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NEW EXPLICIT SOLUTIONS TO THE (2+1)-DIMENSIONAL BROER-KAUP EQUATIONS

Liu Xiqiang
(Department of Mathematics, Liaocheng University, Shandong 252059, China)
(E-mail: liuxiq@sina. com. cn)
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Abstract Applying the homogeneous balance method, we have found the explicit and soliton solutions and given a successive formula of finding explicit solutions to the (2+1)-dimensional Broer-Kaup equations. Moreover, by using the Lie group method, we have discussed the similarity solutions to the (2+1)-dimensional Broer-Kaup equations.
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1. Introduction

For the (1+1)-dimensional Broer–Kaup (BK) system

$$H_t = (H^2 + 2G - H_x)_x, \tag{1}$$

$$G_t = (G_x + 2GH)_x, \tag{2}$$

Matsukidaira and Satsuma obtained their expressions of trilinear forms [1]. Kupershmidt proved the BK system has an infinite number of conserved densities and higher commuting flows [2]. Hirota obtained the N-soliton solutions of the BK system [3,4]. This system is a generalization of the classical dispersiveless long wave equations [2]. Using the Darboux transformation related symmetry constraints of the KP (Kadomtsev-Petviashvili) equation, Lou and Hu have obtained the following Broer-Kaup (BK) equations [5]

$$H_t = H_{xx} - 2HH_x - 2\partial_y^{-1}G_{xx}, \tag{3}$$

$$G_t = -G_{xx} - 2\left(HG\right)_x, \tag{4}$$

where $\partial_y^{-1}f = \int f dy$. Obviously, when $y = x$ and t is replaced by $-t$, the BK equations (3) and (4) can be changed into (1) and (2) respectively. Therefore, the BK equations (3) and (4) are also the generalization of (1) and (2) in (2+1)-dimensions space-time {x,y,t}.

In this paper we consider the following (2+1)-dimensional BK equations

$$H_{ty} = H_{xxy} - 2(HH_x)_y - 2G_{xx}, \tag{5}$$

$$G_t = -G_{xx} - 2\left(HG\right)_x, \tag{6}$$

which is equivalent to (3) and (4). In [6], Ruan and Chen discussed the Plainleve property,symmetries to the BK equations (5) and (6). Our aim is to find the explicit solutions of the BK equations (5) and (6) . In Section 2, by applying the extended homogeneous balance method [7], we first reduce the BK equations (5) and (6) to two linear equations with a constraint condition. In Section 3, based on the reduced equation, we obtain some new explicit solutions to the BK equations (5) and (6). In Section 4, we discuss the optimal system of one- dimensional subalgebras corresponding to the reduced equation. In Section 5, we find some similarity solutions to the reduced equation and can get some new solutions to the BK equations (5) and (6). We generalize the results in [8].

2. Reduction of BK Equations

In this section , we shall get a reduction of the equations (5) and (6) by applying the extended homogeneous balance method . To balance the nonlinear term $(HH_x)_y$ (or $(HG)_x$) and the third order derivative term H_{xxy} (or second order derivative term G_{xx}) in the equation (5) (or (6)), we take the following form solutions of the BK equations

$$H = P\left(x, y, t\right) + h^{'}\left(\phi\right)\phi_x, \tag{7}$$

$$G = Q\left(x, y, t\right) + g^{''}\left(\phi\right)\phi_x\phi_y + g^{'}\phi_{xy}, \tag{8}$$

where the functions $h\left(\phi\right), g\left(\phi\right)$ and $\phi\left(x, y, t\right)$are to be determined, P and Q are the given solutions of the equations (5) and (6), $h^{'}\left(\phi\right) = \partial h/\partial\phi$ and $g^{''}\left(\phi\right) = \partial^2 g/\partial^2\phi$. Substituting (7) , (8) into (5) , (6) and using Maple , we can find that $h\left(\phi\right) = g\left(\phi\right) = \ln\left(\phi\right)$ and yields

