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OBSTACLE PROBLEMS FOR SCALAR GINZBURG-LANDAU EQUATIONS*

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Abstract In this note, we establish some estimates of solutions of the scalar Ginzburg-Landau equation and other nonlinear Laplacian equation $\Delta u = f(x, u)$. This will give an estimate of the Hausdorff dimension for the free boundary of the obstacle problem.
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1. Introduction

Recently there are many interesting results appeared in the study of mathematical theory of super-conductivity. There people considered Dirichlet and Neumann boundary problems. There are a lot of such articles related to bifurcation and stability properties about solutions. People also like to find multiple solutions for complex valued Ginzburg-Landau equations. One interesting problem is the obstacle problem for the scalar Ginzburg-Landau equation. However, the free boundary problems, in particular, obstacle problems, are seldom considered in this theory. Such problem is nature since the Ginzburg-Landau equation has a closed relation with the minimal surface theory. The Obstacle problems for minimal surfaces or for constant mean curvature surfaces have attracted a lot of people. As is well-known, the free boundary problems are very important in science and technology, one may see the article of A. Friedman [1] for more exposition. One such problem for linear elliptic partial differential equations is the obstacle problem, which is considered by many famous mathematicians. In the linear elliptic problem case, L.Caffarelli [2] proved a very beautiful result. In fact, he can show that the solution is $C^{1,1}$ and the free boundary is an n-1 dimensional submanifold. His argument is very delicate. As he pointed out, his method can be used to treat some nonlinear problems. Some of his results and arguments have been extended

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to p-Laplacian problems by K.Lee and H.Shahgholian [3]. One natural question is if a similar result is true for the obstacle problem in the super-conductivity theory.

In this paper, we study the obstacle problem for the scalar Ginzburg-Landau model. Let $D \subset R^n$ be a bounded smooth domain. We are now given a (smooth) bounded function $f(x)$ on ∂D (we assume that f has an extension $f \in C^{2,\mu}(D)$), and a (smooth) function $\phi(x) \in C^{2,\mu}(D)$ with $\varphi(x) < f(x)$ for every $x \in \partial D$. We study the partial differential equation $(GL)_o$:

$$\Delta u + \lambda u(1-u^2) = 0 \quad \text{in } \{u > \varphi\},$$

where $\lambda > 0$ is a (large) constant.

Let $M = |f|_{L^\infty}$. Let $u_0 = inf\{-1, -M\}$ and $u_1 = sup\{1, M\}$. It is clear that u_0 is a sub-solution of $(GL)_o$, and u_1 is a super-solution of $(GL)_o$.

We can get a solution by the direct method. Define K to be the closed convex set

$$K := \{u \in H^1; u_0 \le u \le u_1, u|_{\partial D} = f, u \ge \varphi\}$$

Clearly, since we can extend f on all D such that $f \in K$, K is closed, non-empty convex subset of H^1.

Set

$$J(u) = \int_D |du|^2 + \frac{\lambda}{4}\int_D (u^2-1)^2$$

on K. Then it is easy to see that the infimum is achieved on K. In fact, the minimizer u satisfies the Ginzburg-Landau type equation

$$\Delta u + \lambda u(1-u^2) = 0 \quad \text{in } \{u > \varphi\},$$

where $\lambda > 0$ is a (large) constant. By using a simple comparison argument, it is easy to see that the solution is unique. Let

$$\Omega := \{x; u(x) > \varphi(x)\}$$

Then we meet the question about the regularity of the solution. It is easy to see that since this u is in L^∞, it is $C^{1,\alpha}$, $\alpha \in (0,1)$, (by Theorem 1 in [4]). Furthermore, by adapting the argument in [2], we can show that u is in $C^{1,1}$, and smooth away from the free boundary $S := \{u = \varphi\}$ (see the next section). So the key part is to understand the regularity about the free boundary.

To understand the regularity of this minimizer near the free boundary, without loss of generality, we need only to study the following model problem. In the unit ball of R^n we consider a given function w with the following properties:

(a) $w \ge 0$, $w \in C^{1,1}$;

(b) $\Delta w = g(x)$ in the set $\Omega = \{w > 0\}$;

(c) $0 \in \partial\Omega$, i.e. the point 0 is on the free boundary.

Assume $g(x) = -\Delta\varphi > 0$, where φ is the obstacle function, which is assumed to be given in the class $C^{2,\mu}$. Hence $g \in C^{\mu}$.

