MAXIMUM PRINCIPLES OF NONHOMOGENEOUS SUBELLIPZIC $p$-LAPLACE EQUATIONS AND APPLICATIONS*

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Abstract Maximum principles for weak solutions of nonhomogeneous subelliptic $p$-Laplace equations related to smooth vector fields $\{X_j\}$ satisfying the Hörmander condition are proved by the choice of suitable test functions and the adaption of the classical Moser iteration method. Some applications are given in this paper.

Key Words Subelliptic $p$-Laplacian; maximum principle; Harnack inequality.

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1. Introduction

Over the last decades, the study of nonelliptic equations arising from general families of non-commuting vector fields has made a great development. In spite of the formidable progress, there is still much to discover concerning the basic properties of solutions to these classes of equations.

Consider a family of $C^\infty$ vector fields $X_1, \cdots, X_N$ in $\mathbb{R}^n$, and assume that Hörmander finite rank condition [1]

$$\text{rank Lie } [X_1, \cdots, X_N] = n$$

is satisfied at each $x \in \mathbb{R}^n$. In this paper we are concerned with a kind of the so-called subelliptic $p$-Laplace equation:

$$\sum_{j=1}^{N} X^*_j \left(|Xu|^{p-2} X_j u \right) = 0,$$

where $X^*_j$ denotes the formal adjoint of $X_j$, $Xu = (X_1 u, \cdots, X_N u)$ is the subelliptic gradient of $u$ and $1 \leq p < \infty$ is fixed. It appears in the study of quasiregular mappings

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in stratified Lie groups, also known as Carnot groups \([2]\). We note that (1.2) is the Euler-Lagrange equation of the Sobolev functional

\[
J_p(u) = \int |Xu|^p \, dx. \tag{1.3}
\]

When \(p = 2\), (1.2) is the H¨ omander type equation

\[
\sum_{j=1}^{N} X_j^* X_j u = 0. \tag{1.4}
\]

An important result in the study of (1.4) was given in Nagel, Stein and Wainger’s famous paper \([3]\), in which the following estimates for the Carnot-Carathéodory metric balls were proved: for every \(K \subset \subset \mathbb{R}^n\), there exist positive constants \(C, R_0\) and \(Q\) such that, for any \(x \in K\), \(0 < r < R_0\), and \(0 < t < 1\),

\[
|B_d(x, tr)| \geq Ct^Q |B_d(x, r)|, \tag{1.5}
\]

where \(B_d(x, r) = \{y \in \mathbb{R}^n | d(x, y) < r\}\) is the ball relative to the control distance \(d\) associated to the vector fields \(X_1, \cdots, X_N\). The number \(Q\) plays the role of a dimension in the local analysis of (1.4). It will be called the homogeneous dimension of \(K\) with respect to the family \(X_1, \cdots, X_N\).

In \([2]\), a strong maximum principle of homogeneous subelliptic equations is given with the Hölder estimate. Gutiérrez and Lanconelli in \([4]\) proved a maximum principle and Harnack inequalities for second order uniformly \(X\)-elliptic operators. Xu has studied some subelliptic equations associated with the vector fields satisfying Hörmander condition. He obtained regularity for quasilinear subelliptic equations in \([5]\) and Sobolev inequality of these vector fields in \([6]\). Primarily inspired by \([4]\), our purpose is to establish a maximum principle for the nonhomogeneous equation

\[
L_p u = -\sum_{j=1}^{N} X_j^* (|Xu|^{p-2} X_j u) = f(x), \tag{1.6}
\]

on the bounded open subset in \(\mathbb{R}^n\). Although the method we used is similar to that of \([4]\), the question we discussed here is nonlinear in substance.

We introduce some definitions and results that will be needed in the sequel. Solutions to (1.6) shall be understood in a suitable weak sense. Throughout the paper, \(\Omega\) denotes a bounded open subset in \(\mathbb{R}^n\) and \(Q\) is the homogeneous dimension of \(\Omega\) relative to \(X_1, \cdots, X_N\).