$$A = h^{(3)}R\phi_x\phi_y + h^{(2)}\left[R\phi_{xy} + R_x\phi_y + R_y\phi_x\right] + h^{'}R_{xy} + P_1, \tag{9}$$

$$\begin{aligned} B =& h^{(3)}R\phi_x\phi_y + h^{(2)}\left[R\phi_{xy} + R_x\phi_y + R_y\phi_x\right] + h^{'}R_{xy} \\ &+ 2h^{'}\left[\left(Q - P_y\right)\phi_x\right]_x + Q_1, \end{aligned} \tag{10}$$

where

$$\begin{aligned} A &:= H_{ty} - H_{xxy} + 2\left(HH_x\right)_y + 2G_{xx}, \\ B &:= G_t + G_{xx} + 2\left(HG\right)_x, \\ R &:= \phi_t + \phi_{xx} + 2P\phi_x, \\ P_1 &:= P_{ty} - P_{xxy} + 2\left(PP_x\right)_y + 2Q_{xx}, \\ Q_1 &:= Q_t + Q_{xx} + 2\left(QP\right)_x. \end{aligned}$$

Notice that we use the relations $h^{'}h^{''}=h^{'}g^{''}=g^{'}h^{''}=-2^{-1}h^{(3)}$ and $h^{'2}=h^{'}g^{'}=-h^{''}$. Take $Q=P_y$, it follows from (9) and (10) that

$$A=B=\left(\left(h^{'}R\right)_x+\left(P_t+P_{xx}+2PP_x\right)\right)_y=0. \tag{11}$$

Therefore, the equations (5) and (6) are reduced to the single equation (11) under the following constraint condition

$$Q=P_y. \tag{12}$$

It can be seen that there exist a lot of solutions ϕ and P satisfying the equations (11) and (12). One can get from (11)

$$\left(h^{'}R\right)_x+\left(P_t+P_{xx}+2PP_x\right)=K\left(x,t\right)+M\left(x,t\right), \tag{13}$$

where $K\left(x,t\right)$ and $M\left(x,t\right)$ are arbitrary smooth functions and $h'=\phi^{-1}$. Solving the equation (13) can be changed to solving the following two equations

$$P_t+P_{xx}+2PP_x=K\left(x,t\right), \tag{14}$$

$$\phi_t+\phi_{xx}+2P\phi_x=\left(\int M\left(x,t\right)dx+N\left(y,t\right)\right)\phi, \tag{15}$$

where $N\left(y,t\right)$ is an arbitrary smooth function. We notice that the equation (14) is independent of the function ϕ and (15) is a linear partial differential equation of ϕ . Thus it is convenient to get ϕ and P from the equations (14) and (15). As above, we get the reduced equations (14) and (15) corresponding to the BK equations (5) and (6) by the transformations (7) and (8).

3. Solutions of BK Equations

For the sake of simplicity, we take $K\left(x,t\right)=M\left(x,t\right)=N\left(y,t\right)=0$ in (14) and (15), and get

$$P_t+P_{xx}+2PP_x=0, \tag{16}$$

$$\phi_t+\phi_{xx}+2P\phi_x=0. \tag{17}$$

In addition, we point out that the expressions (7) and (8) can rewriter as

$$H=P+\phi_x/\phi, G=H_y. \tag{18}$$

Whence the solutions P and ϕ are found from the equations(16) and (17) , we can get the solutions of the BK equations from (18) .

It is easy to prove that the constraint condition (12) holds when the functions P and ϕ satisfy (16), (17) and (18).

We sperate the following cases to find the solutions of the equations (16) and (17).

