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SOME NONLINEAR ELLIPTIC EQUATIONS HAVE ONLY CONSTANT SOLUTIONS*

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Dedicated to K. C. Chang with high esteem and warm friendship

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Abstract We study some nonlinear elliptic equations on compact Riemannian manifolds. Our main concern is to find conditions which imply that such equations admit only constant solutions.

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1. Introduction

Motivated by some recent results and questions raised in [1], we study some nonlinear elliptic equations of the form

$$\begin{cases} -\Delta_g u = f(u) & \text{on } M, \\ \quad u > 0 & \text{on } M, \end{cases} \tag{1.1}$$

where (M, g) is a compact Riemannian manifold of dimension $n \geq 2$, without boundary, and $f : (0, +\infty) \to \mathbb{R}$ is a smooth function. Our main concern is to find conditions on M and f which imply that (1.1) admits only constant solutions.

We will present results in two directions:

1) The case where $M = S^n, n \geq 3$, equipped with its standard metric g_0

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In this case our first result is

Theorem 1 *Assume that* $(M,g) = (S^n, g_0), n \geq 3,$ *and*

$$h(t) := t^{-\frac{n+2}{n-2}} \left(f(t) + \frac{n(n-2)}{4} t \right) \text{ is decreasing on } (0, \infty). \tag{1.2}$$

Then any solution of (1.1) *is constant.*

A typical example is the case

$$f(t) = t^p - \lambda t,\ p > 1, \lambda > 0, \tag{1.3}$$

so that (1.1) becomes

$$\begin{cases} -\Delta_g u = u^p - \lambda u & \text{on } S^n, \\ u > 0 & \text{on } S^n. \end{cases} \tag{1.4}$$

Corollary 1 *Assume that* $p \leq (n+2)/(n-2)$ *and* $\lambda \leq n(n-2)/4$, *and at least one of these inequalities is strict. Then the only solution of* (1.4) *is the constant* $u = \lambda^{1/(p-1)}$.

In fact, Corollary 1 is originally due to Gidas-Spruck [2]. But our argument is quite different from theirs; they rely on some remarkable identities while our method uses moving planes.

When $p = (n+2)/(n-2)$ the conclusion of Corollary 1 is sharp. Indeed if $\lambda = n(n-2)/4$ there is a well-known family of nonconstant solutions; moreover all solutions of (1.4) belong to this family. However when $p < (n+2)/(n-2)$, B. Gidas and J. Spruck established a better result which was later sharpened by M.F. Bidaut-Veron and L. Veron. Namely they proved

Theorem 2([2],[3]) *Assume that* $p < (n+2)/(n-2)$ *and* $\lambda \leq n/(p-1)$. *Then the only solution of* (1.4) *is the constant* $u = \lambda^{1/(p-1)}$.

Remark 1 The proof of Theorem 2 in [2] and [3] is based on some remarkable identities. Our proof of Theorem 1 uses the method of moving planes. It would be very interesting to find a proof of Theorem 2 based on moving planes.

On the other hand, bifurcation analysis (see [3] and Section 4 below) yields

Theorem 3 *Assume* $p < (n+2)/(n-2)$ *and* $\lambda > n/(p-1)$ *with* $|\lambda - n/(p-1)|$ *small. Then there exist nonconstant solutions of* (1.4).

Remark 2 When $p > \frac{n+2}{n-2}$, there exist nonconstant solutions of (1.4) for some values of $\lambda < \frac{n(n-2)}{4}$. Indeed bifurcation theory (see Section 4 and Remark 7 there) implies the existence of a branch of nonconstant solutions emanating from the constant solutions at the value $\lambda = \frac{\nu}{p-1}$ where $\nu = n$ is the second eigenvalue of $-\Delta_{g_0}$ on S^n; note that $\frac{\nu}{p-1} < \frac{n(n-2)}{4}$ since $p > \frac{n+2}{n-2}$. These solutions exist for $\lambda < \frac{\nu}{p-1}$ and $|\lambda - \frac{\nu}{p-1}|$ sufficiently small.

Open Problem 1 When $p > \frac{n+2}{n-2}$, we do not know any result asserting that for some value of $\lambda > 0, \lambda$ small, equation (1.4) admits only the constant solution

$u = \lambda^{1/(p-1)}$. In particular, it would be very interesting to decide what happens when $n = 3$, $p > 5$ and $\lambda > 0$ small.

