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THE SELF-SIMILAR SOLUTION FOR GINZBURG-LANDAU EQUATION AND ITS LIMIT BEHAVIOR IN BESOV SPACES

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Abstract In this paper, we study the limit behavior of self-similar solutions for the Complex Ginzburg-Landau (CGL) equation in the nonstandard function space $E_{s,p}$. We prove the uniform existence of the solutions for the CGL equation and its limit equation in $E_{s,p}$. Moreover we show that the self-similar solutions of CGL equation converge, globally in time, to those of its limit equation as the parameters tend to zero.
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1. Introduction

In this paper, we consider the following Cauchy problem for the complex Ginzburg-Landau equation

$$\begin{aligned} &u_t - \varepsilon\Delta u - i\Delta u + (a+ib)|u|^\alpha u = 0, \\ &u(0,x) = u_0(x), \end{aligned} \tag{1}$$

where $\varepsilon > 0$, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $u(x,t)$ is a complex-valued function on $\mathbf{R}^n \times \mathbf{R}^+$. If we set $\varepsilon = 0$ or $\varepsilon = 0$, $a = 0$, the equation (1) formally becomes

$$v_t - i\Delta v + (a+ib)|v|^\alpha v = 0, \qquad v(0,x) = v_0(x), \tag{2}$$

or

$$v_t - i\Delta v + ib|v|^\alpha v = 0, \qquad v(0,x) = v_0(x). \tag{3}$$

An essential problem among (1), (2) and (3) is that: whether the solutions of (1) converge to those of (2), (3) as the parameter $\varepsilon \to 0^+$ or $\varepsilon \to 0^+$, $a \to 0$. Recently in [1], B.Wang gave a positive answer when the initial data in the energy spaces L^2 or H^1. He pointed out that for any fixed $T > 0$, the solutions of (1) converge in $C(0,T;H^s)$,

$s = 0, 1$. In this paper, we consider the case of self-similar solutions for (1)-(3). First of all, we observe that if $u(x,t)$ solves (1)-(3), then

$$D_\lambda u(x,t) = \lambda^{\frac{2}{\alpha}} u\left(\lambda x, \lambda^2 t\right), \tag{4}$$

is also a solution of (1)-(3) with initial data

$$u_{0\lambda}(x) = \lambda^{\frac{2}{\alpha}} u_0(\lambda x). \tag{5}$$

One recalls the solution u is self-similar if it satisfies

$$u(x,t) = D_\lambda u(x,t) \tag{6}$$

for any $(x,t) \in \mathbf{R^n} \times \mathbf{R}^+$ and $\lambda > 0$, it's straightforward to verify $u(x,t)$ is a self-similar solution if and only if

$$u(x,t) = t^{-\frac{1}{\alpha}} u\left(\frac{x}{\sqrt{t}}, 1\right) = t^{-\frac{1}{\alpha}} W\left(\frac{x}{\sqrt{t}}\right), \tag{7}$$

for some $W : \mathbf{R^n} \to \mathbf{C^n}$ called the profile of the self-similar solution. Therefore, the equations (1)-(3) can be studied through a nonlinear elliptic equation on W. But these nonlinear elliptic equations are always complicated and are difficult to solve. On the other hand, one sees from (7) that the initial data of the self-similar solution have to verify

$$u_0(x) = \lambda^{\frac{2}{\alpha}} u_0(\lambda x), \qquad \forall \lambda > 0. \tag{8}$$

For example, $u_0(x) = \frac{\Omega(x')}{|x|^{\frac{2}{\alpha}}}$, $x' = \dfrac{x}{|x|}$, Ω is a function on unit sphere. This leads to another method to treat self-similar solutions of CGL or NLS. Indeed, one chooses a suitable Banach space B as the work space, the well-posedness in B means the initial data in (8) develop into self-similar solutions. However, since such data never belong to any homogeneous Sobolev space, it is not easy to obtain the existence of self-similar solution. Recently, many authors have interests in this area and have done some works as well. For instance, by introducing the nonstandard function space, the existence of the self-similar solution for a class of data in (8) was established. Moreover the self-similar solutions were also used as describing the long-time behavior of the other solutions in a better way. For the details, one refers to [2-5] for nonlinear Schrödinger equation, refers [6-8] for nonlinear wave equation and refers [9], [2] for nonlinear heat equation and NS equation.

In our previous work[10], we dealt with the self-similar solution of CGL equation in the space of the type

$$X_{s,p} = \left\{u; \sup_{t>0} t^{\beta(s,p)} \|u(t)\|_{\dot{H}^{s,p}} < \infty\right\}, \tag{9}$$

and proved the self-similar solutions of CGL equation converge to the corresponding limit Schrödinger equation as the parameters tend to 0 provided that the dimension $1 \le n \le 5$. In (9), $\beta(s,p)$ is chosen to preserve the scaling (4) invariance.

In this paper, we continue the study of the convergence. To avoid the difficulty of nonlinear estimate in the framework of homogeneous Sobolev space, we change the space into

$$E_{s,p} = \left\{ u; \|u\|_{s,p} = \sup_{t>0} t^{\beta(s,p)} \|u(t)\|_{\dot{B}^s_{p,2}} < \infty \right\},$$

where $\beta(s,p)$ is the same as above. Since the space $E_{s,p}$ is a kind of critical space in the sense of scaling transformation, the existence is obtained under some smallness condition on the initial data. So, at first, we should prove the global existence in it for (1)-(3) under the uniform smallness condition with respect to ε and a. Secondly, it seems necessary to obtain some regularities for $u(t)$ in addition to $u(t) \in E_{s,p}$. A uniform existence (or regularity) theorem is stated in Section 2. Initial data satisfying the above small condition are given in Theorem 2. Thus, for a class of self-similar initial data, the equation (1)-(3) will admit self-similar solutions. Finally, by means of the nonlinear estimates in Besov spaces and the multiplier theorems, we show the self-similar solutions of (1) converge to those of (2) or (3) as $\varepsilon \to 0^+$ or $\varepsilon \to 0^+$, $a \to 0$. Theorem3 and Theorem4 correspond to the two different cases.