Cases 1 $P = P(y)$ arbitrary. Take

$$\phi(x,y,t) = v_0 + \sum_{k=1}^{n} v_k \exp\left(a_k x - a_k^2 t - 2Pa_k t + b_k\right), \tag{19}$$

where $v_k = v_k(y), a_k = a_k(y)$ and $b_k = b_k(y)$ are arbitrary functions of y . It is easy to check P and ϕ satisfy the equations (16) and (17). Substituting the functions P and ϕ into the equation (18), we find the solutions H and G to the BK equations (5) and (6)

$$H = P(y) + \frac{\sum_{k=1}^{n} v_k a_k \exp\left(a_k x - a_k^2 t - 2Pa_k t + b_k\right)}{v_0 + \sum_{k=1}^{n} v_k \exp\left(a_k x - a_k^2 t - 2Pa_k t + b_k\right)}, \tag{20}$$

$$G = H_y. \tag{21}$$

In particular, if taking the functions P, v_k and a_k to be appropriate functions, one can obtain the soliton solutions with usual form to the BK equations. For instance, taking $n = 1$ and $v_0 = v_1 =$ const., $P =$ const. and $a_1 =$ const., we get the 1-soliton solution H of the BK equations as follows

$$H = P + 2^{-1}a_1 + 2^{-1}a_1 \tanh\left(2^{-1}\left(a_1 x - a_1^2 t - 2Pa_1 t + b_1\right)\right), \tag{22}$$

$$G = 4^{-1}a_1 b_{1y} \sec^2\left(2^{-1}\left(a_1 x - a_1^2 t - 2Pa_1 t + b_1\right)\right). \tag{23}$$

Cases 2 $P = P(x,y)$.

(a) The equation (16) yields $P_x + P^2 = p_0^2$ and

$$P = p_0 \tanh(p_0 x + p_1), \tag{24}$$

where $p_0 = p_0(y)$ and $p_1 = p_1(y)$ are arbitrary functions. If take $\phi = \phi(t,y)$is an arbitrary function independent of x, then we can get from (15)(taking $M(x,t) = 0$) that

$$\phi = p_2(y) \exp\left(\int N(y,t)\, dt\right). \tag{25}$$

Substituting (24) and (25) into (18),we find the equilibrium solutions of the BK equation. These solutions play an importance role of analyzing the asymptotic property of solutions.

(b) Substituting (24) into (17) yields

$$\phi_t + \phi_{xx} + 2p_0 \tanh(p_0 x + p_1)\phi_x = 0. \tag{26}$$

Setting $\phi = r(t,y)\, s(x,y)$ and substituting it into (26),we have

$$s_{xx} + 2p_0 \tanh(p_0 x + p_1) s_x + \lambda s = 0, \tag{27}$$

$$r_t - \lambda r = 0, \tag{28}$$

where $\lambda = \lambda(y)$ is an arbitrary function. Now we solve the equation (27). Setting $w = s\cosh(p_0 x + p_1)$ and substituting it into (27) yield

$$w_{xx} + \left(\lambda - p_0^2\right) w = 0. \tag{29}$$

Solving the equation (29) yields

$$\begin{aligned} w =& c_1 \cos(\alpha x) + c_2 \sin(\alpha x), \text{for } \alpha^2 = \lambda - p_0^2 > 0, \\ w =& c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x), \text{for } \alpha^2 = -\lambda + p_0^2 > 0, \end{aligned} \tag{30}$$

where $c_1 = c_1(y)$ and $c_2 = c_2(y)$ are arbitrary functions. Substituting (30) into the expression of ϕ yields

$$\phi = \exp(\lambda t)\sec(p_0 x + p_1)\left[c_1 \cos(\alpha x) + c_2 \sin(\alpha x)\right], \text{for } \alpha^2 = \lambda - p_0^2 > 0, \tag{31}$$

$$\phi = \exp(\lambda t)\sec(p_0 x + p_1)\left[c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)\right], \text{for } \alpha^2 = -\lambda + p_0^2 > 0. \tag{32}$$

(c) Solving the equations (16) and (17), we also get the following solutions

$$P = (x + p_1(y))^{-1}, \tag{33}$$

$$\phi(x, y, t) = \left[-2a_1(y)t + a_2(y)\right]\left[x + p_1(y)\right]^{-1} + a_0(y) + a_1(y)\left[x + p_1(y)\right], \tag{34}$$

where $a_k(y)$ and $p_1(y)$ are arbitrary functions.