Remark 3 Theorem 1 is reminiscent of Theorem 1.1 in [4], dealing with (1.1) on $M = \mathbb{R}^n$. One could start with (1.4) on S^n and transport it by stereographic projection to $\mathbb{R}^n$; however the resulting equation does not satisfy the assumptions from [4]. Still there are some analogies.

2) The case of a general manifold

Here our main results are the following

Theorem 4 *Assume $n = 3$. Then there exists some $\lambda^* = \lambda^*(M, g) > 0$ such that* (1.1) *with $f(u) = u^5 - \lambda u, 0 < \lambda < \lambda^*$, admits only the constant solution $u = \lambda^{1/4}$.*

Remark 4 A similar result on a three dimensional smooth convex domain with zero Neumann boundary data was established in [5].

Theorem 5([6]) *Let $n \geq 2$, and assume $1 < p < (n+2)/(n-2)$ (any finite $p > 1$ when $n = 2$). Then there exists some $\lambda^* = \lambda^*(M, g, p) > 0$ such that* (1.1) *with*

$$f(u) = u^p - \lambda u,\ 0 < \lambda < \lambda^*,$$

admits only the constant solution $u = \lambda^{1/(p-1)}$.

Remark 5 A similar result on a smooth domain in the Euclidean space with zero Neumann boundary data was established in [7].

Open Problem 2 Is the conclusion of Theorem 5 valid for $n > 3$ and $p = (n+2)/(n-2)$? If not, identify necessary and sufficient conditions on (M, g), $n \geq 4$, under which the conclusion of Theorem 5 is valid.

The issue concerning Open Problem 2 is whether or not there exist some $\bar{\lambda} > 0$ and $\bar{C} > 0$, depending on (M, g), such that $u \leq \bar{C}$ for all solutions of (1.1) with $f(u) = u^{\frac{n+2}{n-2}} - \lambda u$, $0 < \lambda < \bar{\lambda}$. This is true in dimension $n = 3$ (a consequence of results in [8]), but in dimension $n \geq 4$, we do not expect this to be true for all manifolds. To solve the open problem, efforts can be made in two directions. One is to establish the L^∞ estimates of solutions under appropriate conditions on the manifold. The other is to construct blow-up solutions $\{u_{\lambda_i}\}$ for a sequence of $\lambda_i \to 0^+$ under appropriate conditions on the manifold. Such issues for related problems have been studied, see e.g.[9-11], and the references therein.

Remark 6 A sufficient condition in Open Problem 2 is that the Ricci curvature is positive — this is a consequence of Theorem B.1 in [2]. We have been informed by S.S. Bahoura that he has recently proved that the positivity of the scalar curvature is enough.

2. Proof of Theorem 1

Let u be a solution of (1.1) on $M = S^n$. Let P be an *arbitrary* point on S^n, which we will rename the north pole N. Let S: $S^n \setminus \{N\} \to \mathbb{R}^n$ be the stereographic projection, and let

$$\xi(y) = \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad y \in \mathbb{R}^n. \tag{2.1}$$

Consider the new unknown v, defined on $\mathbb{R}^n$, by

$$v(y) = \xi(y)\, u\,(S^{-1}(y)). \tag{2.2}$$

A standard computation gives

$$-\Delta v = F(y, v), v > 0, \text{ in } \mathbb{R}^n, \tag{2.3}$$

where

$$F(y, v) = \xi(y)^{\frac{n+2}{n-2}} f\left(\frac{v}{\xi(y)}\right) + \frac{n(n-2)}{4}\xi(y)^{\frac{4}{n-2}} v. \tag{2.4}$$

Since ξ depends only on $r = |y|$, we will write $\xi(r)$ and $F(r, v)$.

By (1.2) and (2.4),

$$F(r, v) = v^{\frac{n+2}{n-2}} h\left(\frac{v}{\xi(r)}\right).$$

Thus, by (1.2),

$$\text{for every fixed } v > 0, r \longmapsto F(r, v) \text{ is decreasing in } r > 0. \tag{2.5}$$

Since u is regular at N, it is easy to see from (2.1) and (2.2) that $\frac{1}{|y|^{n-2}} v\left(\frac{y}{|y|^2}\right)$ is smooth and positive near $y = 0$. From the theory of Gidas, Ni and Nirenberg, see [12], we know that any solution v of (2.3), with F satisfying (2.5), must be radially symmetric about the origin. Going back to u, this means that u is constant on every $(n-1)$- sphere $|x - N| =$ constant. Since P is arbitrary on S^n, u must be a constant.