Throughout this paper, C will stand for a positive constant that can be different at different places. We denote by p' the dual number of $p \in [1, \infty]$. We will have occasions to use a variety of function spaces: Lebesgue space L^p, Riesz potential spaces $\dot{H}^{s,p}$, homogeneous Besov spaces $\dot{B}^s_{p,q}$. Some properties of these function spaces can be found in [11]. For $p \in [1, \infty]$, $\mathcal{M}_p$ denotes the multiplier space.

For the sake of convenience, we use the conception of "degree" which was introduced by [12]. Define

$$\deg \dot{H}^{s,p} = s - \frac{n}{p}, \quad \deg L^p = -\frac{n}{p}, \quad \deg \dot{B}^s_{p,2} = s - \frac{n}{p}.$$

In some occasions we may use the simple form $\deg(s,p)$. Another conception is "pair", we call (s,p) a pair if it satisfies $\deg(s,p) = \dfrac{2s-n}{\alpha+2}$. This relation comes from the nonlinear estimate in $\dot{H}^{s,p}$ of the term $|u|^\alpha u$.

As a general way, we write the Cauchy problem (1)(2)(3) in their integral form

$$u(t) = S_\varepsilon(t)u_0 - (a+ib)\int_0^t S_\varepsilon(t-\tau)|u|^\alpha u d\tau, \tag{10}$$

$$v(t) = S_0(t)v_0 - (a+ib)\int_0^t S_0(t-\tau)|v|^\alpha v(\tau) d\tau, \tag{11}$$

$$v(t) = S_0(t)v_0 - (ib)\int_0^t S_0(t-\tau)|v|^\alpha v(\tau) d\tau, \tag{12}$$

where $S_0(t) = e^{it\Delta}$, $S_\varepsilon(t) = e^{it\Delta}e^{\varepsilon t\Delta}$. To preserve the scaling invariance of the space $E_{s,p}$, β is taken as

$$\beta(s,p) = \frac{1}{2}\left(\frac{2}{\alpha} + \deg(s,p)\right).$$

The remains of the paper is arranged as follows: in Section 2, we introduce the theorems. In Section 3, we give some lemmas and establish the nonlinear estimate. Proofs of Theorems are put in Section 4.

2. Preliminaries

Before stating the main theorems, we introduce some notations and conceptions. Some of them derive from [5] and are written out for completeness. For any dimension n, we denote $S_{min} = \frac{n}{2} - \frac{\alpha+2}{\alpha}$, $S_{max} = \frac{n}{2} - \frac{\alpha+2}{\alpha(\alpha+1)}$. Define the set I_α by

$$\begin{array}{lll} 1) & \alpha < 1 & I_\alpha = \{0\} \cap (S_{min}, S_{max}), \\ 2) & \alpha \geq 1, \alpha \notin 2\mathbf{N} & I_\alpha = (0, \alpha) \cap (S_{min}, S_{max}), \\ 3) & \alpha \in 2\mathbf{N} & I_\alpha = (0, \infty) \cap (S_{min}, S_{max}). \end{array} \tag{13}$$

It is easy to show that I_α is not empty when α values in the following sets, we call such α admissible.

$$\begin{array}{lll} 1) & n = 1, 2, \cdots, 6 & \alpha \in (\alpha_0^+, +\infty), \\ 2) & n = 7 & \alpha \in (\alpha_0^+, \frac{4}{5}) \cup [1, +\infty), \\ 3) & n \geq 8 & \alpha \in (\alpha_0^+, \frac{4}{n-2}) \cup (\alpha_+, +\infty), \end{array} \tag{14}$$

where α_0^+ and α_+ are positive root and maximal positive root of the polynomial

$$\begin{array}{l} F_0(\alpha) = n\alpha^2 + \alpha(n-2) - 4, \\ G(\alpha) = 2\alpha^2 + \alpha(2-n) + 4, \end{array} \tag{15}$$

respectively. Similar to [5], we can prove that there exists a unique solution of (10), (11) in the space $E_{s,p}$ under a uniform smallness condition with respect of ε, α for any admissible α, $s \in I_\alpha$ and pair (s, p).

Theorem 1 *Let α be admissible, $\{s_j\}_{j=1}^k \in I_\alpha$, and (s_j, p_j) be pairs. Then there exists $\delta > 0$ such that when*

$$\max_{1 \leq j \leq k} \left(\|S_\varepsilon(t)u_0\|_{E_{s_j,p_j}} \vee \|S_0(t)v_0\|_{E_{s_j,p_j}} \right) < \delta, \tag{16}$$

(10), (11), (12) have unique solutions in $\cap_{j=1}^k E_{s_j,p_j}$ with

$$\max_{1 \leq j \leq k} \left(\|u(t)\|_{E_{s_j,p_j}} \vee \|v(t)\|_{E_{s_j,p_j}} \right) < 2\delta, \tag{17}$$

where $A \vee B$ means the maximum of A and B.

As one expects that, the condition (16) is not easy to verify. However, using Paley-Littlewood theory, one can prove that

Theorem 2 *Assume the conditions in Theorem1 be verified. Let $\Omega(x')$ be a continuous function on the unit sphere of order m, with $m \geq m_0$ and m_0 will be defined in the proof. Let $f = \dfrac{\Omega(x')}{|x|^{\frac{2}{\alpha}}}$, then there exists a constant C independent of ε such that*

$$\max_{1\leq j\leq k}\left(\|S_\varepsilon(t)f\|_{E_{s_j,p_j}} \vee \|S_0(t)f\|_{E_{s_j,p_j}}\right) \leq C\|\Omega\|_{C^{m_0}}. \tag{18}$$

Let us give a brief remark about the proof of Theorem 2. For simpleness and without loss of generality , we consider the case $k = 1$. By the scaling property of Fourier transform and the definition of $E_{s,p}$, it suffices to prove $\|S_0(1)u_0\|_{\dot{B}^s_{p,2}} < C\|\Omega\|_{C^{m_0}}$. In fact, the proof in [5] gives that

$$\|S_0(1)u_0\|_{\dot{B}^s_{p,1}} < C\|\Omega\|_{C^{m_0}}. \tag{19}$$

Recalling the embedding $B^s_{p,1} \hookrightarrow B^s_{p,2}$, we end the proof of Theorem 2.