Substituting (24) , (31) (or (32)) and (33), (34) into (18), we can get the solutions of the BK equations.

Cases 3 $P = P(\eta), \eta = x + b - ct$, where b and c are constants. Substituting P into equation (16) and integrating with respect to η, we obtain

$$P_\eta + P^2 - cP = d, \tag{35}$$

where d is an integrating constant.

(d) Taking $d = -c^2/4$ and solving (35) and (17) ,we find

$$\begin{aligned} P =& c/2 + (\eta + a_0)^{-1} \\ \phi =& - a_1 (\eta + a_0)^{-1} + a_2, \end{aligned} \tag{36}$$

where a_k $(k = 0, 1, 2)$ are constants.

(e) Suppose that α and β are two different roots satisfying the equation $\lambda^2 - c\lambda - d = 0$. From the equation (35), we get

$$P = \left[\alpha - a_0\beta \exp((\beta - \alpha)\eta)\right] / \left[1 - a_0 \exp((\beta - \alpha)\eta)\right]. \tag{37}$$

Set $\phi(x, y, t) = \phi(\eta)$, it follows from (17) that

$$\phi_{\eta\eta} + (2P - c)\phi_\eta = 0. \tag{38}$$

Solve the equation (38) and find

$$\phi = c_0 \int \frac{\exp\left(\left(c-2\alpha\right)\eta\right)}{\left(1-a_0\exp\left(\left(\beta-\alpha\right)\eta\right)\right)^2} d\eta, \tag{39}$$

where c_0 and a_0 are integrating constants. Substituting (36) or (37) and (39) into (18), we also get the solutions of the BK equations (5) and (6).

Cases 4 $P(x, y, t) = \phi(x, y, t)$.

In this case, the equation(16) is equivalent to the equation (17). By using the Cole-Hopf transformation $P = \theta_x/\theta$, the equation (16) is changed to the following linear equation

$$\theta_t + \theta_{xx} = 0. \tag{40}$$

(f) Setting $\theta = v_0 + \sum_{k=1}^{n} v_k \exp\left(a_k x - a_k^2 t + b_k\right)$ and solving the equation(40), we can obtain the solutions P of the equation (16) as follows.

$$P = \phi = \frac{\sum_{k=1}^{n} a_k v_k \exp\left(a_k x - a_k^2 t + b_k\right)}{v_0 + \sum_{k=1}^{n} v_k \exp\left(a_k x - a_k^2 t + b_k\right)}, \tag{41}$$

where v_k, a_k and b_k are functions of y.

(g) By using the method of separation of variables, it turns out from the equation (40) that

$$P = \phi = \exp\left(\lambda^2 t\right)\left[c_1 \cos\left(\lambda x\right) + c_2 \sin\left(\lambda x\right)\right], \tag{42}$$

where c_k and λ are functions of y.

Therefore, substituting (41) or (42) into (18), we also get the corresponding solutions of the BK equations.

Remark 1 Since the functions P and Q are also the solutions of the BK equations, one can get other solutions of the BK equations by given solutions and applying (18). So the expression (18) becomes a successive formula of finding solutions and plays a role of the Backlund transformation..

Remark 2 As taking $P = const.$, $Q = 0, v_k, a_k$ and b_k being constants in (20) and (21), the solutions are the results [8]. Therefore, we generalize the results in [8].

Remark 3 Applying the Lie group method to the equations (16) and (17), we can find the corresponding group-invariant solutions (for example the solutions expressed by the parabolic cylinder functions). These results will be discussed in next section.

4. An Optimal System of One-Dimensional Subalgebras

In this section , we continue to discuss the equation(40), so that one can get other solutions of the BK equations. Let G be the one-parameter (ϵ) Lie group generated by the following infinitesimal operator

$$\vartheta = \xi(x,t)\,\partial_x + \zeta(t)\,\partial_t + v(x,t,\theta)\,\partial_\theta, \tag{43}$$

where infinitesimal elements ξ, ζ and v of G are functions to be determined. Our first task is to determine ξ,ζ and v so that the corresponding one-parameter group $\exp(\varepsilon\vartheta)$ is a symmetry Lie-group of the equation (40).

