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EXISTENCE AND NON-EXISTENCE OF GLOBAL SOLUTIONS OF A DEGENERATE PARABOLIC SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS*

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Abstract In this paper, we study the non-negative solutions to a degenerate parabolic system with nonlinear boundary conditions in the multi-dimensional case. By the upper and lower solutions method, we give the conditions on the existence and non-existence of global solutions.
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1. Introduction and Main Results

Let constants $m > 1$ and p, $q > 0$, and let $R_+^N = \{(x_1, x') \,|\, x_1 > 0,\, x' \in R^{N-1}\}$. In this paper we study the non-negative solutions to the following degenerate parabolic system with nonlinear boundary conditions in half space

$$\begin{cases} u_t = \Delta u^m, \quad v_t = \Delta v^m, & x \in R_+^N, \quad t > 0, \\ -\dfrac{\partial u^m}{\partial x_1} = v^p, \quad -\dfrac{\partial v^m}{\partial x_1} = u^q, & x_1 = 0, \quad t > 0. \end{cases} \tag{1}$$

and initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in R_+^N, \tag{2}$$

where the initial data $u_0(x)$ and $v_0(x)$ are non-negative C^1 functions and satisfy the compatibility conditions

$$-\frac{\partial u_0^m}{\partial x_1} = v_0^p, \quad -\frac{\partial v_0^m}{\partial x_1} = u_0^q, \qquad x_1 = 0.$$

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Moreover, they are compactly supported in R_+^N, and if they are nontrivial, then we assume that they satisfy $u_0(0) > 0$, $v_0(0) > 0$.

Since the pioneering work of Fujita in the 1960's, much work on the global existence and blow-up to the nonlinear parabolic problems has been done, see [1–5] and the references therein. The main aim of this paper is to discuss the global existence and finite time blow-up of solution to the problem (**??**) by constructing self-similar upper solution that exists globally and lower solution that blows up in finite time. This method has been used by many authors, see [**?**, **?**, **?**, **?**, **?**] and the references therein.

For the scalar equation

$$\begin{cases} u_t = \Delta u^m, & x \in R_+^N, \quad t > 0, \\ -\dfrac{\partial u^m}{\partial x_1} = u^p, & x_1 = 0, \quad t > 0, \\ u(x,0) = u_0(x), & x \in R_+^N, \end{cases} \tag{3}$$

where $u_0(x)$ has the similar properties to the functions of (**??**). Huang et al [**?**] obtained

(i) If $p \le p_0 = (m+1)/2$, then all the solutions of the problem (**??**) are global;

(ii) If $p_0 < p < p_c = m + 1/N$, then all the nontrivial solutions of the problem (**??**) blow up in finite time;

(iii) If $p > p_c$, then the solution of the problem (**??**) exists globally for the small initial data u_0, while blows up in finite time for the large initial data u_0.

In the paper [**?**], Quiros and Rossi studied the Fujita type curves of the following problem on the half-line

$$\begin{cases} u_t = (u^m)_{yy}, \quad v_t = (v^n)_{yy}, & y > 0, \quad t > 0, \\ -(u^m)_y(0,t) = v^p(0,t), & t > 0, \\ -(v^n)_y(0,t) = u^q(0,t), & t > 0, \end{cases} \tag{4}$$

with m, $n > 1$ and p, $q > 0$.

Definition 1 *A pair of functions* (u, v) *is called an upper solution* (*lower solution*) *of* (**??**) *if it satisfies*

$$\begin{cases} u_t \ge (\le)\, \Delta u^m, \quad v_t \ge (\le)\, \Delta v^m, & x \in R_+^N, \quad t > 0, \\ -\dfrac{\partial u^m}{\partial x_1} \ge (\le)\, v^p, \quad -\dfrac{\partial v^m}{\partial x_1} \ge (\le)\, u^q, & x_1 = 0, \quad t > 0. \end{cases}$$

Proposition 1 *Let* $(\bar{u}, \bar{v})$ *and* $(\underline{u}, \underline{v})$ *be the upper and lower solutions of* (**??**) *respectively. If there exists a number* $t_0 \ge 0$ *such that*

$$\begin{cases} \underline{u}(x,t_0) \le \bar{u}(x,t_0), \quad \underline{v}(x,t_0) \le \bar{v}(x,t_0), \quad x \in R_+^N, \\ \underline{u}(0,t_0) < \bar{u}(0,t_0), \quad \underline{v}(0,t_0) < \bar{v}(0,t_0), \end{cases}$$

then

$$\underline{u}(x,t) \le \bar{u}(x,t), \quad \underline{v}(x,t) \le \bar{v}(x,t),$$

as long as both pairs of functions exist.