In some occasions below, we may substitute s with ρ. For each $\rho \in \mathbf{N} \cup \{\mathbf{0}\}$, Δ_ρ denotes the set as follows,

$$\Delta_\rho \triangleq \begin{cases} \{2\rho+1, 2\rho+2, \cdots, 2\rho+5\}, & \rho = 0, 1, \\ \{2\rho+1, 2\rho+2, 2\rho+3\}, & \rho = 2, \\ \{2\rho+1, 2\rho+2\}, & \rho \geq 3. \end{cases} \tag{20}$$

It will be clear that when the dimension $n \in \Delta_\rho$, one can choose $E_{\rho,p}$ as a suitable work space to consider the limit behavior of self-similar solutions of (1).

Denote

$$F_\rho(\alpha) = \alpha^2(n-2\rho) + \alpha(n-2\rho-2) - 4, \tag{21}$$

and α^+_ρ the positive root of $F_\rho(\alpha)$, one can check that,

$$\begin{cases} \alpha^+_\rho > 1 & \text{as } n = 2\rho+1, 2\rho+2, \\ \alpha^+_\rho = 1 & \text{as } n = 2\rho+3, \\ \alpha^+_\rho < 1 & \text{as } n \geq 2\rho+4. \end{cases} \tag{22}$$

In view of (22), we have

Proposition 2.1 *Let $n \in \Delta_\rho$, $\alpha \in J$. Then $\rho \in I_\alpha$, and $I_\alpha - \{\rho\} \neq \emptyset$, where J is defined by*

$$J = \begin{cases} \max(\rho, \alpha^+_\rho) < \alpha < \infty, & n = 2\rho+1, 2\rho+2, \\ \rho < \alpha < \frac{4}{n-2\rho-2}, \quad \alpha \geq 1, & n = 2\rho+3, n \in \Delta_\rho, \\ \max(\rho, 1) < \alpha < \frac{4}{n-2\rho-2}, & n = 2\rho+4, 2\rho+5, n \in \Delta_\rho. \end{cases} \tag{23}$$

Proof Noticing $I_\alpha = \{0\}\cap(S_{\min}, S_{\max})$ as $\alpha < 1$, one concludes that the condition $\alpha \geq 1$ is necessary for $\rho \in I_\alpha$ and $I_\alpha - \{\rho\} \neq \emptyset$. One the other hand, one observes that $\rho \in I_\alpha$ means

$$\max(0, S_{min}) < \rho < \min(\alpha, S_{max}),$$

which holds if and only if

$$(n-2-2\rho)\alpha < 4; \quad F_\rho(\alpha) > 0; \quad 0 \leq \rho < \alpha. \tag{24}$$

Since $F_\rho(\alpha) > 0$ is equivalent to $n \geq 2\rho + 1$ and $\alpha > \alpha_\rho^+$, one has,

∗ for $n = 2\rho+1$ and $n = 2\rho+2$, the condition (24) becomes

$$\max(\rho, \alpha_\rho^+) < \alpha < \infty$$

∗ for $n = 2\rho+3$, the condition (24) becomes

$$\alpha \geq 1, \quad \rho < \alpha < \frac{4}{n-2-2\rho}, \qquad \rho \geq 0;$$

∗ for $n \geq 2\rho+4$, the condition (24) becomes

$$\max(1, \rho) < \alpha < \frac{4}{n-2-2\rho}, \qquad \rho \geq 0;$$

Noticing $\max(\rho, 1) < \dfrac{4}{n-2-2\rho}$ holds iff $n \in \Delta_\rho$, we end the proof of Proposition 2.1.

Remark 2.1 The above proposition only deals with the case $\alpha \notin 2\mathbf{N}$. For $\alpha \in 2\mathbf{N}$, one can get a similar result. Denote the set $\{2\rho+1, 2\rho+2\cdots, 2d+5\}$ by Δ'_ρ, then for $n \in \Delta'_\rho$, when $\alpha \in J'$

$$J' = \begin{cases} \alpha_\rho^+ < \alpha < \infty & n = 2\rho+1, 2\rho+2, \\ 1 < \alpha < \frac{4}{n-2\rho-2} & n = 2\rho+3, 2\rho+4, 2\rho+5, \end{cases}$$

ρ belongs to I_α.

Thanks to Proposition 2.1, we can define a nonempty set

$$J_\gamma = (0,1) \cap \left(0, \min\left((S_{\max} \vee \alpha) - \rho, \frac{n}{2} - \rho, \frac{F_\rho(\alpha)}{\alpha+2}\right)\right). \tag{25}$$

Now, we are in a position to state the convergence theorems with two different cases.

Case 1 For fixed a, let $\varepsilon \to 0$.

Theorem 3 *Let $n \in \Delta_\rho$, $\alpha \in J$, $\gamma \in J_\gamma$, and (ρ, p_0), $(\rho + \frac{\gamma}{2}, p_1)$, $(\rho+\gamma, p_2)$ be three pairs. u_0, $v_0$satisfy the conditions of Theorem 2, $u(t), v(t)$ are solutions of (10), (11) respectively, then,*

$$\begin{aligned}\|u(t)-v(t)\|_{E_{\rho,p_0}} \leq & C\|S_0(t)(u_0-v_0)\|_{E_{\rho,p_0}} + C\varepsilon^{\frac{\gamma}{2}}\|S_0(t)u_0\|_{E_{\rho+\gamma,p_0}} \\ & + C(|a|+|b|)\varepsilon^{\frac{\gamma}{2}}\|u\|^\alpha_{E_{\rho+\frac{\gamma}{2},p_1}}\|u\|_{E_{\rho+\gamma,p_2}},\end{aligned} \tag{26}$$

where u_0, v_0 are the same as those in Theorem 1.

Case 2 $\varepsilon \to 0^+$, $a \to 0$

Theorem 4 *Let $n \in \Delta_\rho$, $\alpha \in J$, $\gamma \in J_\gamma$, and (ρ, p_0), $(\rho + \frac{\gamma}{2}, p_1)$, $(\rho+\gamma, p_2)$ be three pairs. u_0, v_0 satisfy the conditions of Theorem 2, $u(t), v(t)$ are solutions of (10), (12) respectively, then,*

$$\begin{aligned}\|u(t)-v(t)\|_{E_{\rho,p_0}} \leq & C\|S_0(t)(u_0-v_0)\|_{E_{\rho,p_0}} + C\varepsilon^{\frac{\gamma}{2}}\|S_0(t)u_0\|_{E_{\rho+\gamma,p_0}} \\ & + C|a|\|u\|_{E_{\rho,p_0}} + C|b|\varepsilon^{\frac{\gamma}{2}}\|u\|^{\alpha}_{E_{\rho+\frac{\gamma}{2},p_1}}\|u\|_{E_{\rho+\gamma,p_2}},\end{aligned} \tag{27}$$

where u_0, v_0 are the same as those in Theorem1.