Our main result is the following result, which is Theorem 5.3 at the end of of this paper.

Theorem *Given f, φ as above. Let u be the solution of the above. Then $u \in C^{1,1}$. Set $g = -\Delta\varphi - \lambda\varphi(1-\varphi^2)$ and assume that $g(x) > 0, x \in \overline{D}$.*

Let $N_\delta = \{x : d(x, \partial\Omega) \le \delta\}$ be the neighborhood of the free boundary $\partial\Omega$. Then

$$|N_\delta \cap B_r| \le C\delta r^{n-1}.$$

In particular, the free boundary has locally finite $(n-1)$–dimensional Hausdorff measure, and

$$H^{n-1}(\partial\Omega \cap B_r) \le C r^{n-1}.$$

We will follow the method of L.Caffarelli [2] (see also [5]), but we treat a little more difficult case than that considered by L. Caffarelli. In fact, we pay more attention to the results applicable to nonlinear elliptic problems. It is not hard to see that our result can be extended to the generalized G-L model

$$\Delta u + \lambda u(b(x) - u^2) = 0 \quad \text{in } \{u > \varphi\},$$

where $b(x)$ is a positive function on $\overline{D}$. It is our belief that this kind of result is also true for p-Laplacian $(p > 2)$ operator, however, we will not consider this.

2. Regularity of the Solution

Let $F(u) = \dfrac{1}{4}(u^2 - 1)^2$.

In this section, we prove by following an argument in [2] the $C^{1,1}$ regularity of the solution u of the obstacle problem for G-L model $(GL)_o$. We may take $\lambda = 1$.

Take any $\xi \in C_0^1(D)$ and $\xi \ge 0$. Let $\epsilon > 0$. Let

$$u_\epsilon = \min\{u_1, u + \epsilon\xi\}.$$

Then

$$u_\epsilon = u + \epsilon\xi - \xi^\epsilon$$

where

$$\xi^\epsilon = \max\{0, u + \epsilon\xi - u_1\} \ge 0.$$

J is differentiable at u in the direction $u_\epsilon - u$. Then, we find

$$0 \le < J'(u), u_\epsilon - u > .$$

Sending $\epsilon \to 0$, we get $0 \leq < J'(u), \xi >$. This means that we have

$$\Delta u + \lambda u(1-u^2) \leq 0 \quad \text{in } D,$$

in H^1-sense, and

$$\Delta u + \lambda u(1-u^2) = 0 \quad \text{in } \{u > \varphi\}.$$

Let V be the solution of the following problem

$$\Delta V - F'(u) = 0, \quad \text{in } D$$

with the Dirichlet boundary condition

$$V = f, \quad \text{in } \partial D.$$

It is well-known that this V is in $C^{2,\mu}(D)$.

Define $U = u - V$, and let $a = \varphi - V$ on D. Then we have

$$\Delta U \leq 0 \quad \text{in } D$$

and

$$\Delta U = 0 \quad \text{in } \{U > a\},$$

Then we are in the situation treated by L.Caffarelli [2]. So, we have the $C^{1,1}$ regularity for U, and then for u.

From here, we may believe (as in [2]) that the free boundary S is an n-1 dimensional $C^{1,\mu}$ sub-manifold. However, we need time to verify this.

3. Gradient Estimates And Maximum Growth

In this section, in the unit ball of R^n we consider a function w in the unit ball of R^n with the following properties:

(a) $w \geq 0$, $w \in C^{1,1}$;

(b) $\Delta w = g(x)$ in the set $\Omega = \{w > 0\}$;

(c) $0 \in \partial\Omega$, i.e. the point 0 is on the free boundary.

Assume $g(x) = -\Delta\varphi > 0$, where φ is the obstacle function, given in the class $C^{2,\mu}$. Hence $g \in C^{\mu}$. In [2], L. Caffarelli considered the special case where $g = 1$.

We begin with well-known estimates:

Lemma 3.1 *Let $\Delta w = g$ in $B_R(x_0)$, then*

$$|\nabla w(x_0)| \leq \frac{n}{R} \operatorname{osc}_{B_R(x_0)} w + R \sup_{B_R(x_0)} g. \tag{3.1}$$

By the $C^{1,1}$ regularity of w, we have

$$|D^2 w| \leq C. \tag{3.2}$$

For a proof of this result, one may see [6].