Proof The proof is analogous to that of the paper [?], we omit the details here.

Before giving the main results, we introduce the following numbers that are useful in the later.

$$\begin{cases} \alpha_1 = \dfrac{1+m+2p}{4pq-(1+m)^2}, \quad \alpha_2 = \dfrac{1+m+2q}{4pq-(1+m)^2}, \\ \beta_i = \{1+(1-m)\alpha_i\}/2, \quad i=1,\,2, \\ p_0 = (m+1)/2, \quad p_c = m+1/N. \end{cases} \tag{5}$$

Our main results read as follows.

Theorem 1 *If $pq \le p_0^2$, then the solution (u,v) of* (**??**) and (**??**) *exists globally.*

Theorem 2 *Assume that $p_0^2 < pq \le p_c^2$, and $p \ne q$ when $pq = p_c^2$. Then every nontrivial solution (u,v) of* (**??**) and (**??**) *blows up in finite time.*

Theorem 3 *Assume that $pq > p_c^2$. Then we have the following results.*

(i) *If $\alpha_1 > N\beta_1$ or $\alpha_2 > N\beta_2$, then every nontrivial solution (u,v) of* (**??**) and (**??**) *blows up in finite time;*

(ii) *If $\alpha_1 \le N\beta_1$ and $\alpha_2 \le N\beta_2$, then the solution (u,v) of* (**??**) and (**??**) *exists globally for the small initial data $(u_0,\,v_0)$, while blows up in finite time for the large initial data $(u_0,\,v_0)$.*

2. Proof of Theorem ??

We first give a lemma which was proved in the paper [?].

Lemma 1 *Assume that $pq = (m+1)(n+1)/4$. Then for any constant $\gamma_1 > 0$, the problem* (**??**) *has a self-similar solution which is global and has the form*

$$u(y,t) = e^{\gamma_1 t} f(ye^{-\lambda_1 t}), \quad v(y,t) = e^{\gamma_2 t} g(ye^{-\lambda_2 t}),$$

where f and g are non-negative functions with compact supports and satisfy $f(0)$, $g(0) > 0$,

$$\gamma_2 = \frac{m+1}{2p}\gamma_1, \quad \lambda_1 = \frac{m-1}{2}\gamma_1, \quad \lambda_2 = \frac{(n-1)(m+1)}{4p}\gamma_1. \tag{6}$$

We divide the proof of Theorem **??** into two lemmas.

Lemma 2 *If $p,\,q < p_0$, then the solution $(u,\,v)$ of* (**??**) *and* (**??**) *exists globally.*

Proof This lemma can be proved by the method that will be used in Lemma **??** below. Here, we will give a direct proof. Let

$$\bar u(x,t) = (T+t)^{-\alpha_1} f(\zeta_1), \quad \zeta_1 = \frac{x_1}{(T+t)^{\beta_1}},$$
$$\bar v(x,t) = (T+t)^{-\alpha_2} g(\xi_1), \quad \xi_1 = \frac{x_1}{(T+t)^{\beta_2}},$$

where the constants α_i, β_i $(i = 1, 2)$ are given by (**??**), and the functions f and g will be determined later. Since $pq < (m+1)^2/4$ and $m > 1$, we have that $\alpha_i < 0 < \beta_i$. To prove that $(\bar{u}, \bar{v})$ is an upper solution of (**??**), it suffices to verify that

$$(f^m)'' + \beta_1\zeta_1 f' + \alpha_1 f \le 0, \qquad \zeta_1 > 0, \tag{7}$$

$$(g^m)'' + \beta_2\xi_1 g' + \alpha_2 g \le 0, \qquad \xi_1 > 0, \tag{8}$$

$$-\frac{df^m(0)}{d\zeta_1} \ge g^p(0), \qquad -\frac{dg^m(0)}{d\xi_1} \ge f^q(0). \tag{9}$$