Remark 2.2 One can easily prove $\|S_0(t)u_0\|_{E_{\rho+\gamma,p_0}} < \infty$ in (26), (27). In fact, by applying the results in [5], it is sufficient to verify that

$$\rho+\gamma+\frac{2}{\alpha}-\frac{n}{p_0'}<0, \qquad \rho+\gamma+\frac{2}{\alpha}-\frac{n}{p_0}>0. \tag{28}$$

Using the fact that $\rho+\gamma \in I_\alpha$, $S_{\min} < \rho+\gamma < S_{\max}$ and $(\rho+\gamma, p_2)$ is a pair, one deduces

$$\rho+\gamma+\frac{2}{\alpha}-\frac{n}{p_2'}<0, \qquad \rho+\gamma+\frac{2}{\alpha}-\frac{n}{p_2}>0. \tag{29}$$

Thus, (28) is valid by (29) and $p_0 > p_2$ (here we use $\rho < \rho+\gamma$).

3. Some Lemmas

First of all, we need the $L^p - L^{p'}$ estimate of Schrödinger operator in Besov spaces.

Lemma 3.1 *Let $S_0(t) = e^{it\Delta}$, then for $s \in \mathbf{R}^+$, $p \geq 2$ and $u_0 \in \dot{B}^s_{p',2}$, we have*

$$\|S_0(t)u\|_{\dot{B}^s_{p,2}} \leq C|t|^{-n(\frac{1}{2}-\frac{1}{p})}\|u_0\|_{\dot{B}^s_{p',2}}. \tag{30}$$

Proof By virtue of the well-known decay result

$$\|S_0(t)u_0\|_{\dot{H}^{\gamma,p}} \leq C|t|^{-n(\frac{1}{2}-\frac{1}{p})}\|u_0\|_{\dot{H}^{\gamma,p'}}, \tag{31}$$

and setting $\gamma = s-\varepsilon, s+\varepsilon$, we have the desired result by the real interpolation.

Lemma 3.2 *Let T be a bounded translation invariant operator on $L^p(1 \leq p \leq \infty$ then T is bounded on $\dot{B}^s_{p,r}$ for any $s \in \mathbf{R}$, $1 \leq r \leq \infty$.*

Proof We recall that if T is a translation invariant operator, then there exists a distribution A in $\mathcal{S}'(\mathbf{R^n})$ such that for any $f \in L^p(\mathbf{R^n})$

$$Tf = A * f \tag{32}$$

Let $\{\psi_j\}_{j=-\infty}^{+\infty}$ be dyadic decomposition on $\mathbf{R^n} - \{\mathbf{0}\}$, then it is obvious that

$$T(\psi_j * f) = A * \psi_j * f = \psi_j * Af = \psi_j * Tf. \tag{33}$$

By the boundness of T and (33), we have

$$\|\psi_j * (Tf)\|_p \leq C\|\psi_j * f\|_p,$$

from which Lemma 3.2 follows.

The next lemma concerns the property of the multiplier $m(\xi) = |\xi|^{-\gamma}(e^{-|\xi|^2} - 1)$.

Lemma 3.3 *For any $\gamma \in [0, 2]$, and $1 < p < \infty$, $m(\xi) \in \mathcal{M}_p(R^n)$.*

Proof By Mihlin-Hörmander multiplier theorem (see [11] for details), it suffices to show that for some $k > \frac{n}{2}$, there exists a constant B such that

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} m(x)\right| \leq B|x|^{-|\alpha|} \qquad \forall \quad |\alpha| \leq k. \tag{34}$$

By a simple calculation, one sees that it is enough to prove

$$\frac{d^j}{d|\xi|^j} m(|\xi|) \leq B|\xi|^{-j} \qquad \forall \quad 0 \leq j \leq k. \tag{35}$$

Because $m(|\xi|) \in L^\infty$, (35) holds for $j = 0$. When $j > 0$, we calculate the jth derivative by Newton-Leibnitz formula that

$$\begin{aligned} \frac{d^j}{d|\xi|^j} m(|\xi|) &= \frac{d^j}{d|\xi|^j}\left[\left(e^{-|\xi|^2} - 1\right)|\xi|^{-\gamma}\right] \\ &= \sum_{i=0}^{j} C_i^j F_i(|\xi|) G_{j-i}(|\xi|), \end{aligned}$$

where

$$\begin{aligned} F_i(|\xi|) &= \frac{d^i}{d|\xi|^i}\left(e^{-|\xi|^2} - 1\right) \\ &\leq C \begin{cases} e^{-|\xi|^2} - 1, & \text{for } i = 0 \\ |\xi| e^{-|\xi|^2}, & \text{for } i = 1 \\ e^{-|\xi|^2} \sum_{s=0}^{i} |\xi|^s & \text{for } i \geq 2. \end{cases} \\ G_{j-i}(|\xi|) &\leq C|\xi|^{-\gamma-(j-i)}, \end{aligned}$$

Observing $\frac{d^j}{d|\xi|^j} m(|\xi|) = O(|\xi|^{-\gamma-i})$ as $|\xi| \to \infty$ and $\frac{d^j}{d|\xi|^j} m(|\xi|) = O(|\xi|^{-\gamma-j+2})$ as $|\xi| \to 0$, we have a constant $C > 0$ such that when $\gamma \in [0, 2]$,

$$\frac{d^j}{d|\xi|^j} m(|\xi|) \leq C|\xi|^{-j}.$$

Thus, we end the proof of Lemma3.3.

Remark 3.1 From Lemma 3.2 and Lemma 3.3, we see that the operator $(\mathcal{F}^{-1} m(\xi))*$ is bounded on $\dot{B}^s_{p,2}$, for $1 < p < \infty$, $s \in \mathbf{R}$.