3. Proof of Theorem 4

To prove Theorem 4, we first apply the results in [8] to establish

Lemma 1 *Assume $n = 3$. Then there exist some constants $C_1, \varepsilon_1 > 0$ such that for $0 < \lambda < \varepsilon_1$, any solution u of* (1.1)*, with $f(u) = u^5 - \lambda u$, satisfies*

$$u \leq C_1.$$

Proof Suppose the contrary; then there exist $\lambda_i \to 0^+, u_i$ satisfies (1.1) with $f(u) = u^5 - \lambda_i u$, such that

$$\max_M u_i \to \infty.$$

By the results in [8] (see in particular Theorem 0.2, Proposition 5.2, Proposition 4.1 and Proposition 3.1), there exist distinct points $p_1, \ldots, p_m$ on M, $m \geq 1$, and $p_\ell^{(i)} \to p_\ell$ as $i \to \infty$, and $\ell = 1, \ldots, m$, such that

$$u_i(p_1^{(i)})u_i \to \eta \text{ in } C^2_{\text{loc}}(M \setminus \{p_1, \ldots p_m\}), \quad \text{as } i \to \infty,$$

where η satisfies

$$\eta > 0 \text{ in } M \setminus \{p_1, \ldots, p_m\},$$

$$\Delta_g \eta = 0 \text{ in } M \setminus \{p_1, \ldots, p_m\},$$

$$\lim_{p \to p_\ell} \eta(p) = \infty \ , \ \ell = 1, 2, \ldots, m.$$

But this violates the maximum principle, since η clearly has an interior minimum point in $M \setminus \{p_1, \ldots, p_m\}$.

Proof of Theorem 4 Integrating equation (1.1) on M leads to, using Hölder inequality,

$$\|u\|_{L^5(M)} \leq C\lambda^{1/4}. \tag{3.1}$$

Here and in the following, C denotes some positive constant depending only on (M, g).

By Lemma 1 and the equation satisfied by u,

$$|\Delta_g u| \leq Cu.$$

By elliptic estimates, in view of (3.1),

$$\|u\|_{L^\infty(M)} \leq C\lambda^{1/4}. \tag{3.2}$$

Next, we use an argument due to J.R. Licois and L. Veron [6]. From (1.4) we have

$$\int_M \nabla u \nabla (u - \bar{u}) + \lambda \int_M u(u - \bar{u}) = \int_M u^5 (u - \bar{u}) \tag{3.3}$$

where $\bar{u} = \fint_M u$. Clearly

$$\int_M \bar{u}(u - \bar{u}) = \int_M \bar{u}^5 (u - \bar{u}) = 0. \tag{3.4}$$

By (3.3) and (3.4) we have

$$\int_M |\nabla (u - \bar{u})|^2 + \lambda \int_M |u - \bar{u}|^2 = \int_M (u^5 - \bar{u}^5)(u - \bar{u}). \tag{3.5}$$

Let ν_1 be the first positive eigenvalue of $-\Delta_g$. From (3.5) we deduce that

$$(\nu_1 + \lambda)\|u - \bar{u}\|^2_{L^2} \leq 5\|u\|^4_{L^\infty} \|u - \bar{u}\|^2_{L^2}. \tag{3.6}$$

Combining (3.2) and (3.6) yields $u = \bar{u} = \lambda^{1/4}$ when λ is sufficiently small.

4. Bifurcation Analysis. Proof of Theorem 3

We now return to equation (1.1) with f given by (1.3), i.e.,

$$\begin{cases} -\Delta_g u = u^p - \lambda u & \text{on } M, \\ u > 0 & \text{on } M, \end{cases} \tag{4.1}$$

where $1 < p < \infty$ and $\lambda > 0$.

