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SHORT COMMUNICATION SECTION

AN INITIAL VALUE PROBLEM FOR PARABOLIC MONGE-AMPÈRE EQUATION FROM INVESTMENT THEORY

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The author of [1] raised an optimal investment problem in time interval $[0, T]$, in which the financial market is characterized by the parameters r, b, σ, the attitude of the investor to the risk versus the gain at the final time is described by a utility function $g(y)$, the purpose is to find out an optimal portfolio to maximize the profit of the investor. To this end, in [1] the following initial value problem is derived:

$$\begin{cases} V_s V_{yy} + ryV_y V_{yy} - \theta V_y^2 = 0, \ \ V_{yy} < 0, \ \ (s, y) \in [0, T) \times \mathbb{R}, \\ V(T, y) = g(y), \ \ g'(y) \geq 0, \ \ y \in \mathbb{R}, \end{cases} \tag{1}$$

where $V = V(s, y)$ is the unknown function, constants $r \geq 0, \sigma > 0, b - r > 0, \theta = \dfrac{b - r}{\sigma}$, and

$$g(y) = 1 - e^{-\lambda y} \tag{2}$$

is a typical case, where λ is a positive constant. The relation between the optimal investment problem and (1) lies in

Lemma 1 *Suppose (1) admits a classical solution $V(s, y)$ such that the function*

$$\tilde{\pi}(s, y) \stackrel{\text{def}}{=} -\frac{\theta V_y(s, y)}{\sigma V_{yy}(s, y)}, \quad (s, y) \in [0, T] \times \mathbb{R} \tag{3}$$

is Lipschitz continuous in y. Then $V(s, y)$ is the value function of the optimal problem with the optimal portfolio given by

$$\overline{\pi}(t) \equiv \tilde{\pi}(t, \overline{Y}(t)), \quad t \in [s, T], \tag{4}$$

where $\overline{Y}(\cdot)$ *is the solution of*

$$\begin{cases} d\overline{Y}(t) = r\overline{Y}(t) + (b-r)\tilde{\pi}(t,\overline{Y}(t))]dt, \\ +\sigma\tilde{\pi}(t,\overline{Y}(t))dW(t), \quad t \in [s,T], \\ \overline{Y}(s) = y. \end{cases} \tag{5}$$

Which is proved in [1].

The equation in (1) is called parabolic Monge-Ampère equation in [1], which is indeed a nonlinear and un-uniformly parabolic equation. But there is not any existence result for it in [1].

We obtain a general approach to both the solution to (1) and the optimal portfolio to the optimal investment problem, which goes like this:

Let $f(s,y)$ be a smooth function, which is Lipschitz continuous in y. Insert

$$\frac{V_y}{V_{yy}} = f(s,y) \tag{6}$$

into (1), then (1) becomes a Cauchy problem for a homogeneous linear partial differential equation of first order, which is our key observation. And, by the known result, the unique solution of this Cauchy problem is $g(Y(T;s,y))$, where $y = Y(s;s_0,y_0)$ is the solution of the initial value problem for its characteristic equation

$$\begin{cases} \dfrac{dy}{ds} = f(s,y) - ry, \\ y|_{s=s_0} = y_0. \end{cases} \tag{7}$$

Now it is obvious that, in order that the function $g(Y(T;s,y))$ can be the solution to (1), we need and only need that the function $V(s,y) = g(Y(T;s,y))$ satisfies (6), which can be expressed in a formula; since, by the theorem of differentiability of the solution of (7) w.r.t. the initial value, we can calculate $\dfrac{\partial g(Y(T;s,y))}{\partial y}$ and $\dfrac{\partial^2 g(Y(T;s,y))}{\partial y^2}$.