Lemma 3.4 *For* $\gamma \in (0,1)$, $0 \le s < \alpha$, $1 \le p < \frac{n}{s+\frac{\gamma}{2}}$, $r \ge 2$, $1 < q \le 2$ *and* $\frac{1}{q} = \alpha\left(\frac{1}{p} - \frac{s+\frac{\gamma}{2}}{n}\right) + \frac{1}{r}$, *we have*

$$\||f|^\alpha f\|_{\dot{B}^{s+\gamma}_{q,2}} \le C\|f\|^\alpha_{\dot{B}^{s+\frac{\gamma}{2}}_{p,2}}\|f\|_{\dot{B}^{s+\gamma}_{r,2}}. \tag{36}$$

Proof Without loss of generality, we assume f is a real valued function. We first introduce the following equivalent norm in homogeneous Besov space $\dot{B}^s_{p,q}$,

$$\|f\|_{\dot{B}^s_{p,q}} \approx \sum_{|\beta|=[s]}\left(\int_0^\infty t^{-q(s-[s])}\sup_{|h|\le t}\|\Delta_h D^\beta f\|_p^q \frac{dt}{t}\right)^{\frac{1}{q}}, \tag{37}$$

where $\Delta_h f = f(\cdot + h) - f(\cdot)$, $[s]$ stands for the maximal integer which is less than or equal to s. We distinguish between the following two cases,

Case 1 $0 < s+\gamma < 1$. Observe

$$|\Delta_h(|f|^\alpha f)(x)| \le (|f|^\alpha(x+h) + |f|^\alpha(x))|\Delta f(x)|.$$

Denote $b = \frac{2pn}{\alpha(2n - p(2s+\gamma))}$, thus $\frac{1}{b} + \frac{1}{r} = \frac{1}{q}$. From the translation invariance of L^p and Hölder inequality, we have

$$\begin{aligned}\||f|^\alpha f\|_{\dot{B}^{s+\gamma}_{q,2}} &= \left(\int_0^\infty t^{-2(s+\gamma)}\sup_{|h|\le t}\|\Delta_h|f|^\alpha f\|_q^2\frac{dt}{t}\right)^{\frac{1}{2}} \\ &\le C\left(\int_0^\infty t^{-2(s+\gamma)}\sup_{|h|\le t}\||f|^\alpha\|_b\|\Delta_h f\|_r\frac{dt}{t}\right)^{\frac{1}{2}} \\ &\le C\|f\|^\alpha_{b\alpha}\|f\|_{\dot{B}^{s+\gamma}_{r,2}}.\end{aligned} \tag{38}$$

By noting $b\alpha > 2$, (36) follows from the embedding

$$\dot{B}^{s+\frac{\gamma}{2}}_{p,2} \hookrightarrow L^{b\alpha}$$

Case 2 $s+\gamma \ge 1$. Let β denote the multiple index, $|\beta| = [s+\gamma]$, $\lambda = s+\gamma-[s+\gamma]$. It follows from the induction that:

$$\begin{aligned}|\Delta_h D^\beta(|f|^\alpha f)| \le{}& C\sum_{k=1}^{|\beta|}\sum_{\wedge_k^\beta}\Bigg\{\left(|f_h|^{\alpha-k} + |f|^{\alpha-k}\right)|\Delta_h f|\prod_{i=1}^k|D^{\beta_i}f| \\ &+ \sum_{i=1}^k\left(|f_h|^{\alpha+1-k} + |f|^{\alpha+1-k}\right)\prod_{j=1}^{i-1}|D^{\beta_j}f_h|\prod_{j=i+1}^k|D^{\beta_j}f||D^{\beta_i}\Delta_h f|\Bigg\} \\ :={}& \sum_{k=1}^{|\beta|}\sum_{\wedge_k^\beta}\left(J_k + \sum_{i=1}^k I_{ik}\right),\end{aligned} \tag{39}$$

where

$$\wedge_k^\beta = \left\{ \sum_{j=1}^{k} \beta_j = \beta, 1 \le |\beta_1| \le |\beta_2| \cdots \le |\beta_k| \right\}.$$

For detail, one refers to [1]. We give the estimates of $\|J_k\|_q$ and $\|I_{ik}\|_q$, respectively. Choosing

$$\begin{aligned} a &= \frac{pn}{(n-p(s+\gamma/2))(\alpha-k)}, \\ y_i &= \frac{pnr(s+\gamma)}{r(n-p(s+\gamma/2)(s+\gamma-|\beta_i|))+pn|\beta_i|} \quad i=1,\cdots,k, \\ z &= \frac{pnr(s+\gamma)}{r(s+\gamma-\lambda)(n-p(s+\gamma/2))+pn\gamma}, \end{aligned} \tag{40}$$

one easily sees

$$\frac{1}{a} + \sum_{i=1}^{k} \frac{1}{y_i} + \frac{1}{z} = \frac{1}{q}.$$

Letting $B_k = \left(\int_0^\infty t^{-2\lambda} (\sup_{|h|\le t} \|J_k\|_q^2) \frac{dt}{t} \right)^{\frac{1}{2}}$, we have

$$B_k \le \|f\|^{\alpha-k}_{\frac{pn}{n-p(s+\gamma/2)}} \prod_{i=1}^{k} \|f\|_{\dot H^{|\beta_i|,y_i}} \|f\|_{\dot B^\lambda_{z,2}}, \tag{41}$$

by using Hölder inequality. By virtue of the interpolation inequality,

$$\begin{aligned} \|f\|_{\dot H^{|\beta_i|,y_i}} &\le C\|f\|^{1-\theta}_{L^{\frac{pn}{n-p(s+\gamma/2)}}} \|f\|^\theta_{\dot H^{s+\gamma,r}}, \qquad \theta = \frac{|\beta_i|}{s+\gamma} \\ \|f\|_{\dot B^\lambda_{z,2}} &\le C\|f\|^{1-\theta}_{\dot B^0_{\frac{pn}{n-p(s+\gamma/2)},2}} \|f\|^\theta_{\dot B^{s+\gamma}_{r,2}}, \qquad \theta = \frac{\lambda}{s+\gamma}, \end{aligned}$$

