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UPWIND DISCONTINUOUS GALERKIN METHODS FOR TWO DIMENSIONAL NEUTRON TRANSPORT EQUATIONS*

Yuan Guangwei, Shen Zhijun and Yan Wei
(Laboratory of Computational Physics
Division of Applied Scientific Computing
Institute of Applied Physics and Computational Mathematics
P. O. Box 8009, Beijing, 100088, China)
(E-mail: yuan_guangwei@iapcm.ac.cn; shen_zhijun@iapcm.ac.cn; yan_wei@iapcm.ac.cn)
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Abstract In this paper the upwind discontinuous Galerkin methods with triangle meshes for two dimensional neutron transport equations will be studied. The stability for both of the semi-discrete and full-discrete method will be proved.

Key Words Upwind discontinuous Galerkin methods; Neutron transport equations; Stability.

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1. Introduction

Many papers have been devoted to the discontinuous finite element methods for neutron transport equations. In [1] the neutron transport equations with coordinates in cylinderical symmetry geometries were considered, and in [2] the Galerkin methods for both space and angular variables for nonlinear neutron transport equations were studied and the uniform priori estimate for Galerkin approximate solutions was proved. In [3] the discontinuous finite element methods were first introduced for steady neutron transport equations. In [4] the generalized difference method of "upwind" type for hyperbolic equations was discussed, and the stability and convergence of the method were proved. Upwind scheme is an important numerical scheme for solving hyperbolic equations of first-order and has been studied extensively (e.g., see [5]).

In this paper the discontinuous Galerkin methods of "upwind type" for two dimensional neutron transport equations will be introduced, where the space variables are discretized by Galerkin methods with triangle meshes, and angular variables by discrete

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ordinates and source iterate methods, and time variable by implicit or Crank-Nicolson scheme. The stability of the methods will be proved.

Consider the following two dimensional neutron transport equations

$$\frac{\partial u}{\partial t} + \mu\nabla \cdot u + \Sigma_t(x)u(x,\mu,t) = \frac{\Sigma_s(x)}{2\pi}\int_D \frac{u(x,\mu',t)}{(1-|\mu'|^2)^{\frac{1}{2}}}d\mu' + f(x,t) \quad (x,\mu)\in\Omega\times D,\ \ t\in(0,T) \tag{1}$$

with the boundary and initial conditions

$$u(x,\mu,t) = g(x,\mu,t),\ \ on\ \Gamma_\mu^-$$
$$u(x,\mu,0) = u_0(x,\mu),\ \ in\ \Omega\times D$$

where $\Sigma_t(x)$ and $\Sigma_s(x)$ are total cross section and scattering cross section respectively, $u(x,\mu,t)$ is an angular flux. $\Omega\subset\mathbf{R}^2$ is a convex polygon, $\Gamma=\partial\Omega$, $D=\{\mu\in\mathbf{R}^2 : |\mu|\le 1\}, \mu=(\mu_1,\mu_2)$, $\Gamma_\mu^- = \{x\in\Gamma:\ \mu\cdot\vec{n}(x)<0\}$, where $\vec{n}(x)$ is an outer unit normal vector.

Assume the following conditions hold:

$$g(x,\cdot,t)\in L^2(\Gamma_\mu^-\times[0,T]);\quad u_0(x,\cdot,t)\in L^2(\Omega\times[0,T]). \tag{A1}$$

$$\Sigma_t(x),\Sigma_s(x)\in L^\infty(\Omega), f(x,t)\in L^2(\Omega\times[0,T]). \tag{A2}$$

2. Discontuous Galerkin Methods and Main Results

First the angular variable in (1) is discretized. Let's introduce the discrete set $\Delta=\{\mu^1,\mu^2,\cdots\mu^N\}$, and approximate the integral term at the right hand of (1) with quadrature formular

$$\int_D u(\mu)(1-|\mu|^2)^{-\frac{1}{2}}d\mu \sim \sum_{\mu\in\Delta} u(\mu)\omega_\mu$$

where ω_μ is a positive weight function.

Then the discretization of the spatial variable is introduced. Let $E_h=\{K\}$ be a quasi-uniform triangulation of Ω, $diamK\le h$. Denote $V_h=\{v\in L_2(\Omega): v|_K$ is linear$\}$.

Now we construct the discontuous Galerkin methods of "upwind" type.

Semi-discrete scheme: for $\mu\in\Delta$, find$u_N^h = u_N^h(\cdot,\mu,t)\in V_h$ such that

$$\frac{d}{dt}(u_N^h,v) + \sum_{K\in E_h}\left[(\mu\cdot\nabla u_N^h + \Sigma_t u_N^h, v)_K + \int_{\partial K_-}[u_N^h]v_+|\mu\cdot\vec{n}|d\sigma\right] = \frac{1}{2\pi}(\Sigma_s U_N, v)_K + (f,v),\quad \forall v\in V_h \tag{2}$$

where $U_N(x,t) = \sum\limits_{\mu\in\Delta} u_N^h(x,\mu,t)\omega_\mu$, $\quad (u,v) = \int_\Omega uvdx$, $(u,v)_K = \int_K uvdx$, $\partial K_- = \{x \in \partial K;\ \ \mu\cdot\vec{n}(x) < 0\}$, $[v] = v_+ - v_-$, $v_\pm(x) = \lim\limits_{s\to 0\pm} v(x\pm s\mu)$, $x\in\partial K$. Here $\vec{n}(x)$ is an outer unit normal vector at $x \in \partial K$, $u_{N-}^h = g$ on Γ_μ^h.

