SOME NONLINEAR ELLIPTIC EQUATIONS HAVE ONLY CONSTANT SOLUTIONS*

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Dedicated to K. C. Chang with high esteem and warm friendship
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Abstract We study some nonlinear elliptic equations on compact Riemannian manifolds. Our main concern is to find conditions which imply that such equations admit only constant solutions.

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1. Introduction

Motivated by some recent results and questions raised in [1], we study some nonlinear elliptic equations of the form

\[ \begin{cases}
-\Delta_g u = f(u) & \text{on } M, \\
u > 0 & \text{on } M,
\end{cases} \]

where \((M, g)\) is a compact Riemannian manifold of dimension \(n \geq 2\), without boundary, and \(f : (0, +\infty) \to \mathbb{R}\) is a smooth function. Our main concern is to find conditions on \(M\) and \(f\) which imply that (1.1) admits only constant solutions.

We will present results in two directions:

1) The case where \(M = S^n, n \geq 3\), equipped with its standard metric \(g_0\)

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In this case our first result is

**Theorem 1** Assume that \((M, g) = (S^n, g_0), n \geq 3, \) and
\[
h(t) := t^{-\frac{n+2}{n-2}} \left( f(t) + \frac{n(n-2)}{4} t \right)
\]
is decreasing on \((0, \infty). \) \hfill (1.2)

Then any solution of (1.1) is constant.

A typical example is the case
\[
f(t) = t^{p} - \lambda t, \quad p > 1, \lambda > 0,
\]
so that (1.1) becomes
\[
\begin{cases}
-\Delta_g u = u^p - \lambda u & \text{on } S^n, \\
u > 0 & \text{on } S^n.
\end{cases}
\]

**Corollary 1** Assume that \( p \leq \frac{(n+2)/(n-2)}{n-2(2)} \) and \( \lambda \leq n(n-2)/4, \) and at least one of these inequalities is strict. Then the only solution of (1.4) is the constant \( u = \lambda^{1/(p-1)}. \)

In fact, Corollary 1 is originally due to Gidas-Spruck [2]. But our argument is quite different from theirs; they rely on some remarkable identities while our method uses moving planes.

When \( p = \frac{(n+2)/(n-2)}{n-2} \) the conclusion of Corollary 1 is sharp. Indeed if \( \lambda = n(n-2)/4 \) there is a well-known family of nonconstant solutions; moreover all solutions of (1.4) belong to this family. However when \( p < \frac{(n+2)/(n-2)}{n-2} \), B. Gidas and J. Spruck established a better result which was later sharpened by M.F. Bidaut-Veron and L. Veron. Namely they proved

**Theorem 2** Assume that \( p < \frac{(n+2)/(n-2)}{n-2}/\) and \( \lambda > n/(p-1). \) Then the only solution of (1.4) is the constant \( u = \lambda^{1/(p-1)}. \)

**Remark 1** The proof of Theorem 2 in [2] and [3] is based on some remarkable identities. Our proof of Theorem 1 uses the method of moving planes. It would be very interesting to find a proof of Theorem 2 based on moving planes.

On the other hand, bifurcation analysis (see [3] and Section 4 below) yields

**Theorem 3** Assume \( p < \frac{(n+2)/(n-2)}{n-2}/\) and \( \lambda > n/(p-1) \) with \( |\lambda - n/(p-1)| \) small. Then there exist nonconstant solutions of (1.4).

**Remark 2** When \( p > \frac{n+2}{n-2} \), there exist nonconstant solutions of (1.4) for some values of \( \lambda < \frac{n(n-2)}{4} \). Indeed bifurcation theory (see Section 4 and Remark 7 there) implies the existence of a branch of nonconstant solutions emanating from the constant solutions at the value \( \lambda = \frac{\nu}{p-1} \) where \( \nu = n \) is the second eigenvalue of \(-\Delta_{g_0}\) on \( S^n; \) note that \( \frac{\nu}{p-1} < \frac{n(n-2)}{4} \) since \( p > \frac{n+2}{n-2}. \) These solutions exist for \( \lambda < \frac{\nu}{p-1} \) and \( |\lambda - \frac{\nu}{p-1}| \) sufficiently small.

**Open Problem 1** When \( p > \frac{n+2}{n-2}, \) we do not know any result asserting that for some value of \( \lambda > 0, \lambda \) small, equation (1.4) admits only the constant solution
$u = \lambda^{1/(p-1)}$. In particular, it would be very interesting to decide what happens when $n = 3$, $p > 5$ and $\lambda > 0$ small.