Set $f(\zeta_1) = Ae^{-\sigma_1\zeta_1}$ and $g(\xi_1) = Ae^{-\sigma_2\xi_1}$, where A and $\sigma_i\,(i = 1, 2)$ are positive constants will be specified. Then (**??**), (**??**), (**??**) become

$$m^2\sigma_1^2 A^{m-1} + \alpha_1 e^{(m-1)\sigma_1\zeta_1} - \beta_1\sigma_1 e^{(m-1)\sigma_1\zeta_1}\zeta_1 \le 0, \quad \zeta_1 > 0, \tag{10}$$

$$m^2\sigma_2^2 A^{m-1} + \alpha_2 e^{(m-1)\sigma_2\xi_1} - \beta_2\sigma_2 e^{(m-1)\sigma_2\xi_1}\xi_1 \le 0, \quad \xi_1 > 0, \tag{11}$$

$$m\sigma_1 A^m \ge A^p, \qquad m\sigma_2 A^m \ge A^q \tag{12}$$

respectively. By putting $\sigma_1 = A^{(p-m)}/m$ and $\sigma_2 = A^{(q-m)}/m$, it is easy to see that (**??**) are in fact two equalities. The assumption $p, q < p_0$ implies that

$$\lim_{A\to\infty} \sigma_i^2 A^{m-1} = 0, \quad i = 1, 2.$$

Recalling that $\alpha_i < 0 < \beta_i$, we may choose A large enough such that (**??**) and (**??**) hold. The above arguments show that $(\bar{u}, \bar{v})$ is an upper solution of (**??**).

Furthermore, since $u_0(x)$ and $v_0(x)$ have compact supports, we may choose T sufficiently large such that

$$\begin{cases} \bar{u}(x, 0) \ge u_0(x), \quad \bar{v}(x, 0) \ge v_0(x), \quad x \in R_+^N, \\ \bar{u}(0, 0) > u_0(0), \quad \bar{v}(0, 0) > v_0(0). \end{cases}$$

Applying Proposition **??**, we get that the solution (u, v) exists globally.

Lemma 3 *Assume that $pq \le p_0^2$, and $p \ge p_0$ or $q \ge p_0$, then every solution (u, v) of* (**??**) and (**??**) *is global.*

Proof We discuss only the case of $q \ge p_0$. As $pq \le p_0^2$, $p \le p_0$ must hold. Choose $\tilde{p} \ge p$ such that $\tilde{p}q = p_0^2 = (m+1)^2/4$. For any $T > 0$, according to Lemma **??**, there exists a global solution $(u^*(y, t), v^*(y, t))$ of (**??**) which has the following form

$$u^*(y, t) = e^{\gamma_1(t+T)} f(ye^{-\lambda_1(t+T)}), \qquad v^*(y, t) = e^{\gamma_2(t+T)} g(ye^{-\lambda_2(t+T)}),$$

where the constant $\gamma_1 > 0$ is arbitrary, and the constants γ_2, $\lambda_i\,(i = 1, 2)$ are determined by (**??**) with the number p being replaced by $\tilde{p}$ and $m = n$. Set

$$\bar{u}(x, t) = u^*(x_1, t), \qquad \bar{v}(x, t) = v^*(x_1, t).$$

Observe that $(\bar{u},\ \bar{v})$ is an upper solution of (**??**) as long as $v^*(0,t) \geq 1$. Since $f(0),\ g(0) > 0$, and $u_0(x)$ and $v_0(x)$ have compact supports, we may choose T large enough such that $v^*(0,t) \geq 1$ and

$$u_0(x) \leq e^{\gamma_1 T} f(x_1 e^{-\lambda_1 T}) = \bar{u}(x,0), \quad v_0(x) \leq e^{\gamma_2 T} g(x_1 e^{-\lambda_2 T}) = \bar{v}(x,0),$$
$$u_0(0) < e^{\gamma_1 T} f(0) = \bar{u}(0,0), \qquad v_0(0) < e^{\gamma_2 T} g(0) = \bar{v}(0,0).$$

By propsition1, $(u,v) \leq (\bar{u},\bar{v})$. Hence, the solution (u,v) of (**??**) and (**??**) exists globally.

