.} \quad a_1 - \frac{a_2c_2}{b_1} > \pi^2. \qquad (4.4)$$

For the condition (ii) in (2.10)

$$\sup_{x\in\Omega,\,0\le t\le T}(a - 2b\,\Theta - c\,\Theta_\tau) + \sup_{x\in\Omega,\,0\le t\le T}(c\,\theta_0) < \pi^2 \qquad (4.5)$$

is not easily checked because $\theta_0$ and $\Theta$ can not be present directly. We give some analysis below. Given $a(t,x) > \pi^2$ and any $b(t,x)$, we can choose a suitable small $c(t,x)$ to satisfy the inequality (4.4). From the problem (1.8) we know $\theta_0$ is confirmed only by $a(t,x)$ and $b(t,x)$. If $c^{'}(t,x) \le c(t,x)$, then

$$a - c^{'}\,\theta_{0\,\tau} \ge a - c\,\theta_{0\,\tau}, \qquad (4.6)$$
$$u[a - c^{'}\,\theta_{0\tau} - bu] \ge u[a - c\,\theta_{0\tau} - bu], \qquad (4.7)$$

which imply that $c^{'}(t,x)$ also satisfies (4.4) and $\Theta^{'}(t,x) \ge \Theta(t,x)$, where $\Theta^{'}$ is the $T$-periodic solution of (1.8) respect to $c^{'}$. So

$$\begin{aligned}&\sup_{x\in\Omega,\,0\le t\le T}(a - 2b\,\Theta^{'} - c^{'}\,\Theta^{'}_\tau) + \sup_{x\in\Omega,\,0\le t\le T}(c^{'}\,\theta_0)\\ &\le \sup_{x\in\Omega,\,0\le t\le T}(a - 2b\,\Theta^{'}) + \sup_{x\in\Omega,\,0\le t\le T}(c^{'}\,\theta_0)\\ &\le \sup_{x\in\Omega,\,0\le t\le T}(a - 2b\,\Theta) + \sup_{x\in\Omega,\,0\le t\le T}(c\,\theta_0). \end{aligned} \qquad (4.8)$$

In this sense, if $c(t,x)$ gets smaller, then

$$\sup_{x\in\Omega,\ 0\le t\le T}(a-2b\,\Theta)+\sup_{x\in\Omega,\ 0\le t\le T}(c\,\theta_0)$$

will gets smaller along. So it is possible to select a $c(t,x)$ small enough to satisfy the inequality (4.5). In this example we choose

$$\begin{aligned} &a(t,x)=20+4\sin(2\pi t), \qquad b(t,x)=10,\\ &c(t,x)=1+\cos(2\pi t), \qquad \phi(t,x)=\sin(\pi x). \end{aligned} \tag{4.9}$$

And the simulation for the problem (4.1)–(4.3) in Fig-1 shows the asymptotic behavior in Theorem 2.1 as below

$$\lim_{t\to\infty}[u(t,x)-\theta(t,x)]=0 \quad \text{for} \quad \forall\, x\in[0,1]. \tag{4.10}$$

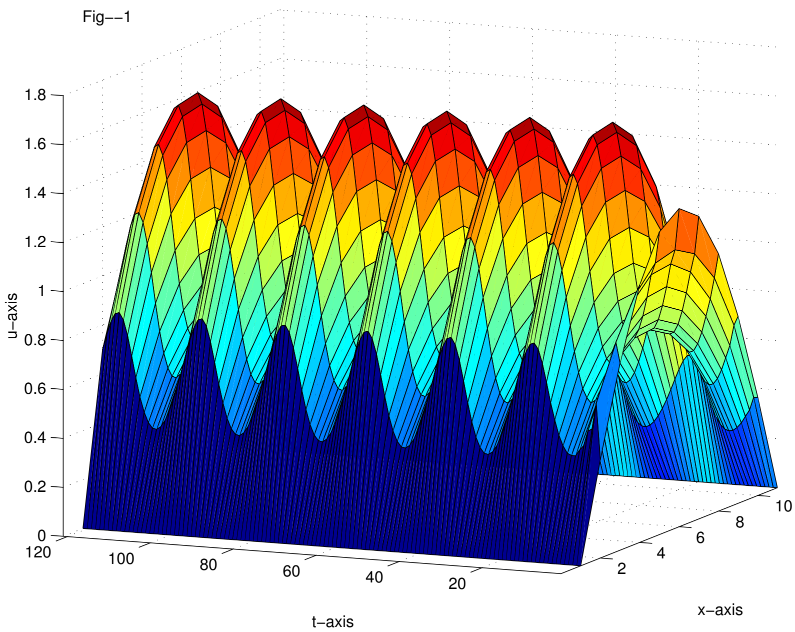


Fig-1 : To show the Asymptotic periodicity of Dirichlet boundary problem (4.1)–(4.3).