Lemma 3.2 (Gradient Bound) *There exists $C > 0$ such that*

$$|\nabla w(x)| \le C\sqrt{w(x)}, \quad x \in \Omega.$$

Proof Let $x_0 \in \Omega$ and $w(x_0) = h > 0$ and $R = c\sqrt{h}$ with $c > 0$ small enough. Then from the estimate (3.2),$B_R(x_0) \subset \Omega$.

First, let v solve the problem

$$\begin{cases} -\Delta v &= g(x) \quad \text{in } B_R(x_0), \\ v &= 0 \quad\quad \text{on } \partial B_R(x_0). \end{cases}$$

Since $0 < g \le C$, we may verify that $0 \le v \le C(R^2 - |x - x_0|^2)$.

Consider the function

$$u(x) = w(x) + v(x),$$

then u is harmonic, and positive in $B_R(x_0)$. By the Harnack inequality

$$\sup_{B_R(x_0)} u \le Cu(x_0),$$

and so

$$\sup w \le \sup u \le Cu(x_0) \le C[w(x_0) + R^2].$$

Hence it follows from (3.1)that

$$|\nabla w(x_0)| \le \frac{C}{R} w(x_0) + CR.$$

To conclude, we note that $w(x_0) = h$ and $R = c\sqrt{h}$, which completes the proof.

Lemma 3.3 (Maximum Growth) *There exists $c > 0$ such that for every $x_0 \in \Omega$ and $R_0 = \operatorname{dist}(x_0, \partial D \cap \Omega)$,*

$$\sup_{B_R(x_0)} w \ge cR^2, \quad 0 < R < R_0.$$

Proof Let $g_0 = \min g$. Then

$$u = w - \frac{g_0}{2n}|x - x_0|^2$$

is subharmonic in $B_R(x_0) \cap \Omega$ and $u(x_0) = w(x_0) > 0$. Thus u reaches a positive maximum on $\partial(B_R(x_0) \cap \Omega)$. But on $B_R(x_0) \cap \partial\Omega$

$$u = -\frac{g_0}{2n}|x - x_0|^2 < 0.$$

Hence, u takes its positive maximum at some $x_1 \in \partial B_R(x_0) \cap \Omega$ (which is nonempty). Namely,

$$u(x_1) = w(x_1) - \frac{g_0}{2n} R^2 > 0.$$

Therefore,

$$\max_{B_R(x_0)} w \ge w(x_1) \ge \frac{g_0}{2n} R^2.$$

Remark 3.4 i) From the proof we may give the constants explicitly:

$$c = \frac{\min g}{2n}.$$

ii) The same results are also true for a function w in the unit ball B_1 of R^n with the following properties:

(a) $w \geq 0$, $w \in C^{1,1}$;

(b) $\Delta w(x) - w(x) = g(x)$ in the set $\Omega = \{w > 0\}$;

(c) $0 \in \partial\Omega$, i.e. the point 0 is on the free boundary.

Here we assume $g(x) > 0$ and $g \in C^{\alpha}(B_1)$. One can see [5] for more comments.

4. Nonlinear Equations

By using the standard comparison argument we have

$$\min\{-1, \min_{\Gamma} u\} \leq u \leq \max\{1, \max_{\Gamma} u\},$$

where Γ is the boundary of the domain D. The function $w = u - \varphi$ satisfies

$$\Delta w = -\lambda u(1 - u^2) - \Delta\varphi(x) \quad \text{in } \{w > 0\}. \tag{4.1}$$

and a similar gradient estimate is true:

$$|\nabla w(x)| \leq (\lambda C_1 + C_2)\sqrt{w(x)}, \quad x \in \Omega.$$

The maximum growth estimates for the solution $w = u - \varphi > 0$ is also true near the free boundary $\{w = 0\}$ provided the right side of the equation (4.1) is positive:

Assumption $g = -\Delta\varphi - \lambda\varphi(1 - \varphi^2) > 0$ over the closure of D.

Remark This condition is nature. In fact, if $-\Delta\varphi > 0$, and $-1 \leq \varphi < 0$, then

$$g = -\Delta\varphi - \lambda\varphi(1 - \varphi^2) > 0$$

for all $\lambda > 0$.

Under this assumption we may write (4.1) as

$$\Delta w = g(x) + \lambda[\varphi(1 - \varphi^2) - u(1 - u^2)],$$

and then

$$G(x, \varphi, u) = g(x) + \lambda[\varphi(1 - \varphi^2) - u(1 - u^2)] \geq \frac{1}{2}g(x) > 0$$

if $w = u - \varphi > 0$ is small enough.