Theorem **??** follows from Lemmas **??** and **??**.

3. Proof of Theorem ??

To begin with, we give the following preliminary lemma.

Lemma 4 *Assume that* $pq > p_0^2$, *then the solution* $(u,\ v)$ *of* (**??**) *and* (**??**) *blows up in finite time for the large initial data* $(u_0,\ v_0)$.

Proof Let α_i and $\beta_i (i=1,\ 2)$ be given by (**??**). As $pq > p_0^2$, it holds $\alpha_i > 0$. For $T > 0$, we construct

$$\underline{u}(x,t) = (T-t)^{-\alpha_1} f(r), \qquad r = |\zeta|, \qquad \zeta = \frac{x}{(T-t)^{\beta_1}},$$
$$\underline{v}(x,t) = (T-t)^{-\alpha_2} g(s), \qquad s = |\xi|, \qquad \xi = \frac{x}{(T-t)^{\beta_2}},$$

where $f(r)$ and $g(s)$ are non-negative functions. To prove that $(\underline{u},\ \underline{v})$ is a lower solution of (**??**), it is sufficient to show that in the domain where $f(r) > 0$ and $g(s) > 0$, the following hold

$$m(m-1) f^{m-2} (f')^2 + m f^{m-1} f'' + \frac{m(N-1)}{r} f^{m-1} f' - \alpha_1 f - \beta_1 r f' \geq 0, \qquad (13)$$
$$m(m-1) g^{m-2} (g')^2 + m g^{m-1} g'' + \frac{m(N-1)}{s} g^{m-1} g' - \alpha_2 g - \beta_2 s g' \geq 0, \qquad (14)$$
$$-\frac{\partial f^m}{\partial \zeta_1} \leq g^p \ \text{ at } \ \zeta_1 = 0, \qquad -\frac{\partial g^m}{\partial \xi_1} \leq f^q \ \text{ at } \ \xi_1 = 0. \qquad (15)$$

Set

$$f(r) = A(b-r)_+^{\frac{1}{m-1}} (r-a)_+^{\frac{1}{m-1}}, \quad g(s) = A(b-s)_+^{\frac{1}{m-1}} (s-a)_+^{\frac{1}{m-1}},$$

where A, a and b are positive constants with $a < b$. From the structure of r and s, it is obvious that (**??**) is valid. By a series of computations it is easy to see that if we can prove the following inequalities (**??**), then (**??**) and (**??**) also hold.

$$\tilde{d}_1 z^3 + \tilde{d}_{2i} z^2 + \tilde{d}_{3i} z + \tilde{d}_4 \geq 0, \quad i = 1,2, \quad 0 < a < z < b, \qquad (16)$$

where

$$\tilde{d}_1 = \Big(\frac{2m+2m^2}{(m-1)^2} + \frac{2m(N-1)}{m-1}\Big)A^{m-1} + \frac{1}{m-1},$$
$$\tilde{d}_{2i} = -\Big(\frac{2m+2m^2}{(m-1)^2} + \frac{3m(N-1)}{m-1}\Big)(a+b)A^{m-1} - \frac{1}{2}(\alpha_i + \frac{1}{m-1})(a+b), \quad i=1,2,$$
$$\tilde{d}_{3i} = \Big(\frac{m}{(m-1)^2}(a^2+b^2) + \frac{2m^2}{(m-1)^2}ab + \frac{m(N-1)}{m-1}(a^2+b^2+4ab)\Big)A^{m-1} + \alpha_i ab, \quad i=1,2,$$
$$\tilde{d}_4 = -\frac{m(N-1)}{m-1}ab(a+b)A^{m-1}.$$

Here we have used

$$\alpha_i + 2\beta_i/(m-1) = 1/(m-1), \quad \alpha_i + \beta_i/(m-1) = (1/(m-1)+\alpha_i)/2, \ i=1,2.$$