**Remark 3** Theorem 1 is reminiscent of Theorem 1.1 in [4], dealing with (1.1) on $M = \mathbb{R}^n$. One could start with (1.4) on $S^n$ and transport it by stereographic projection to $\mathbb{R}^n$; however the resulting equation does not satisfy the assumptions from [4]. Still there are some analogies.

**2) The case of a general manifold**

Here our main results are the following

**Theorem 4** Assume $n = 3$. Then there exists some $\lambda^* = \lambda^*(M, g) > 0$ such that (1.1) with $f(u) = u^5 - \lambda u, 0 < \lambda < \lambda^*$, admits only the constant solution $u = \lambda^{1/4}$.

**Remark 4** A similar result on a three dimensional smooth convex domain with zero Neumann boundary data was established in [5].

**Theorem 5** ([6]) Let $n \geq 2$, and assume $1 < p < (n+2)/(n-2)$ (any finite $p > 1$ when $n = 2$). Then there exists some $\lambda^* = \lambda^*(M, g, p) > 0$ such that (1.1) with

$$f(u) = u^p - \lambda u, 0 < \lambda < \lambda^*,$$

admits only the constant solution $u = \lambda^{1/(p-1)}$.

**Remark 5** A similar result on a smooth domain in the Euclidean space with zero Neumann boundary data was established in [7].

**Open Problem 2** Is the conclusion of Theorem 5 valid for $n > 3$ and $p = (n+2)/(n-2)$? If not, identify necessary and sufficient conditions on $(M, g), n \geq 4$, under which the conclusion of Theorem 5 is valid.

The issue concerning Open Problem 2 is whether or not there exist some $\tilde{\lambda} > 0$ and $\tilde{C} > 0$, depending on $(M, g)$, such that $u \leq \tilde{C}$ for all solutions of (1.1) with $f(u) = u^{\frac{n+2}{n-2}} - \lambda u, 0 < \lambda < \tilde{\lambda}$. This is true in dimension $n = 3$ (a consequence of results in [8]), but in dimension $n \geq 4$, we do not expect this to be true for all manifolds. To solve the open problem, efforts can be made in two directions. One is to establish the $L^\infty$ estimates of solutions under appropriate conditions on the manifold. The other is to construct blow-up solutions $\{u_{\lambda_i}\}$ for a sequence of $\lambda_i \to 0^+$ under appropriate conditions on the manifold. Such issues for related problems have been studied, see e.g.[9-11], and the references therein.

**Remark 6** A sufficient condition in Open Problem 2 is that the Ricci curvature is positive — this is a consequence of Theorem B.1 in [2]. We have been informed by S.S. Bahoura that he has recently proved that the positivity of the scalar curvature is enough.
2. Proof of Theorem 1

Let \( u \) be a solution of (1.1) on \( M = S^n \). Let \( P \) be an arbitrary point on \( S^n \), which we will rename the north pole \( N \). Let \( S: S^n \setminus \{N\} \to \mathbb{R}^n \) be the stereographic projection, and let
\[
\xi(y) = \left( \frac{2}{1 + |y|^2} \right)^{\frac{n-2}{2}}, \quad y \in \mathbb{R}^n. \tag{2.1}
\]

Consider the new unknown \( v \), defined on \( \mathbb{R}^n \), by
\[
v(y) = \xi(y) u(S^{-1}(y)). \tag{2.2}
\]

A standard computation gives
\[
-\Delta v = F(y, v), \quad v > 0, \quad \text{in } \mathbb{R}^n, \tag{2.3}
\]
where
\[
F(y, v) = \xi(y)^{\frac{n+2}{n-2}} f \left( \frac{v}{\xi(y)} \right) + \frac{n(n-2)}{4} \xi(y)^{\frac{4}{n-2}} v. \tag{2.4}
\]

Since \( \xi \) depends only on \( r = |y| \), we will write \( \xi(r) \) and \( F(r, v) \).

By (1.2) and (2.4),
\[
F(r, v) = v^{\frac{n+2}{n-2}} h \left( \frac{v}{\xi(r)} \right). \tag{2.5}
\]

Thus, by (1.2),
\[
\text{for every fixed } v > 0, r \mapsto F(r, v) \text{ is decreasing in } r > 0.
\]

Since \( u \) is regular at \( N \), it is easy to see from (2.1) and (2.2) that \( \frac{1}{|y|^{n-2}} v \left( \frac{y}{|y|^2} \right) \) is smooth and positive near \( y = 0 \). From the theory of Gidas, Ni and Nirenberg, see [12], we know that any solution \( v \) of (2.3), with \( F \) satisfying (2.5), must be radially symmetric about the origin. Going back to \( u \), this means that \( u \) is constant on every \( (n-1) \)-sphere \( |x - N| = \text{constant} \). Since \( P \) is arbitrary on \( S^n \), \( u \) must be a constant.