Further introduce the discretization of the time variable t in (2). Denote $u_{N,h}^n(x,\mu) = u_h(x,\mu,n\Delta t)$, Δt is time step.

Fully-discrete scheme:

$$(u_{N,h}^{n+1} - u_{N,h}^n, v) + \Delta t \sum_{K\in E_h}\left[(\mu\cdot\nabla u_{N,h}^{n+1} + \Sigma_t u_{N,h}^{n+1}, v)_K + \int_{\partial K_-} u_{N,h}^{n+1} v_+ |\mu\cdot\vec{n}|d\sigma\right]$$
$$= \frac{\Delta t}{2\pi}\left(\Sigma_s U_{N,h}^{n+1}, v\right) + \Delta t\left(f^{n+1}, v\right), \tag{3}$$

where $v\in V_h$, $U_{N,h}^{n+1}(x) = \sum\limits_{\mu\in\Delta} u_{N,h}^{n+1}(x,\mu)\omega_\mu$, $f^{n+1} = f(x,(n+1)\Delta t$.

The boundary and initial conditions are

$$u_{N,h}^{n+1}(x,\mu) = g(x,\mu,t)\ \ \textit{on}\ \Gamma_\mu^-,$$
$$u(x,\mu,0) = u_0(x,\mu)\ \ \textit{in}\ \Omega\times D.$$

Also the Crank-Nicolson center discrete approximation is intrduced:

$$\left(u_{N,h}^{n+1} - u_{N,h}^n, v\right) + \frac{1}{2}\Delta t \sum_{K\in E_h}\left[\left(\mu\cdot\nabla(u_{N,h}^{n+1} + u_{N,h}^n) + \Sigma_t(u_{N,h}^{n+1} + u_{N,h}^n), v\right)_K\right.$$
$$\left. + \frac{1}{2}\int_{\partial K_-}(u_{N,h}^{n+1} + u_{N,h}^n)v_+|\mu\cdot\vec{n}|d\sigma\right]$$
$$= \frac{\Delta t}{4\pi}\left(\Sigma_s(U_{N,h}^{n+1} + U_{N,h}^n), v\right) + \Delta t\left(f^{n+1}, v\right). \tag{4}$$

Our main results are as follows.

Theorem 1 *Under the assumptions (A1) and (A2), the semi-discrete scheme (2) is stable in the norm of* L^2.

Theorem 2 *Under the assumptions (A1) and (A2), both of the fully-discrete schemes (3) and (4) are stable in the norm of* L^2.

3. Proof of Theorems 1 and 2

First we prove Theorem 1. In (2), by taking $v = u_N^h$, we get

$$\frac{d}{dt}(u_N^h, u_N^h) + \sum_{K\in E_h}\left[\left(\mu\cdot\nabla u_N^h + \Sigma_t u_N^h, u_N^h\right)_K + \int_{\partial K_-}\left[u_N^h\right] u_{N+}^h |\mu\cdot\vec{n}|d\sigma\right]$$
$$= \frac{1}{2\pi}\left(\Sigma_s U_N, u_N^h\right)_K + \left(f, u_N^h\right). \tag{5}$$

From integration by part it follows

$$\sum_{K\in E_h}\left[\left(\mu\cdot\nabla u_N^h+\Sigma_t u_N^h,u_N^h\right)+\int_{\partial K_-}\left[u_N^h\right]u_{N+}^h|\mu\cdot\vec{n}|d\sigma\right]$$
$$=\int_\Omega \Sigma_t|u_N^h|^2dx+\sum_{K\in E_h}\left[\int_{\partial K}\frac{1}{2}|u_N^h|^2\mu\cdot\vec{n}d\sigma+\int_{\partial K_-}\left[u_N^h\right]u_{N+}^h|\mu\cdot\vec{n}|d\sigma\right]$$
$$\equiv I+II. \tag{6}$$

Term II can be decomposed into two parts

$$II=\sum_{\substack{K\in E_h\\ \partial K\cap\Gamma\neq\phi}}+\sum_{\substack{K\in E_h\\ \partial K\cap\Gamma=\phi}}\equiv II_1+II_2. \tag{7}$$