The corresponding second prolongation of ϑ is

$$pr^{(2)}\vartheta = \vartheta + v^x\partial_{\theta_x} + v^t\partial_{\theta_t} + v^{xx}\partial_{\theta_{xx}}, \tag{44}$$

where

$$\begin{aligned}
v^x &= v_x + (v_\theta - \xi_x)\,\theta_x,\\
v^t &= v_t + (v_\theta - \zeta_t)\,\theta_t - \zeta_t\theta_x,\\
v^{xx} &= v_{xx} + (2v_{x\theta} - \xi_{xx})\,\theta_x + v_{\theta\theta}\theta_x^2 + (v_\theta - 2\xi_x)\,\theta_{xx}.
\end{aligned} \tag{45}$$

Substituting (45) to the equation (40) yields the infinitesimal criterion

$$v^t + v^{xx} = 0. \tag{46}$$

From the equation (46), we find the following determining equations

$$\theta_x^2 : v_{\theta\theta} = 0; \theta_{xx} : \zeta_t - 2\xi_x = 0; \theta_x : \xi_t - 2v_{x\theta} + \xi_{xx} = 0; v_t + v_{xx} = 0. \tag{47}$$

Solving the equation (47), one gets the infinitesimal elements ξ, ζ and v as follows

$$\begin{aligned}
\xi =& c_1 + c_4x + 2c_5t + 4c_6xt,\\
\zeta =& c_2 + 2c_4t + 4c_6t^2,\\
v =& \left(c_3 + c_5x - 2c_6t + c_6x^2\right)\theta + \alpha,
\end{aligned} \tag{48}$$

where $c_{1,\cdots,}c_6$ are six arbitrary group constants and $\alpha(x,y,t)$ satisfies the equation $\alpha_t + \alpha_{xx} = 0$.

In general , to each subgroup of the symmetry group, there will correspond a family of group-invariant solutions of the adjoint heat equation. It is very complicated to list all possible group-invariant solutions. So it is necessary to lead to an "optimal system" of group-invariant solutions. For simplicity, we will find an optimal system [9] of one-dimensional subalgebras.

The knowledge of the infinitesimal elements ξ, ζ and v detainment by (48) enable us to construct the following six operators v_k.

$$\begin{aligned}
v_1 =& \partial_x, v_2 = \partial_t, v_3 = \theta\partial_\theta, v_4 = x\partial_x + 2t\partial_t,\\
v_5 =& 2t\partial_x + x\theta\partial_\theta, v_6 = 4xt\partial_x + 4t^2\partial_t + \left(x^2 - 2t\right)\theta\partial_\theta.
\end{aligned} \tag{49}$$

Here, we ignore the discussion of the infinite-dimensional subalgebra $v_\alpha = \alpha(x,y,t)\,\partial_\theta$. Note that the operators (48) differ from those of the heat equation. Applying the commutator operators $[v_i, v_j] = v_j v_i - v_i v_j$, we get

Proposition 1 *The operators v_k (k =1,$\cdots$,6) form a Lie algebra σ, which is a six-dimensional symmetry algebra.*

Note the facts that (1) each element $g\ (\in G)$ can be represented by the products of $\exp(\varepsilon_k v_k)$ and $\exp(v_\alpha)$; (2) the problem of classifying group-invariant solutions reduces to the problem of classifying subgroup of G under conjugation; (3) the problem of finding an optimal system of subgroup is equivalent to that of finding an optimal system of subalgebras. Thus we shall find an optimal system of one-dimensional subalgebras of σ. Applying the formula

$$Ad\left(\exp\left(\varepsilon\vartheta\right)\right)w_0 = w_0 - \varepsilon\left[\vartheta, w_0\right] + \frac{1}{2}\varepsilon^2\left[\vartheta,\left[\vartheta, w_0\right]\right] - \cdots, \tag{50}$$