and the embedding theorem

$$\begin{aligned} &\dot H^{s+\gamma/2,p} \hookrightarrow L^{\frac{pn}{n-p(s+\gamma/2)}} \quad \dot B^{s+\gamma/2}_{p,2} \hookrightarrow \dot B^0_{\frac{pn}{n-p(s+\gamma/2)},2}, \\ &\dot B^{s+\gamma/2}_{p,2} \hookrightarrow \dot H^{s+\gamma/2,p} \quad \dot B^{s+\gamma}_{r,2} \hookrightarrow \dot H^{s+\gamma,r}, \text{ for } p \ge 2, \quad r \ge 2. \end{aligned}$$

it follows that,

$$\begin{aligned} B_k &\le C\|f\|^{\alpha-k}_{\dot H^{s+\gamma/2,p}} \prod_{i=1}^{k} \left(\|f\|^{\frac{s+\gamma-|\beta_i|}{s+\gamma}}_{\dot H^{s+\gamma/2,p}} \|f\|^{\frac{|\beta_i|}{s+\gamma}}_{\dot H^{s+\gamma,r}} \right) \times \|f\|^{\frac{s+\gamma-\lambda}{s+\gamma}}_{\dot B^{s+\gamma/2}_{p,2}} \|f\|^{\frac{\lambda}{s+\gamma}}_{\dot B^{s+\gamma}_{r,2}} \\ &\le C\|f\|^\alpha_{\dot B^{s+\gamma/2}_{p,2}} \|f\|_{\dot B^{s+\gamma}_{r,2}}. \end{aligned} \tag{42}$$

As for the estimate of I_{ik}, we take

$$a = \frac{pn}{(n-p(s+\gamma/2))(\alpha+1-k)},$$
$$y_j = \begin{cases} \dfrac{pnr(s+\gamma)}{r(n-p(s+\gamma/2)(s+\gamma-|\beta_j|))+pn|\beta_j|} & j \neq i \quad 1 \le j \le k \\ \dfrac{pnr(s+\gamma)}{r(n-p(s+\gamma/2))(s+\gamma-|\beta_j|-\lambda)+pn(|\beta_j|+\lambda)} & j = i. \end{cases} \tag{43}$$

Since

$$\frac{1}{a} + \sum_{j=1}^{k} \frac{1}{y_j} = \frac{1}{q}, \tag{44}$$

we obtain

$$\|I_{ik}\|_q \le C \prod_{j\neq i, j=1}^{k} \|f\|_{\dot{H}^{|\beta_j|, y_j}} \|\Delta_h D_{\beta_i} f\|_{y_i} \|f\|^{\alpha+1-k}_{\frac{pn}{n-p(s+\gamma/2)}} \tag{45}$$

by using Hölder inequality. Set $A_{ik} = \left(\int_0^\infty t^{-2\lambda} \|I_{ik}\|_q^2 \dfrac{dt}{t}\right)^{\frac{1}{2}}$, we have

$$A_{ik} \le C \prod_{j\neq i} \|f\|_{\dot{H}^{|\beta_j|, y_j}} \|f\|_{\dot{B}^{|\beta_i|+\lambda}_{y_i,2}} \|f\|^{\alpha+1-k}_{\frac{pn}{n-p(s+\gamma/2)}} \tag{46}$$

by (45). Using the interpolation inequality

$$\|f\|_{\dot{B}^{|\beta_i|+\lambda}_{y_i,2}} \le C \|f\|^{1-\theta}_{\dot{B}^0_{\frac{pn}{n-p(s+\gamma/2)},2}} \|f\|^{\theta}_{\dot{B}^{s+\gamma}_{r,2}}, \qquad \theta = \frac{|\beta_j|+\lambda}{s+\gamma}$$
$$\|f\|_{\dot{H}^{|\beta_j|, y_j}} \le C \|f\|^{1-\theta}_{L^{\frac{pn}{n-p(s+\gamma/2)}}} \|f\|^{\theta}_{\dot{H}^{s+\gamma, r}}, \qquad \theta = \frac{|\beta_j|}{s+\gamma}, j \neq i$$

and the embedding theorem

$$\dot{H}^{s+\gamma/2,p} \hookrightarrow L^{\frac{pn}{n-p(s+\gamma/2)}} \quad \dot{B}^{s+\gamma/2}_{p,2} \hookrightarrow \dot{B}^0_{\frac{pn}{n-p(s+\gamma/2)},2},$$
$$\dot{B}^{s+\gamma}_{r,2} \hookrightarrow \dot{H}^{s+\gamma,r} \quad \dot{B}^{s+\gamma/2}_{p,2} \hookrightarrow \dot{H}^{s+\gamma/2,p} \quad \text{for } r \ge 2, p \le 2,$$

we arrive at

$$\begin{aligned} A_{ik} &\le C \prod_{j\neq i} \|f\|^{1-\frac{|\beta_j|}{s+\gamma}}_{\dot{H}^{s+\gamma/2,p}} \|f\|^{\frac{|\beta_i|}{s+\gamma}}_{\dot{H}^{s+\gamma,r}} \times \|f\|^{\frac{s+\gamma-|\beta_j|-\lambda}{s+\gamma}}_{\dot{B}^{s+\gamma/2}_{p,2}} \|f\|^{\frac{|\beta_i|+\lambda}{s+\gamma}}_{\dot{B}^{s+\lambda}_{r,2}} \|f\|^{\alpha+1-k}_{\dot{H}^{p,s+\gamma/2}} \\ &\le C \|f\|^{\alpha}_{\dot{B}^{s+\gamma/2}_{p,2}} \|f\|_{\dot{B}^{s+\gamma}_{r,2}}. \end{aligned} \tag{47}$$

Collecting the estimates (47),(42) and recalling (37), we obtain

$$\||f|^\alpha f\|_{\dot{B}^{s+\gamma}_{q,2}} \le C \|f\|^{\alpha}_{\dot{B}^{s+\gamma/2}_{p,2}} \|f\|_{\dot{B}^{s+\gamma}_{r,2}}, \tag{48}$$

which is the desired result.