Note that

$$\begin{aligned}II_1&=\frac{1}{2}\int_\Gamma|u_N^h|^2\mu\cdot\vec{n}d\sigma+\int_{\Gamma_\mu^-}\left(u_{N+}^h-u_{N-}^h\right)u_{N+}^h|\mu\cdot\vec{n}|d\sigma\\&=\frac{1}{2}\int_{\Gamma_\mu^+}|u_{N-}^h|^2|\mu\cdot\vec{n}|d\sigma+\frac{1}{2}\int_{\Gamma_\mu^-}|u_{N+}^h|^2|\mu\cdot\vec{n}|d\sigma\\&\quad-\int_{\Gamma_\mu^-}u_{N-}^hu_{N+}^h|\mu\cdot\vec{n}|d\sigma\\&\geq\frac{1}{2}\int_{\Gamma_\mu^-}|u_{N+}^h-u_{N-}^h|^2|\mu\cdot\vec{n}|d\sigma-\frac{1}{2}\int_{\Gamma_\mu^-}|\mu\cdot\vec{n}||u_{N-}^h|^2d\sigma,\end{aligned} \tag{8}$$

and

$$\begin{aligned}II_2&=\sum_{\partial K\cap\Gamma=\phi}\left(\frac{1}{2}\int_{\partial K_+}|u_{N-}^h|^2\mu\cdot\vec{n}d\sigma+\frac{1}{2}\int_{\partial K_-}|u_{N+}^h|^2\mu\cdot\vec{n}d\sigma\right.\\&\quad\left.+\int_{\partial K_-}\left(u_{N+}^h-u_{N-}^h\right)u_{N+}^h|\mu\cdot\vec{n}|d\sigma\right)\\&=\sum_{\partial K\cap\Gamma=\phi}\frac{1}{2}\int_{\partial K_-}|u_{N+}^h-u_{N-}^h|^2|\mu\cdot\vec{n}|d\sigma.\end{aligned} \tag{9}$$

Combining (3)-(7), and integrating with respect to t, and multiplying with ω_μ and then summing up for $\mu\in\Delta$, we can obtain the stability estimate

$$\begin{aligned}&\sum_{\mu\in\Delta}\omega_\mu\int_\Omega|u_N^h(x,\mu,t)|^2dx+\sum_{\mu\in\Delta}\sum_{K\subset E_h}\omega_\mu\int_{\partial K_-}\int_0^t\left[u_N^h\right]^2|\mu\cdot\vec{n}|d\sigma\\&\quad\leq C\left[\sum_{\mu\in\Delta}\omega_\mu\int_\Omega|u_0(x,\mu)|^2dx+\int_0^t\int_\Omega|f(x,s)|^2dxds\right.\\&\qquad\left.+\int_0^t\sum_{\mu\in\Delta}\omega_\mu\int_{\Gamma_\mu^-}|g(x,\mu,s)|^2|\mu\cdot\vec{n}|d\sigma ds\right],\end{aligned} \tag{10}$$

where the Cauchy inequality and Gronwall ineqquality are used. The proof of Theorem 1 is completed.

Finally we prove Theorem 2. In (3), letting $v = u_{N,h}^{n+1}$ and noticing that

$$\left(u_{N,h}^{n+1} - u_{N,h}^{n}, u_{N,h}^{n+1}\right) = \frac{1}{2}\left(u_{N,h}^{n+1}, u_{N,h}^{n+1}\right) + \frac{1}{2}\left(u_{N,h}^{n+1} - u_{N,h}^{n}, u_{N,h}^{n+1} - u_{N,h}^{n}\right) - \frac{1}{2}\left(u_{N,h}^{n}, u_{N,h}^{n}\right)$$

and proceeding the similar argument as above, we have

$$\sum_{K\in E_h}\left[\left(\mu\cdot\nabla u_{N,h}^{n+1}, u_{N,h}^{n+1}\right)_K + \int_{\partial K_-}\left[u_{N,h}^{n+1}\right] u_{N,h}^{n+1}|\mu\cdot\vec{n}|d\sigma\right] \geq -\frac{1}{2}\int_{\Gamma_\mu^-}|\mu\cdot\vec{n}|\cdot|u_{N,h}^{n+1}|^2 d\sigma.$$

Multipying the above inequality with ω_μ, and summing up the resulting inequality with respect to $\mu\in\Delta$, and then using Cauchy inequality and Gronwall inequality, we obtain

$$\begin{aligned}\sum_{\mu\in\Delta}\omega_\mu\int_\Omega|u_{N,h}^{n+1}|^2dx &\leq \sum_{\mu\in\Delta}\omega_\mu\int_\Omega|u_0(x,\mu)|^2dx + \sum_{m=1}^{n+1}\int_\Omega|f(x,m\Delta t)|^2dx\Delta t\\ &+\textstyle\sum_{m=1}^{n+1}\Delta t\sum_{\mu\in\Delta}\omega_\mu\int_{\Gamma_\mu^-}|g(x,\mu,n\Delta t)|^2|\mu\cdot\vec{n}|d\sigma\end{aligned}$$

It follows that (3) is stable in L^2 norm.

If take $v = \frac{1}{2}(u_{N,h}^{n+1} + u_{N,h}^{n})$ in (4), then the stability of (4) can be proved similarly. The proof of Theorem 2 is completed.

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