We may summarize the above general approach into the following

Theorem 1 *Suppose* $f(s,y)$ *is a smooth function, which is Lipschitz continuous in* y*. Then (1) will have a solution* $V(s,y)$ *with the property*

$$\tilde{\pi}(s,y) \overset{\text{def}}{=} -\frac{\theta V_y(s,y)}{\sigma V_{yy}(s,y)} = -\frac{\theta}{\sigma} f(s,y), \quad (s,y) \in [0,T] \times \mathbb{R} \tag*{$(3)_f$}$$

if and only if the following condition holds:

$$\frac{g'(Y(T;s,y)) \exp\left\{\displaystyle\int_T^s \left[r - \theta \frac{\partial f}{\partial Y}(\xi, Y(\xi,y))\right] d\xi\right\}}{g''(Y(T;s,y)) + g'(Y(T;s,y))\displaystyle\int_T^s \theta \frac{\partial^2 f}{\partial Y^2}(\tau, Y(\tau;s,y)) \exp\int_T^\tau \left[(r - \theta \frac{\partial f}{\partial Y}(\xi, Y(\xi;s,y))\right] d\xi d\tau} = f(s,y), \tag{8}$$

where $Y(s;s_0,y_0)$ *is defined by the general solution to (6). When (8) is valid, (1) has a solution satisfying* $(3)_f$*, which is*

$$V(s,y) = g(Y(T;s,y)) \tag{9}$$

and the optimal portfolio can be obtained via Lemma 1.

For some particular f or g, we may get further results. For example, in $(3)_f$ taking $f(s,y)=\alpha(s)y+\beta(s)$, which is a special case of (3), i.e.

$$\begin{cases} \tilde{\pi}(s,y) \stackrel{\text{def}}{=} -\dfrac{\theta V_y(s,y)}{\sigma V_{yy}(s,y)} = -\dfrac{\theta}{\sigma}\left[\alpha(s)y+\beta(s)\right], \quad (s,y)\in[0,T]\times\mathbb{R}, \\ \text{where } \alpha(s) \text{ and } \beta(s) \text{ are smooth functions depend only on } s. \end{cases} \tag{$(3)_{\alpha\beta}$}$$

then, from Theorem 1, we may derive the following two interesting corollaries:

Corollary 1 *If the $g(y)$ in (1) is taken by (2), then the problem (1) so obtained has a solution $V(s,y)$ with $(3)_{\alpha\beta}$ satisfied, in this case*

$$V(s,y) = 1 - e^{-\theta(T-s)}e^{-\lambda y e^{r(T-s)}}, \tag{$(9)'$}$$

which is the value function of the related optimal investment problem, and the corresponding optimal portfolio is

$$\overline{\pi}(t) = \frac{\theta}{\sigma\lambda}e^{-r(T-t)}, \quad t\in[s,T]. \tag{$(4)'$}$$

Corollary 2 *If (1) has a solution $V(s,y)$ satisfying $(3)_{\alpha\beta}$, then the $g(y)$ in (1) must take the following form*

$$g(y) = C_1 - C_2e^{-C_3y}, \quad C_1, C_2, C_3 \text{ are constants, with } C_2>0, C_3>0. \tag{$(2)'$}$$

In this case,

$$V(s,y) = C_1 - C_2e^{-\theta(T-s)}e^{-C_3 y e^{r(T-s)}}, \tag{$(9)''$}$$

which is the value function of the related optimal investment problem, and the corresponding optimal portfolio is

$$\overline{\pi}(t) = \frac{\theta}{\sigma C_3}e^{-r(T-t)}, \quad t\in[s,T]. \tag{$(4)''$}$$

The reason that the conclusion of these two corollaries is so concise lies in: after taking $f(s,y)=\alpha(s)y+\beta(s)$ in $(3)_f$, the corresponding equation in (7) becomes a linear ordinary differential equation of first order, whose solution can be written down in an explicit expression, from which, by noting $g''<0$ and $g'>0$, it is easy to prove the two corollaries.

References

[1] Yong Jiongmin, Introduction to Mathematical Finance. In "Mathematical Finance—Theory and Applications", Jiongmin Yong, Rama Cont, eds., Beijing, High Education Press, (2000), 19-137.