4. Proofs of Theorems

Proof of Theorem 1 By recalling the boundedness of the heat semigroup in L^p, that is

$$\|S_{\varepsilon t\Delta}(t)u_0\|_p = \|e^{i\varepsilon t}S_0(t)u_0\|_p \le \|S_0(t)u_0\|_p, \quad t\ge 0,$$

the existence of the Cauchy problem for CGL can be reduced to the Cauchy problem with respective to the Schrödinger equation, which can be proved by using the contraction method in a small ball in the cap space $\cap_{j=1}^k E_{s_j,p_j}$. The proof is parallel except verifying the nonlinear estimates

$$\begin{aligned} \||u|^\alpha u\|_{\dot B^s_{p',2}} &\le C\|u\|^{\alpha+1}_{\dot B^s_{p,2}} \\ \||u|^\alpha u - |v|^\alpha v\|_{\dot B^s_{p',2}} &\le C(\|u\|^\alpha_{\dot B^s_{p,2}} + \|v\|^\alpha_{\dot B^s_{p,2}})\|u-v\|^\alpha_{\dot B^s_{p,2}}, \end{aligned} \tag{49}$$

for pair (s,p) and $0\le s\le\alpha$, $p>2$. By invoking the Proposition2 in [5], the above nonlinear estimates are immediate results of the embedding

$$\dot B^s_{p,2}\hookrightarrow \dot H^{s,p} \qquad \dot H^{s,p'}\hookrightarrow \dot B^s_{p',2} \quad p\ge 2.$$

Thus we complete the proof of Theorem 1.

Proof of Theorem 3 Subtracting (11) from(10), we have

$$\begin{aligned} u(t)-v(t) &= S_\varepsilon(t)u_0 - S_0(t)v_0 \\ &\quad -(a+ib)\int_0^t (S_\varepsilon(t-\tau)|u|^\alpha u(\tau) - S_0(t-\tau)|v|^\alpha v(\tau))\,d\tau. \end{aligned} \tag{50}$$

We first estimate the linear term. We can verify

$$\begin{aligned} \|S_\varepsilon(t)u_0 - S_0(t)u_0\|_{E_{\rho,p_0}} &= \left\|\frac{e^{\varepsilon t\Delta}-1}{((\varepsilon t)^{1/2}\nabla)^\gamma}(\varepsilon t)^{\gamma/2}\nabla^\gamma S_0(t)u_0\right\|_{E_{\rho,p_0}} \\ &= \sup_{0<t<\infty}(\varepsilon t)^{\gamma/2}t^{\beta(\rho,p_0)}\left\|\frac{e^{\varepsilon t\Delta}-1}{((\varepsilon t)^{1/2}\nabla)^\gamma}\nabla^\gamma S_0(t)u_0\right\|_{\dot B^\rho_{p_0,2}}, \end{aligned} \tag{51}$$

by Remark 3.1 and the affine invariance of M_p. Notice that

$$\beta(\rho,p)+\frac{\gamma}{2} = \beta(\rho+\gamma,p_0).$$

One easily sees

$$\|S_\varepsilon(t)u_0 - S_0(t)u_0\|_{E_{\rho,p_0}} \le C\varepsilon^{\gamma/2}\|S_0(t)u_0\|_{E_{\rho+\gamma,p_0}}. \tag{52}$$

Thus, we get

$$\begin{aligned} \|S_\varepsilon(t)u_0 - S_0(t)v_0\|_{E_{\rho,p_0}} &\le \|S_\varepsilon(t)u_0 - S_0(t)u_0\|_{E_{\rho,p_0}} + \|S_0(t)(u_0-v_0)\|_{E_{\rho,p_0}} \\ &\le C\varepsilon^{\gamma/2}\|S_0(t)u_0\|_{E_{\rho+\gamma,p_0}} + \|S_0(t)(u_0-v_0)\|_{E_{\rho,p_0}}. \end{aligned} \tag{53}$$

The nonlinear estimate For the last term in (50), we have

$$\begin{aligned}
&\left\|(a+ib)\int_0^t S_\varepsilon(t-\tau)|u|^\alpha u(\tau) - S_0(t-\tau)|v|^\alpha v(\tau)d\tau\right\|_{E_{\rho,p_0}} \\
&\quad \le (|a|+|b|)\left\|\int_0^t S_0(t-\tau)(|u|^\alpha u(\tau) - |v|^\alpha v(\tau))d\tau\right\|_{E_{\rho,p_0}} \\
&\quad \le (|a|+|b|)\left\|\int_0^t (S_\varepsilon(t-\tau) - S_0(t-\tau))|u|^\alpha u(\tau)d\tau\right\|_{E_{\rho,p_0}} \\
&\quad = I + II. \qquad (54)
\end{aligned}$$

Using Lemma 3.1, the definition of E_{ρ,p_0} and the nonlinear estimate (49), one has

$$\begin{aligned}
I \le\ & C(|a|+|b|)\sup_{t>0} t^{\beta(\rho,p_0)}\int_0^t |t-\tau|^{-n(1/2-1/p_0)} \\
&\times\left(\|u\|^\alpha_{\dot{B}^\rho_{p_0,2}} + \|v\|^\alpha_{\dot{B}^\rho_{p_0,2}}\right)\|u-v\|_{\dot{B}^\rho_{p_0,2}}d\tau \\
\le\ & C(|a|+|b|)\sup_{t>0} t^{\beta(\rho,p_0)}\int_0^t |t-\tau|^{-n(1/2-1/p_0)}\tau^{(\alpha+1)\beta(\rho,p_0)}d\tau \\
&\times\left(\|u\|^\alpha_{E_{\rho,p_0}} + \|v\|^\alpha_{E_{\rho,p_0}}\right)\|u-v\|_{E_{\rho,p_0}} \\
\le\ & C(|a|+|b|)\int_0^1 |1-\tau|^{-n(1/2-1/p_0)}\tau^{-(\alpha+1)\beta(\rho,p_0)}d\tau \\
&\times\sup_{t>0} t^{1+\beta(\rho,p_0)-n(1/2-1/p_0)-(\alpha+1)\beta(\rho,p_0)} \\
&\times\left(\|u\|^\alpha_{E_{\rho,p_0}} + \|v\|^\alpha_{E_{\rho,p_0}}\right)\|u-v\|_{E_{\rho,p_0}}. \qquad (55)
\end{aligned}$$