3. Proof of Theorem 4

To prove Theorem 4, we first apply the results in [8] to establish

**Lemma 1** Assume \( n = 3 \). Then there exist some constants \( C_1, \varepsilon_1 > 0 \) such that for \( 0 < \lambda < \varepsilon_1 \), any solution \( u \) of (1.1), with \( f(u) = u^5 - \lambda u \), satisfies
\[
u \leq C_1.
\]

**Proof** Suppose the contrary; then there exist \( \lambda_i \to 0^+, u_i \) satisfies (1.1) with \( f(u) = u^5 - \lambda_i u \), such that
\[
\max_M u_i \to \infty.
\]
By the results in [8] (see in particular Theorem 0.2, Proposition 5.2, Proposition 4.1 and Proposition 3.1), there exist distinct points $p_1, \ldots, p_m$ on $M$, $m \geq 1$, and $p_{\ell}^{(i)} \to p_{\ell}$ as $i \to \infty$, and $\ell = 1, \ldots, m$, such that

$$u_i(p_{\ell}^{(i)}) u_i \to \eta \text{ in } C^2_0(M \setminus \{p_1, \ldots, p_m\}), \text{ as } i \to \infty,$$

where $\eta$ satisfies

$$\eta > 0 \text{ in } M \setminus \{p_1, \ldots, p_m\},$$

$$\Delta_g \eta = 0 \text{ in } M \setminus \{p_1, \ldots, p_m\},$$

$$\lim_{p \to p_\ell} \eta(p) = \infty, \ell = 1, 2, \ldots, m.$$

But this violates the maximum principle, since $\eta$ clearly has an interior minimum point in $M \setminus \{p_1, \ldots, p_m\}$.

**Proof of Theorem 4** Integrating equation (1.1) on $M$ leads to, using Hölder inequality,

$$\|u\|_{L^5(M)} \leq C \lambda^{1/4}. \quad (3.1)$$

Here and in the following, $C$ denotes some positive constant depending only on $(M, g)$.

By Lemma 1 and the equation satisfied by $u$,

$$|\Delta_g u| \leq Cu.$$

By elliptic estimates, in view of (3.1),

$$\|u\|_{L^\infty(M)} \leq C \lambda^{1/4}. \quad (3.2)$$

Next, we use an argument due to J.R. Licois and L. Veron [6]. From (1.4) we have

$$\int_M \nabla u \nabla (u - \bar{u}) + \lambda \int_M u(u - \bar{u}) = \int_M u^5(u - \bar{u}) \quad (3.3)$$

where $\bar{u} = \int_M u$. Clearly

$$\int_M \bar{u}(u - \bar{u}) = \int_M \bar{u}^5(u - \bar{u}) = 0. \quad (3.4)$$

By (3.3) and (3.4) we have

$$\int_M |\nabla (u - \bar{u})|^2 + \lambda \int_M |u - \bar{u}|^2 = \int_M (u^5 - \bar{u}^5)(u - \bar{u}). \quad (3.5)$$

Let $\nu_1$ be the first positive eigenvalue of $-\Delta_g$. From (3.5) we deduce that

$$(\nu_1 + \lambda)\|u - \bar{u}\|_{L^2}^2 \leq 5\|u\|_{L^\infty}^2 \|u - \bar{u}\|_{L^2}^2. \quad (3.6)$$

Combining (3.2) and (3.6) yields $u = \bar{u} = \lambda^{1/4}$ when $\lambda$ is sufficiently small.
4. Bifurcation Analysis. Proof of Theorem 3

We now return to equation (1.1) with \( f \) given by (1.3), i.e.,

\[
\begin{aligned}
-\Delta_g u &= u^p - \lambda u & \text{on } M, \\
\quad u > 0 & & \text{on } M,
\end{aligned}
\]

(4.1)

where \( 1 < p < \infty \) and \( \lambda > 0 \).