and Proposition 1, we can get the following proposition

Proposition 2 $\eta\left(Adg\left(\vartheta\right)\right) = \eta\left(\vartheta\right)$, *where* $g \in G$, $\eta = \eta\left(\vartheta\right) = \left(a_4\right)^2 - 4a_2a_6$ *and* $\vartheta = \sum_1^6 a_k v_k$. *The symbol* $Ad\left(\exp\left(\varepsilon\vartheta\right)\right)$ *denotes the adjoint representation of one-parameter Lie group* $\exp\left(\varepsilon\vartheta\right)$.

The proof of Proposition 1 or 2 is omitted (see [9] for detail). By using Proposition 2, we get the following results.

Theorem 3 *An optimal system S of one-dimensional subalgebras of σ is generated by the basis operators*

$(a)\, v_4 + 2av_3, \eta > 0, a \in \mathbf{R}$; $(b)\, v_2 + v_6 + av_3, \eta < 0, a \in \mathbf{R}$; $(c_1)\, v_2 - v_5, \eta = 0$;
$(c_2)\, v_2 + v_5, \eta = 0$; $(d)\, v_2 + av_3, \eta = 0, a \in \mathbf{R}$; $(e)\, v_1, \eta = 0$; $(f)\, v_3, \eta = 0$.

The proof of the theorem is the same as that of the heat equation [10]. Here is omitted.

5. Group-Invariant and Similarity Solutions

To find invariants s and $m(s)$ determined by the corresponding optimal system S, one needs to solve the following characteristic equations

$$\frac{dx}{\xi} = \frac{dt}{\zeta} = \frac{d\theta}{v}.$$

(a) $v_4 + 2av_3 = x\partial_x + 2t\partial_t + 2a\theta\partial_\theta$.

Solve the corresponding characteristic equations and obtain the invariants $s = x/\sqrt{t}, w = t^{-a}\theta$. The reduced equation of the equation (40) is $w_{ss} - (1/2)sw_s + aw = 0$. Under the transformation $m = w\exp\left(-\frac{1}{8}s^2\right)$, we get the equation $m_{ss} + \left(-\frac{1}{16}s^2 + \left(a + \frac{1}{4}\right)\right)m = 0$. Solve this equation and get the general solutions

$$m\left(s\right) = b_1\left(y\right)U\left(-2a - \frac{1}{2}, \frac{s}{\sqrt{2}}\right) + b_2\left(y\right)V\left(-2a - \frac{1}{2}, \frac{s}{\sqrt{2}}\right),$$

where U and V are the parabolic cylinder functions [11], $b_1(y)$ and $b_2(y)$ are arbitrary functions of y. Thus we obtain the general invariant solutions of the equation (40) as follows.

$$\theta(x,y,t)=t^a\exp\left(\frac{x^2}{8t}\right)\left[b_1(y)U\left(-2a-\frac{1}{2},\frac{x}{\sqrt{2t}}\right)+b_2(y)V\left(-2a-\frac{1}{2},\frac{x}{\sqrt{2t}}\right)\right] \tag{51}$$

By using the relations (51) and $U\left(-n-1/2,x/\sqrt{2t}\right)=\exp\left(-x^2/8t\right)He_n\left(x/\sqrt{2t}\right)$, $V\left(n+1/2,x/\sqrt{2t}\right)=(-i)^n\sqrt{2\pi}\exp\left(x^2/8t\right)He_n\left(ix/\sqrt{2t}\right)$, it follows that the following solutions

$$\theta(x,y,t)=t^{n/2}b_1(y)He_n\left(\frac{x}{\sqrt{2t}}\right),a=\frac{n}{2},b_2(y)=0, \tag{52}$$

$$\theta(x,y,t)=(-i)^n\sqrt{2\pi}t^{-\frac{n+1}{2}}b_2(y)\exp\left(x^2/4t\right)He_n\left(\frac{ix}{\sqrt{2t}}\right),a=-\frac{n+1}{2},b_1(y)=0, \tag{53}$$

where He_n is the n-th Hermite polynomial.