**Example 2** We consider the Neumann boundary problem below

$$\frac{\partial}{\partial t}u(t,x)-\Delta u(t,x)=u(t,x)[a(t,x)-b(t,x)u(t,x)-c(t,x)u(t-\frac{1}{2},x)], \quad (t,x)\in(0,\infty)\times(0,1), \tag{4.11}$$

$$\frac{\partial u(t,0)}{\partial x}=\frac{\partial u(t,1)}{\partial x}=0, \qquad t\in[0,\infty), \tag{4.12}$$

$$u(t,x)=\phi(t,x), \qquad (t,x)\in[-\frac{1}{2},0]\times[0,1]. \tag{4.13}$$

If we set

$$a(t,x) = 14 + 2\sin(2\pi t), \quad b(t,x) = 30,$$
$$c(t,x) = 1 + \cos(2\pi t), \quad \phi(t,x) = \frac{1}{3}[\sin(\pi t) + 1.2],$$

then $a_1 = 12,\ a_2 = 16,\ b_1 = b_2 = 30,\ c_1 = 0,\ c_2 = 2$, and $\phi(0,x) = 0.4$ which satisfies the concordant condition. We check that the conditions (i)(ii) in (2.16) are satisfied as the following: As $a_1 b_1 = 12 \times 30 = 360$, and $a_2 c_2 = 16 \times 2 = 32$, so the condition (i) is satisfied. From (2.12) we know $k_2 = 8/15$, $k_1 = 82/225$, and

$$a_2 - (2b_1 + c_1)k_1 + c_2 k_2 = 16 - 60 \times \frac{82}{225} + \frac{16}{15} \approx -4.8 < 0,$$

so the condition (ii) is satisfied. According to Theorem 2.2, the problem (4.11)–(4.12) has a unique 1-periodic solution $\theta(t,x)$ on $(0,\infty) \times [0,1]$, and the time-dependent solution $u(t,x)$ of the problem (4.11)-(4.13) has the asymptotic behavior

$$\lim_{t\to\infty} [u(t,\cdot) - \theta(t,\cdot)] = 0 \qquad \text{in } C([0,1]).$$

Accordingly, our numerical simulation for the problem (4.11)–(4.13) is given by Fig-2.

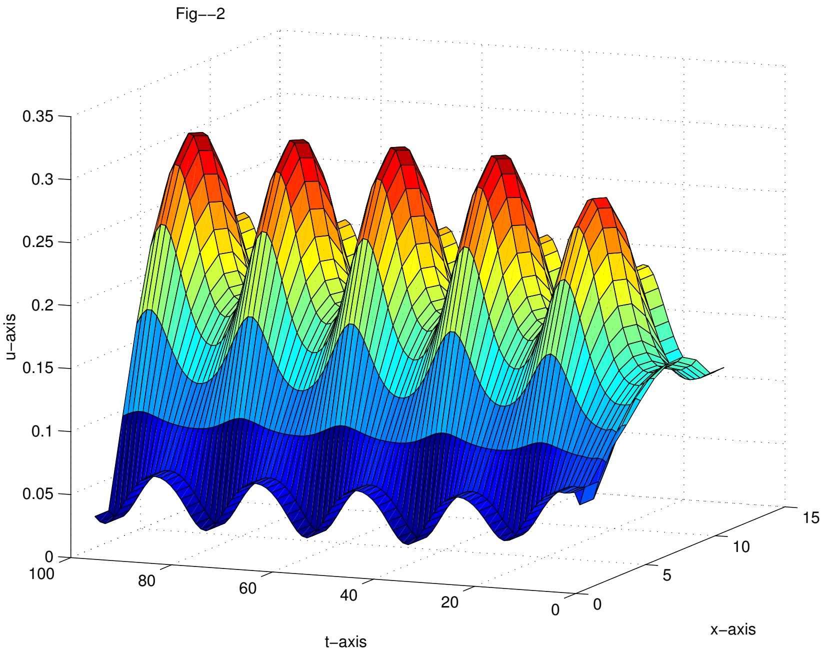


Fig-2 : To show the Asymptotic periodicity of Neumann boundary problem (4.11)–(4.13).

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