Taking A large enough we see that, for proving (**??**) it is sufficient to prove

$$H(z) = d_1z^3 + d_2z^2 + d_3z + d_4 > 0, \quad 0 < a < z < b, \tag{17}$$

where

$$d_1 = \frac{2+2m}{m-1} + 2(N-1),$$
$$d_2 = -\Big(\frac{2+2m}{m-1} + 3(N-1)\Big)(a+b),$$
$$d_3 = \frac{1}{m-1}(a^2+b^2) + \frac{2m}{m-1}ab + (N-1)(a^2+b^2+4ab),$$
$$d_4 = -(N-1)ab(a+b).$$

Let $b = ca,\ c > 1$. We claim that there exists a constant $\delta > 0$ such that $H(z) \geq \delta$ for $z \in [a,b]$ as c is close to 1. In fact, we have $\lim_{z\to\infty} H(z) = \infty$ since $d_1 > 0$. On the other hand, it is easy to know

$$H(a) = \frac{1}{m-1}(c-1)^2a^3 > 0, \quad H(b) = \frac{1}{m-1}(1-1/c)^2b^3 > 0.$$

If there exists a number $z_0 \in (a,b)$ such that $H(z_0) = 0$, then by Lemma 2.1 in the paper [**?**], $H(z) = 0$ has one real root and two conjugate complex roots as c is close to 1. From the above arguments we easily obtain $H(z_0) = \min_{z\in[a,b]} H(z)$, which implies that z_0 is at least a double-multiplicity real root, a contradiction. Therefore, (**??**) holds, and in turn (**??**) holds by taking A sufficiently large. This shows that $(\underline{u},\ \underline{v})$ is a lower solution of (**??**). If the initial datum $(u_0,\ v_0)$ is large enough such that $u_0(x) \geq \underline{u}(x,0)$, $v_0(x) \geq \underline{v}(x,0)$ in R^N_+, and $u_0(0) > \underline{u}(0,0)$, $v_0(0) > \underline{v}(0,0)$, then Proposition **??** asserts our conclusion.

Proof of Theorem ??

Let $\theta(s) = B(\sigma - s^2)_+^{1/(m-1)}$, where $\sigma > 0$ and $B = [\frac{m-1}{2m(N(m-1)+2)}]^{1/(m-1)}$. For any $\tau > 0$, the function

$$w(x,\,t) = (\tau+t)^{-kN}\theta(s), \quad s = |\zeta|, \quad \zeta = \frac{x}{(\tau+t)^k},$$

with $k = 1/\{N(m-1)+2\}$, is the so-called Zel'dovich-Kompanetz-Barenblatt (ZKB) solution of the porous medium equation (see [?, ?, ?]). As $\frac{\partial w^m}{\partial x_1}\Big|_{x_1=0} = 0$ and $w(x,\, t) \geq 0$, we see that $(\bar{u},\, \bar{v}) = (w,\, w)$ is a lower solution of (**??**).

From the assumption of u_0 and v_0, we claim that there is a $t_0 > 0$ such that $u(0,\, t_0) > 0$ and $v(0,\, t_0) > 0$. Since $u(x,\, t_0)$ and $v(x,\, t_0)$ are continuous functions, we may first find a $\tau > 0$ large enough and then determine a $\sigma > 0$ sufficiently small so that

$$u(0,\, t_0),\; v(0,\, t_0) > w(0, t_0); \quad u(x,\, t_0),\; v(x,\, t_0) \geq w(x, t_0), \quad x \in R^N_+.$$

By Proposition 1,

$$u(x,\, t),\; v(x,\, t) \geq w(x,\, t), \quad t \geq t_0, \quad x \in R^N_+. \tag{18}$$