(b) $v_2+v_6+av_3=v_6=4xt\partial_x+(1+4t^2)\partial_t+\left(x^2-2t+a\right)\theta\partial_\theta$.

Solve the corresponding characteristic equations and get the invariants $s=x\left(1+4t^2\right)^{-1/2}$, $w=\left(1+4t^2\right)^{1/4}\theta\exp\left[-\left(1+4t^2\right)^{-1}tx^2-\frac{a}{2}\arctan(2t)\right]$. Further we get the reduced equation $m_{ss}+\left(a+s^2\right)m=0$. The corresponding invariant solutions are

$$\begin{aligned}\theta(x,y,t)=&\left(1+4t^2\right)^{-1/4}\left[b_1(y)W\left(-\frac{a}{2},\frac{x}{\sqrt{8t^2+2}}\right)+b_2(y)W\left(-\frac{a}{2},\frac{-x}{\sqrt{8t^2+2}}\right)\right]\\&\times\exp\left[\frac{tx^2}{4t^2+1}+\frac{a}{2}\arctan(2t)\right],\end{aligned} \tag{54}$$

where W is the parabolic cylinder function [11].

(c_1) $v_2-v_5=-2t\partial_x+\partial_t-x\theta\partial_\theta$.

One can get the invariants $s=x+t^2$, $w=\theta\exp\left(\frac{8}{3}t^3+xt\right)$. It follows from the equation (40) that the Airy′s equation $w_{ss}-sw=0$. Solving the Airy′ equation , we can find the corresponding invariant solutions of Equation (40) are

$$\theta(x,y,t)=\left[b_7(y)Ai\left(x+t^2\right)+b_2(y)Bi\left(x+t^2\right)\right]\exp\left(-xt-\frac{2}{3}t^3\right), \tag{55}$$

where Ai and Bi are the Airy functions [11].

(c_2) $v_4-v_5=-2t\partial_x+\partial_t-x\theta\partial_\theta$.

The corresponding invariants solution θ can be obtained by replacing x by $-x$ in (52).

(d) $v_2+av_3=\partial_t+a\theta\partial_\theta$.

$x, w = \theta \exp(-at)$ are the corresponding invariant. It follows the group invariants solutions

$$\theta(x,y,t) = b_1, \tag{56}$$

$$\theta(x,y,t) = b_1(y)\,x + b_2(y), a = 0, \tag{57}$$

$$\theta(x,y,t) = b_1(y)\exp(at)\cos\left(\sqrt{-a}x + c_7(y)\right), a < 0. \tag{58}$$

For the two subalgebras, one generated by v_1 has only constant for its invariant solutions, and the other generated by v_3 has no invariant solutions. Thus we find the group invariant solutions of the optimal system S determined by Theorem 3. Other invariant solutions of the equation (7) can be obtained by a couple of subgroups not appearing in the optimal system in Theorem 3.

Substituting (16)- (23) and $\phi = P = \theta_x/\theta$ into (3), one can get the corresponding similarity solutions of the BK equations (1) and (2),respectively.

In this paper, we find the one-parameter Lie group with six group constants and the optimal system of one-dimensional subalgebras of the adjoint heat equation. We also find the group invariant solutions of the adjoint heat equation and the similarity solutions of the BK equations. By our results, we also can get the corresponding new solutions of the Kadomtsev-Petviashvili (KP) equation.

6. Conclusions

In this paper,we find the new explicit solutions of the (2+1)-dimensional BK equations (5) and (6), and obtain the successive formula (19) of finding the solutions. When these solutions become seed-solutions, we can find other solutions by using the successive formula (19). The results of this paper denote that there exist a lot of solutions of the BK equations. If the equations (15) and (16) are obtained, then other solutions of the BK equations will be found. We generalize the results in [8].

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