Writing the solution \( u \) as

\[ u = \lambda^{1/(p-1)} v, \]

equation (4.1) becomes

\[
\begin{aligned}
-\Delta_g v &= \lambda (v^p - v) & \text{on } M, \\
\quad v > 0 & & \text{on } M.
\end{aligned}
\]

Next we set

\[ w = v - 1 \]

and we are led to

\[
\begin{aligned}
-\Delta_g w &= \lambda F(w) & \text{on } M, \\
\quad w > -1 & & \text{on } M,
\end{aligned}
\]

(4.2)

where

\[ F(w) = (w + 1)^p - w - 1. \]

Clearly,

\[ F(0) = 0, \quad F'(0) = p - 1, \quad F''(0) = p(p - 1), \quad F'''(0) = p(p - 1)(p - 2). \]

Bifurcation theory asserts that, under some assumptions, a branch of solutions of (4.2), parametrized as \((\lambda(t), w(t))\), bifurcates from the 0-solution with

\[ \lambda(0) F''(0) = \lambda(0)(p - 1) = \nu \]

(4.3)

and \( \nu \) is an eigenvalue of \(-\Delta_g\). In particular, if \( \nu \) is a simple eigenvalue the result of Crandall-Rabinowitz [13, Theorem 1.7] applies and yields the existence of a smooth branch of solutions of (4.2) of the form \((\lambda(t), w(t))\), \( t \in (-\alpha, +\alpha) \) satisfying (4.3) and

\[ w(t) = t\varphi + \psi(t) \]

where

\[ -\Delta_g \varphi = \nu \varphi, \varphi \neq 0, \]
\[ \psi(0) = 0, \quad \psi'(0) = 0, \]
\[
\int_M \varphi \psi(t) = 0 \quad \forall t \in (-a, +a).
\]

We now differentiate (4.2) with respect to \(t\) and obtain
\[
-\Delta_g w' = \lambda F'(w)w' + \lambda' F(w),
\]
\[
-\Delta_g w'' = \lambda[F''(w)(w')^2 + F'(w)w''] + 2\lambda' F'(w)w' + \lambda'' F(w). \tag{4.4}
\]
Taking \(t = 0\) in (4.4) yields
\[
-\Delta_g \psi''(0) - \nu \psi''(0) = \nu p \varphi'^2 + 2\lambda'(0)(p - 1) \varphi
\]
and thus

**Lemma 2** We have
\[
\lambda'(0) = -\frac{\nu p}{2(p - 1)} \int \varphi^3.
\]

When \(\int \varphi^3 \neq 0\) we may be satisfied with the information \(\lambda'(0) \neq 0\) which gives the existence of nonconstant solutions of (4.1), close to the constant solution \(u = \lambda^{1/(p-1)}\), for all values of \(\lambda\) with \(|\lambda - \nu/(p - 1)|\) sufficiently small.

However when
\[
\int \varphi^3 = 0 \tag{4.5}
\]
we have \(\lambda'(0) = 0\) and we must study \(\lambda''(0)\). First observe that if (4.5) holds then \(\psi''(0)\) is uniquely determined by the relations
\[
-\Delta_g \psi''(0) - \nu \psi''(0) = \nu p \varphi'^2 \tag{4.6}
\]
\[
\int \varphi \psi''(0) = 0. \tag{4.7}
\]
Differentiating (4.4) with respect to \(t\) once more gives
\[
-\Delta_g w''' = \lambda[F'''(w)(w')^3 + 3F''(w)w' w'' + F'(w)w'''] +
+ 3\lambda' F''(w)(w')^2 + F'(w)w'] + 3\lambda'' F'(w)w' + \lambda''' F(w). \tag{4.8}
\]
Evaluating (4.8) at \(t = 0\) yields
\[
-\Delta_g \psi'''(0) - \nu \psi'''(0) = \nu[p(p - 2) \varphi^3 + 3p \varphi \psi''(0)] + 3\lambda''(0)(p - 1) \varphi
\]
and thus

**Lemma 3** We have
\[
\lambda''(0) = -\frac{\nu p[(p - 2) \int \varphi^4 + 3 \int \varphi^2 \psi''(0)]}{3(p - 1) \int \varphi^2}. \tag{4.9}
\]
We are now more specific and take $M = S^n$ equipped with its standard metric $g_0$. The first positive eigenvalue of $-\Delta_{g_0}$ is $\nu_1 = n$. Its multiplicity is $(n + 1)$ and the corresponding eigenvalues are the functions $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$ restricted to $S^n$. We are going to look for solutions of (1.4) which are radial about a point $N$ on $S^n$, say $N = (0, 0, \ldots, 1)$. Restricted to the class of radial functions the eigenvalue $\nu_1 = n$ becomes simple and the corresponding eigenfunction is

$$\varphi = x_{n+1}.$$ 

It is convenient to work with the variable $\theta = d_{S^n}(x, N) = \text{geodesic distance between } x \text{ and } N \text{ on } S^n$. In the $\theta$-variable we have

$$\varphi(\theta) = \cos \theta$$ 

so that

$$\int_{S^n} \varphi^3 = C_n \int_0^\pi \cos^3 \theta d\theta = 0,$$

and thus $\lambda'(0) = 0$ by Lemma 2. We now proceed to compute $\lambda''(0)$ using Lemma 3.