Writing the solution u as

$$u = \lambda^{1/(p-1)} v,$$

equation (4.1) becomes

$$\begin{cases} -\Delta_g v = \lambda(v^p - v) & \text{on } M, \\ v > 0 & \text{on } M. \end{cases}$$

Next we set

$$w = v - 1$$

and we are led to

$$\begin{cases} -\Delta_g w = \lambda F(w) & \text{on } M, \\ w > -1 & \text{on } M, \end{cases} \tag{4.2}$$

where

$$F(w) = (w+1)^p - w - 1.$$

Clearly,

$$F(0) = 0, \quad F'(0) = p-1, \quad F''(0) = p(p-1), \quad F'''(0) = p(p-1)(p-2).$$

Bifurcation theory asserts that, under some assumptions, a branch of solutions of (4.2), parametrized as $(\lambda(t), w(t))$, bifurcates from the 0-solution with

$$\lambda(0)F'(0) = \lambda(0)(p-1) = \nu \tag{4.3}$$

and ν is an eigenvalue of $-\Delta_g$. In particular, if ν is a simple eigenvalue the result of Crandall-Rabinowitz [13, Theorem 1.7] applies and yields the existence of a smooth branch of solutions of (4.2) of the form $(\lambda(t), w(t)), t \in (-a, +a)$ satisfying (4.3) and

$$w(t) = t\varphi + \psi(t)$$

where

$$-\Delta_g \varphi = \nu\varphi, \varphi \neq 0$$

$$\psi(0) = 0, \quad \psi'(0) = 0,$$

$$\int_M \varphi\psi(t) = 0 \quad \forall t \in (-a, +a).$$

We now differentiate (4.2) with respect to t and obtain

$$-\Delta_g w' = \lambda F'(w)w' + \lambda' F(w),$$

$$-\Delta_g w'' = \lambda[F''(w)(w')^2 + F'(w)w''] + 2\lambda' F'(w)w' + \lambda'' F(w). \tag{4.4}$$

Taking $t = 0$ in (4.4) yields

$$-\Delta_g \psi''(0) - \nu\psi''(0) = \nu p\varphi^2 + 2\lambda'(0)(p-1)\varphi$$

and thus

Lemma 2 *We have*

$$\lambda'(0) = -\frac{\nu p \int \varphi^3}{2(p-1)\int \varphi^2}.$$

When $\int \varphi^3 \neq 0$ we may be satisfied with the information $\lambda'(0) \neq 0$ which gives the existence of nonconstant solutions of (4.1), close to the constant solution $u = \lambda^{1/(p-1)}$, for all values of λ with $|\lambda - \nu/(p-1)|$ sufficiently small.

However when

$$\int \varphi^3 = 0 \tag{4.5}$$

we have $\lambda'(0) = 0$ and we must study $\lambda''(0)$. First observe that if (4.5) holds then $\psi''(0)$ is uniquely determined by the relations

$$-\Delta_g \psi''(0) - \nu\psi''(0) = \nu p\varphi^2 \tag{4.6}$$

$$\int \varphi\psi''(0) = 0. \tag{4.7}$$

Differentiating (4.4) with respect to t once more gives

$$\begin{aligned} -\Delta_g w''' &= \lambda[F'''(w)(w')^3 + 3F''(w)w'w'' + F'(w)w'''] + \\ &\quad + 3\lambda'[F''(w)(w')^2 + F'(w)w''] + 3\lambda'' F'(w)w' + \lambda''' F(w). \end{aligned} \tag{4.8}$$

Evaluating (4.8) at $t = 0$ yields

$$-\Delta_g \psi'''(0) - \nu\psi'''(0) = \nu[p(p-2)\varphi^3 + 3p\varphi\psi''(0)] + 3\lambda''(0)(p-1)\varphi$$

and thus

Lemma 3 *We have*

$$\lambda''(0) = -\frac{\nu p[(p-2)\int \varphi^4 + 3\int \varphi^2\psi''(0)]}{3(p-1)\int \varphi^2}. \tag{4.9}$$

We are now more specific and take $M = S^n$ equipped with its standard metric g_0. The first positive eigenvalue of $-\Delta_{g_0}$ is $\nu_1 = n$. Its multiplicity is $(n+1)$ and the corresponding eigenvalues are the functions $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$ restricted to S^n. We are going to look for solutions of (1.4) which are radial about a point N on S^n, say $N = (0, 0, \ldots, 1)$. Restricted to the class of radial functions the eigenvalue $\nu_1 = n$ becomes simple and the corresponding eigenfunction is

$$\varphi = x_{n+1}.$$

It is convenient to work with the variable $\theta = d_{S^n}(x, N) =$ geodesic distance between x and N on S^n. In the θ-variable we have

$$\varphi(\theta) = \cos\theta$$

so that

$$\int_{S^n} \varphi^3 = C_n \int_0^\pi \cos^3\theta d\theta = 0,$$

and thus $\lambda'(0) = 0$ by Lemma 2. We now proceed to compute $\lambda''(0)$ using Lemma 3.