More general nonlinearities Consider the bounded solution of the following equation

$$\Delta u = f(u, x) \quad \text{in} \quad \{u > \varphi\},$$

where f is a proper function, and we impose the condition that $\Delta\varphi < f(\varphi, x)$. Then we get for $w = u - \varphi$ the equation

$$\Delta w = f(u, x) - \Delta\varphi > f(u, x) - f(\varphi, x) \quad \text{in} \quad \{w > 0\}.$$

Note that on the free boundary $\{w = 0\}$ we have

$$\Delta w > f(u, x) - f(\varphi, x) = 0.$$

Hence the gradient bound and the maximum growth are true for this problem.

Remark 4.1 Consider the equations of type

$$\Delta u = f(Du, u, x) \quad \text{in } \{u > \varphi\}.$$

There are some difficulties, even we use $f(|Du|, u, x)$ to replace $f(Du, u, x)$. At this moment, we leave this open.

5. Estimates for Free Boundaries

We derive the free boundary estimates by following L.Caffarelli's idea [5].

Lemma 5.1 ("Gradient Strip" Estimates) *Let $w_e = D_e w$ be the directional derivative of w in the direction e. Assume that $g \in W^{1,n}$. Then there exists $C > 0$ such that*

$$\int_{\{0 \le w_e \le h\} \cap B_r} |\nabla w_e|^2 \le Chr^{n-1}.$$

Proof Truncate w_e at levels ϵ and h: $\overline{w}_e = \min[(w_e - \epsilon)^+, h]$. Then from Green formula and $\Delta w_e = D_e g$ it follows that

$$\int_{B_r} \nabla\overline{w}_e \cdot \nabla w_e dx = -\int_{B_r} \overline{w}_e D_e g dx + \int_{\partial B_r} \overline{w}_e D_\nu w_e ds.$$

Since $\epsilon \le \overline{w}_e \le h$, $|D_\nu w| \le |D^2 w| \le C$, $D_e g \in L^n(B_r)$, we have

$$\int_{\{\epsilon \le w_e \le h\} \cap B_r} |\nabla w_e|^2 dx \le h \int_{B_r} |D_e g| dx + h \int_{\partial B_r} |D_\nu w_e| ds \le Chr^{n-1},$$

which leads to the conclusion by letting $\epsilon \to 0$.

Lemma 5.2 ("Level Strip" Estimates) *Let $S_h = \{0 < w < h^2\}$. Then there exists $C > 0$ such that*

$$|S_h \cap B_r| \le Chr^{n-1}.$$

Proof It follows from the Gradient Bound that

$$S_h \subset \{|\nabla w| \le h\} \subset \bigcup_{i=1}^{n} \{|w_{e_i}| \le h\}.$$

Thus,

$$\begin{aligned} |S_h \cap B_r| &\le \frac{1}{\min g^2} \int_{S_h \cap B_r} (\Delta w)^2 dx \le C \int_{S_h \cap B_r} |D^2 w|^2 dx \\ &\le C \sum_{i,j=1}^{n} \int_{\{|w_{e_i}| \le h\}} |\nabla w_{e_j}|^2 dx \le C h r^{n-1}. \end{aligned}$$

Theorem 5.3 *Let $N_\delta = \{x : d(x, \partial\Omega) \le \delta\}$ be the neighborhood of the free boundary $\partial\Omega$. Then*

$$|N_\delta \cap B_r| \le C\delta r^{n-1}.$$

In particular, the free boundary has locally finite $(n-1)$–dimensional Hausdorff measure, and

$$H^{n-1}(\partial\Omega \cap B_r) \le C r^{n-1}.$$

Proof From $|D^2 w| \le C$ it follows that

$$0 < w(x) < C\delta^2, \quad x \in N_\delta \cap \Omega,$$

that is, $(N_\delta \cap \Omega) \subset S_\delta$. Thus

$$|N_\delta \cap \Omega \cap B_r| \le C\delta r^{n-1}.$$

Then we get a proof of the result by the definition of Hausdorff measure.

We remark that with a little more effort, we can follow [2] to prove that the free boundary $S := \{u = \varphi\} = \partial\Omega$ is an $n-1$ dimensional $C^{1,1}$ sub-manifold in D. This may be a very useful fact for further study related to scalar G-L models.

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