**Lemma 4** We have

$$\lambda''(0) = K_{p,m} \left[ -p + \frac{(n + 2)}{(n - 2)} \right]$$

(4.10)

where $K_{p,m}$ is a positive constant depending only on $p$ and $n$.

**Proof** For simplicity we write $\Delta$ instead of $\Delta_{g_0}$. We first determine $\psi''(0)$ using (4.6) - (4.7). Note that

$$\Delta \varphi^2 = 2\varphi \Delta \varphi + 2|\nabla \varphi|^2 = -2n\varphi^2 + 2|\nabla \varphi|^2.$$

(4.11)

On the other hand

$$|\nabla \varphi| = |\varphi_\theta| = \sin \theta$$

and therefore

$$|\nabla \varphi|^2 = 1 - \varphi^2$$

(4.12)

Inserting this into (4.11) yields

$$\Delta \varphi^2 = -2(n + 1)\varphi^2 + 2.$$

Thus the solution $\psi''(0)$ of (4.6)-(4.7) is given by

$$\psi''(0) = a\varphi^2 + b$$

with

$$a = \frac{np}{n + 2}$$

(4.13)
\[ b = \frac{-2p}{n + 2}. \] (4.14)

Going back to (4.9) we find
\[
\lambda''(0) = -np\frac{[(p - 2) + 3a]}{3(p - 1)} \int \frac{\varphi^4}{\varphi^2} - \frac{npb}{(p - 1)}. \] (4.15)

It remains to compute \( \int \varphi^4 / \int \varphi^2 \). For this purpose we write
\[
\Delta \varphi^4 = 4\varphi^3 \Delta \varphi + 12\varphi^2 |\nabla \varphi|^2
= -4n\varphi^4 + 12\varphi^2(1 - \varphi^2) \text{ by (4.12)}. \] (4.16)

Integrating (4.16) gives
\[
\int \frac{\varphi^4}{\int \varphi^2} = \frac{3}{n + 3}. \] (4.17)

Combining (4.15) with (4.13), (4.14) and (4.17) we are led to
\[
\lambda''(0) = -\frac{3np}{(n + 3)} \left[ \frac{(p - 2)}{3(p - 1)} + \frac{np}{(p - 1)(n + 2)} \right] + \frac{2np^2}{(p - 1)(n + 2)}
= \frac{np}{(p - 1)(n + 2)(n + 3)} \left[ - (p - 2)(n + 2) - 3np + 2p(n + 3) \right] \]
\[
= \frac{2np(n - 2)}{(p - 1)(n + 2)(n + 3)} \left[ -p + \frac{(n + 2)}{(n - 2)} \right]. \]

**Proof of Theorem 3**  When \( p < (n + 2)/(n - 2) \) we obtain from Lemmas 3 and 4 that \( \lambda'(0) = 0 \) and \( \lambda''(0) > 0 \). Hence the branch of solutions of (4.2) (and thus (1.4)) emanating from \((\lambda(0), w(0)) = \left( \frac{n}{p - 1}, 0 \right) \) bends to the right of \( \lambda(0) \). This was already observed in [3] based on Theorem 2.

**Remark 7**  When \( p > (n + 2)/(n - 2) \) we have \( \lambda'(0) = 0 \) and \( \lambda''(0) < 0 \). In this case the branch of solutions of (4.3) emanating from \( \left( \frac{n}{p - 1}, 0 \right) \) bends to the left of \( \lambda(0) \).

**Remark 8**  When \( p = (n + 2)/(n - 2) \) we have \( \lambda'(0) = 0 \) and \( \lambda''(0) = 0 \). In fact the branch of solutions of (4.2) emanating from \( \left( \frac{n}{p - 1}, 0 \right) \) satisfies \( \lambda(t) \equiv \lambda(0) = \frac{n(n - 2)}{4} \), i.e., the branch is vertical and it corresponds to the standard solutions of (1.4) with \( \lambda = n(n - 2)/4 \).

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References