We now prove there exists a $t_* \geq t_0$ and T large enough such that

$$u(x,\, t_*) \geq \underline{u}(x,\, 0), \quad \text{or} \quad v(x,\, t_*) \geq \underline{v}(x, 0), \quad x \in R^N_+, \tag{19}$$

where the pair of functions $(\underline{u},\, \underline{v})$ are determined in the proof of Lemma **??**. Indeed, resulting from the assumption of p and q, we can easily prove that at least one of the following two inequalities holds

$$2pq + (1-m)p - 2p/N < (m+1)(m+1/N), \tag{20}$$

$$2pq + (1-m)q - 2q/N < (m+1)(m+1/N). \tag{21}$$

Without loss of generality we assume (**??**) holds, which is equivalent to $N\beta_1 < \alpha_1$. It follows that there exist $t_* \geq t_0$ and $T \gg 1$ such that

$$(\tau + t_*)^{-kN} \gg T^{-\alpha_1}, \quad (\tau + t_*)^k \gg T^{\beta_1}. \tag{22}$$

Therefore, (**??**) holds. As (**??**) holds for any nontrivial initial data, thus every nontrivial solution (u, v) of (**??**) and (**??**) blows up in finite time.

4. Proof of Theorem ??

Firstly, from the assumption of (i), we have (**??**) or (**??**) holds, so the proof of (i) is the same as that of Theorem **??**. Secondly, the second claim of (ii) follows from Lemma 4. Therefore, the remain work is to show the first claim of (ii).

To this end, we consider our problem only on the interval $[0, T]$ for arbitrary $T > 0$. We claim $\beta_1 \neq \beta_2$. In fact, if $\beta_1 = \beta_2$, then from (**??**), we obtain $\alpha_1 = \alpha_2$. Therefore, both $\alpha_1 = N\beta_1,\ \alpha_2 < N\beta_2$ and $\alpha_1 < N\beta_1,\ \alpha_2 = N\beta_1$ can not hold, the unique possibility is that $\alpha_1 = N\beta_1,\ \alpha_2 = N\beta_2$ holds. But from this we get $p = q = p_c$, which contradicts with $pq > p_c^2$.

We may assume that $\beta_1 < \beta_2$. Choose a positive constant $d \geq 1/[(pq/m^2)^{1/(\beta_2-\beta_1)} - 1]$, and set

$$\bar{u}(x,t) = (dT+t)^{-\alpha_1} f(\zeta_1), \qquad \zeta_1 = \frac{x_1 + b}{(dT+t)^{\beta_1}},$$
$$\bar{v}(x,t) = (dT+t)^{-\alpha_2} g(\xi_1), \qquad \xi_1 = \frac{x_1 + b}{(dT+t)^{\beta_2}},$$

where α_i, β_i are given by (**??**), the nonnegative functions $f,\ g$ and constant $b > 0$ will be determined later. By the assumption of $pq > p_c^2$, we get $\beta_i \geq \alpha_i/N > 0,\ i = 1,2$. To prove that $(\bar{u},\ \bar{v})$ is an upper solution of (**??**), we need to verify

$$(f^m)'' + \beta_1\zeta_1 f' + \alpha_1 f \leq 0, \qquad \zeta_1 > b_1, \tag{23}$$
$$(g^m)'' + \beta_2\xi_1 g' + \alpha_2 g \leq 0, \qquad \xi_1 > b_2, \tag{24}$$
$$-\frac{df^m(b_1)}{d\zeta_1} \geq g^p(b_2), \qquad -\frac{dg^m(b_2)}{d\xi_1} \geq f^q(b_1), \tag{25}$$

where $b_i = b/(dT+t)^{\beta_i} (i = 1,2)$. Choose $f(\zeta_1) = A_1 e^{-\sigma_1\zeta_1},\ g(\xi_1) = A_2 e^{-\sigma_2\xi_1}$ with $A_i,\ \sigma_i$ being positive constants to be fixed. (**??**), (**??**), (**??**) become

$$m^2\sigma_1^2 A_1^{m-1} + \alpha_1 e^{(m-1)\sigma_1\zeta_1} - \beta_1\sigma_1 e^{(m-1)\sigma_1\zeta_1}\zeta_1 \leq 0,\ \ \zeta_1 > b_1, \tag{26}$$
$$m^2\sigma_2^2 A_2^{m-1} + \alpha_2 e^{(m-1)\sigma_2\xi_1} - \beta_2\sigma_2 e^{(m-1)\sigma_2\xi_1}\xi_1 \leq 0,\ \ \xi_1 > b_2, \tag{27}$$
$$m\sigma_1 A_1^m \geq A_2^p e^{(m\sigma_1 b_1 - p\sigma_2 b_2)}, \qquad m\sigma_2 A_2^m \geq A_1^q e^{(m\sigma_2 b_2 - q\sigma_1 b_1)} \tag{28}$$