Since

$$\begin{aligned}
&n\left(\frac{1}{2}-\frac{1}{p_0}\right) < 1, \qquad (\alpha+1)\beta(\rho,p_0) < 1, \\
&1+\beta(\rho,p_0) - n\left(\frac{1}{2}-\frac{1}{p_0}\right) - (\alpha+1)\beta(\rho,p_0) = 0,
\end{aligned}$$

for $\alpha \in J$ and the pair (ρ, p_0), we get a constant $C > 0$ such that

$$I \le C(|a|+|b|)(\|u\|^\alpha_{E_{\rho,p_0}} + \|v\|^\alpha_{E_{\rho,p_0}})\|u-v\|_{E_{\rho,p_0}}. \qquad (56)$$

We may as well take δ such that $C(|a|+|b|)(2\delta)^\alpha < \frac{1}{2}$, hence I can be estimated as

$$I \le C(|a|+|b|)(2\delta)^\alpha\|u-v\|_{E_{\rho,p_0}} < \frac{1}{2}\|u-v\|_{E_{\rho,p_0}}. \qquad (57)$$

At last, we estimate the term II. In exactly the same method as in(51), we see that

$$\begin{aligned}
S_\varepsilon(t-\tau) - S_0(t-\tau) =\ & S_0(t-\tau)\left(e^{\varepsilon(t-\tau)\Delta} - 1\right) \\
=\ & (\varepsilon(t-\tau))^{\frac{\gamma}{2}} S_0(t-\tau)\frac{e^{\varepsilon(t-\tau)\Delta}-1}{\left(\sqrt{\varepsilon(t-\tau)}\nabla\right)^\gamma}\cdot\nabla^\gamma,
\end{aligned}$$

thus, we can get

$$II \le C(|a|+|b|)\sup_{t>0} t^{\beta(\rho,p_0)} \int_0^\infty |t-\tau|^{-n(1/2-1/p_0)} \||u|^\alpha u(\tau)\|_{\dot{B}^{\rho+\gamma}_{p_0',2}}, d\tau \tag{58}$$

by using $L^p - L^{p'}$ estimate of Schrödinger group, Lemma 3.3 and the affine invariance of $\mathcal{M}_p$. Taking $p_1 = \dfrac{n(\alpha+2)}{n+\alpha(\rho+\gamma/2)}$, $p_2 = \dfrac{n(\alpha+2)}{n+\alpha(\rho+\gamma)}$, one can verify that $(\rho+\dfrac{\gamma}{2}, p_1)$, $(\rho+\gamma, p_2)$ are two pairs with

$$\left(\frac{1}{p_1} - \frac{\rho+\gamma/2}{n}\right)\alpha + \frac{1}{p_2} = \frac{1}{p_0'}.$$

Notifying $\gamma < \dfrac{n}{2} - \rho$ means

$$p_0' < 2, \qquad p_1, \quad p_2 > 2,$$

we obtain

$$\||u|^\alpha u\|_{\dot{B}^{\rho+\gamma}_{p_0',2}} \le C\|u\|^\alpha_{\dot{B}^{\rho+\gamma/2}_{p_1,2}} \|u\|_{\dot{B}^{\rho+\gamma}_{p_2,2}}, \tag{59}$$

by Lemma 3.4. Applying the method in (55), we get

$$\begin{aligned} II \; &\le C\varepsilon^{\gamma/2}(|a|+|b|)\sup_{t>0} t^{\beta(\rho,p_0)} \int_0^t |t-\tau|^{-n(1/2-1/p_0)+\gamma/2} \\ &\qquad \times \tau^{-\alpha\beta(\rho+\gamma/2,p_1)} \tau^{-\beta(\rho+\gamma,p_2)} d\tau \, \|u\|^\alpha_{E_{\rho+\gamma/2,p_1}} \|u\|_{E_{\rho+\gamma,p_2}} \\ &\le C\varepsilon^{\gamma/2}(|a|+|b|)\sup_{t>0} t^{1+\beta(\rho,p_0)-n(1/2-1/p_0)+\gamma/2-\alpha\beta(\rho+\gamma/2,p_1)-\beta(\rho+\gamma,p_2)} \\ &\qquad \times \int_0^1 |1-\tau|^{-n(1/2-1/p_0)+\gamma/2} \tau^{-\alpha\beta(\rho+\gamma/2,p_1)-\beta(\rho+\gamma,p_2)} d\tau \\ &\qquad \times \|u\|^\alpha_{E_{\rho+\gamma/2,p_1}} \|u\|_{E_{\rho+\gamma,p_2}}. \end{aligned} \tag{60}$$

Since $\gamma < \dfrac{F_\rho(\alpha)}{\alpha+2}$ and $\alpha < \dfrac{4}{n-2\rho-2}$ mean

$$\begin{aligned} &\alpha\beta\left(\rho+\frac{\gamma}{2}, p_1\right) + \beta(\rho+\gamma, p_2) < 1, \\ &n\left(\frac{1}{2} - \frac{1}{p_0}\right) < 1, \end{aligned}$$

there exists a constant such that

$$II \le C\varepsilon^{\gamma/2}(|a|+|b|)\|u\|^\alpha_{E_{\rho+\gamma/2,p_1}} \|u\|_{E_{\rho+\gamma,p_2}}. \tag{61}$$

(26) follows from (53), (57) and (61).

Proof of Theorem 4 Subtracting (12) from (10), we get

$$\begin{aligned} u(t)-v(t) &= S_\varepsilon(t)u_0 - S_0(t)v_0 - a\int_0^t S_\varepsilon(t-\tau)|u|^\alpha u(\tau)d\tau \\ &\quad -(ib)\int_0^t \left(S_\varepsilon(t-\tau)|u|^\alpha u(\tau) - S_0(t-\tau)|v|^\alpha v(\tau)\right)d\tau. \end{aligned}$$

The estimates of other terms are similar except that of

$$a\int_0^t S_0(t-\tau)|u|^\alpha u(\tau)d\tau,$$

for which we follow the way as in (55) to get

$$\left|a\int_0^t S_0(t-\tau)|u|^\alpha u(\tau)d\tau\right| \le C|a|\|u\|^{\alpha+1}_{E_{\rho,p_0}}. \tag{62}$$

This concludes the proof of Theorem4.

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