Lemma 4 *We have*

$$\lambda''(0) = K_{p,m}\left[-p + \frac{(n+2)}{(n-2)}\right] \tag{4.10}$$

where $K_{p,m}$ is a positive constant depending only on p and n.

Proof For simplicity we write Δ instead of Δ_{g_0}. We first determine $\psi''(0)$ using (4.6) - (4.7). Note that

$$\Delta\varphi^2 = 2\varphi\Delta\varphi + 2|\nabla\varphi|^2 = -2n\varphi^2 + 2|\nabla\varphi|^2. \tag{4.11}$$

On the other hand

$$|\nabla\varphi| = |\varphi_\theta| = \sin\theta$$

and therefore

$$|\nabla\varphi|^2 = 1 - \varphi^2 \tag{4.12}$$

Inserting this into (4.11) yields

$$\Delta\varphi^2 = -2(n+1)\varphi^2 + 2.$$

Thus the solution $\psi''(0)$ of (4.6)-(4.7) is given by

$$\psi''(0) = a\varphi^2 + b$$

with

$$a = \frac{np}{n+2} \tag{4.13}$$

$$b = \frac{-2p}{n+2}. \tag{4.14}$$

Going back to (4.9) we find

$$\lambda''(0) = -np\frac{[(p-2)+3a]}{3(p-1)}\frac{\int \varphi^4}{\int \varphi^2} - \frac{npb}{(p-1)}. \tag{4.15}$$

It remains to compute $\int \varphi^4 / \int \varphi^2$. For this purpose we write

$$\begin{aligned} \Delta\varphi^4 &= 4\varphi^3\Delta\varphi + 12\varphi^2|\nabla\varphi|^2 \\ &= -4n\varphi^4 + 12\varphi^2(1-\varphi^2) \text{ by (4.12).} \end{aligned} \tag{4.16}$$

Integrating (4.16) gives

$$\frac{\int \varphi^4}{\int \varphi^2} = \frac{3}{n+3} \tag{4.17}$$

Combining (4.15) with (4.13), (4.14) and (4.17) we are led to

$$\begin{aligned} \lambda''(0) &= \frac{-3np}{(n+3)}\left[\frac{(p-2)}{3(p-1)} + \frac{np}{(p-1)(n+2)}\right] + \frac{2np^2}{(p-1)(n+2)} \\ &= \frac{np}{(p-1)(n+2)(n+3)}[-(p-2)(n+2) - 3np + 2p(n+3)] \\ &= \frac{2np(n-2)}{(p-1)(n+2)(n+3)}\left[-p + \frac{(n+2)}{(n-2)}\right]. \end{aligned}$$

Proof of Theorem 3 When $p < (n+2)/(n-2)$ we obtain from Lemmas 3 and 4 that $\lambda'(0) = 0$ and $\lambda''(0) > 0$. Hence the branch of solutions of (4.2) (and thus (1.4)) emanating from $(\lambda(0), w(0)) = \left(\frac{n}{p-1}, 0\right)$ bends to the right of $\lambda(0)$. This was already observed in [3] based on Theorem 2.

Remark 7 When $p > (n+2)/(n-2)$ we have $\lambda'(0) = 0$ and $\lambda''(0) < 0$. In this case the branch of solutions of (4.3) emanating from $\left(\frac{n}{p-1}, 0\right)$ bends to the left of $\lambda(0)$.

Remark 8 When $p = (n+2)/(n-2)$ we have $\lambda'(0) = 0$ and $\lambda''(0) = 0$. In fact the branch of solutions of (4.2) emanating from $\left(\frac{n}{p-1}, 0\right)$ satisfies $\lambda(t) \equiv \lambda(0) = \frac{n(n-2)}{4}$, i.e., the branch is vertical and it corresponds to the standard solutions of (1.4) with $\lambda = n(n-2)/4$.

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