respectively. Since $\alpha_i \leq N\beta_i,\ \zeta_1 > b_1 \geq b/(dT+T)^{\beta_1}$ and $\xi_1 > b_2 \geq b/(dT+T)^{\beta_2}$, to prove (**??**) and (**??**), it is enough to show that

$$m^2\sigma_1^2 A_1^{m-1} + (N - \sigma_1 b/(dT+T)^{\beta_1})\beta_1 e^{(m-1)\sigma_1\zeta_1} \leq 0,\ \ \zeta_1 > b_1, \tag{29}$$
$$m^2\sigma_2^2 A_2^{m-1} + (N - \sigma_2 b/(dT+T)^{\beta_2})\beta_2 e^{(m-1)\sigma_2\xi_1} \leq 0,\ \ \xi_1 > b_2. \tag{30}$$

For any fixed constants σ_1 and σ_2, we let

$$b > N \max\left\{(d+1)^{\beta_i} T^{\beta_i}/\sigma_i,\ \ i = 1,2\right\},$$

and take A_1, A_2 small enough. It is easy to check that (**??**) and (**??**) hold. Furthermore, we demand $A_1 = A_2^{p/m}$. Due to $pq > p_c^2 > m^2$, we have $p/m > m/q$. Then, by taking A_1, A_2 sufficiently small, we obtain $A_1 = A_2^{p/m} < A_2^{m/q}$, i.e., $A_1^q < A_2^m$.

Consider the function

$$G(t) = (\sigma_1/\sigma_2)(dT+t)^{(\beta_2-\beta_1)}, \quad t \in [0,T].$$

It easily verifies that $G(t)$ increases in $[0,T]$. We choose $\sigma_1 \geq 1/m$, $\sigma_2 \geq 1/m$ such that

$$\frac{\sigma_1}{\sigma_2} = \frac{p}{m}[(d+1)T]^{\beta_1-\beta_2}.$$

Then

$$\begin{aligned} G(T) &= \frac{p}{m}[(d+1)T]^{\beta_1-\beta_2}(dT+T)^{\beta_2-\beta_1} = \frac{p}{m}, \\ G(0) &= \frac{p}{m}(d+1)^{\beta_1-\beta_2}T^{\beta_1-\beta_2}(dT)^{\beta_2-\beta_1} = \frac{p}{m}\Big(\frac{d}{1+d}\Big)^{\beta_2-\beta_1} \\ &\geq \Big(\frac{1/((pq/m^2)^{1/(\beta_2-\beta_1)}-1)}{1+1/((pq/m^2)^{1/(\beta_2-\beta_1)}-1)}\Big)^{\beta_2-\beta_1}\frac{p}{m} = \frac{m^2}{pq}\times\frac{p}{m} = \frac{m}{q}, \end{aligned}$$

which implies that $m/q \leq G(t) \leq p/m, \ \forall\, t \in [0,T]$. Thus

$$m\sigma_1 b_1(t) - p\sigma_2 b_2(t) \leq 0, \quad m\sigma_2 b_2(t) - q\sigma_1 b_1(t) \leq 0, \quad \forall\, t \in [0,T].$$

These two inequalities and the choices of A_i, σ_i assure that (**??**) holds. The above arguments show that $(\bar{u},\bar{v})$ is an upper solution of the problem (**??**). Moreover, we may choose $u_0(x), v_0(x)$ small enough so that

$$\begin{aligned} u_0(x) \leq \bar{u}(x,0), \quad v_0(x) \leq \bar{v}(x,0), \\ u_0(0) < \bar{u}(0,0), \quad v_0(0) < \bar{v}(0,0). \end{aligned}$$

By proposition **??**, we get the conclusion.

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