Let \(S^{1,p}(\Omega)\) be the closure of \(\{u \in C^\infty(\Omega) : u, X_j u \in L^p(\Omega), \text{ for } 1 \leq j \leq N\}\) under the norm

\[
||u||_{S^{1,p}(\Omega)} = \left[ \int_\Omega (|u|^p + |Xu|^p) \, dx \right]^{\frac{1}{p}}. \tag{1.7}
\]
Denote $\hat{S}^{1,p}(\Omega)$ the closure of $C^1_0(\Omega)$ with respect to the norm. We say that $u \in S^{1,p}(\Omega)$ is a weak solution of the nonhomogeneous equation (1.6) if
\[
\int_\Omega \sum_{j=1}^N |Xu|^{p-2}X_juX_j \varphi dx = \int_\Omega f \varphi dx
\] (1.8)
for each $\varphi \in \hat{S}^{1,p}(\Omega)$. A function $u \in S^{1,p}(\Omega)$ is a weak subsolution of (1.6) if
\[
\int_\Omega \sum_{j=1}^N |Xu|^{p-2}X_juX_j \varphi dx \leq \int_\Omega f \varphi dx
\] (1.9)
for each $\varphi \in \hat{S}^{1,p}(\Omega)$ with $\varphi \geq 0$. Analogously, we define a weak supersolution.

Let $u^+ = \max(u, 0)$, $u^- = \min(u, 0)$ and $u = u^+ + u^-$. The main results of our paper are the following

**Theorem 1.1 (Maximum Principle)** If $u$ is a weak subsolution of (1.6) on $\Omega$, then we have
\[
\sup_{\Omega} u^+ \leq \sup_{\partial \Omega} u^+ + C\|f\|_{\frac{Q}{Q-1}}^{\frac{1}{p-1}}
\] (1.10)
where $C > 0$ only depended on $X, \Omega, p > Q$ and $m > \frac{Q}{p}$.

As the consequence of the maximum principle theorem, we have

**Theorem 1.2 (Harnack Inequality)** Let $1 < p \leq Q$ and $u \in S^{1,p}(\Omega)$ be a non-negative solution to (1.2). Then there exist $C > 0$ and $R_0 > 0$ such that for each $B_R = B(x, R)$, $B_{4R} \subseteq \Omega$ and $R \leq R_0$,
\[
\sup_{B_R} u \leq C \inf_{B_R} u.
\] (1.11)

In the proof of the results, we need the following inequality (see [7], see also [6]).

**Theorem 1.3 (Sobolev Embedding Theorem)** Let $\Omega \in \mathbb{R}^n$ be a bounded open set and denote by $Q$ the homogeneous dimension relative to $\Omega$. Let $1 < p < Q$. Then there exist $C > 0$ and $R_0 > 0$ such that for any $x \in U$, $B_R = B(x, R)$, with $R \leq R_0$, we have
\[
\left(\frac{1}{|B_R|} \int_{B_R} |u|^p dx\right)^{\frac{1}{p}} \leq CR \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx\right)^{\frac{1}{p}}
\] (1.12)
for any $u \in \hat{S}^{1,p}(B_R)$. Here, $1 \leq \kappa \leq \frac{Q}{Q-p}$.

The result asserts that, if $1 < p < Q$, then $\hat{S}^{1,p}(B_R) \hookrightarrow L^q(B_R)$ for $1 \leq q \leq \frac{Op}{Q-p}$. Furthermore, we have for $u \in \hat{S}^{1,p}(B_R)$
\[
\left(\frac{1}{|B_R|} \int_{B_R} |u|^q dx\right)^{\frac{1}{q}} \leq CR \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx\right)^{\frac{1}{p}},
\] (1.13)
for some $C = C(\Omega, X_1, \ldots, X_N) > 0$. (1.13) and a standard partition of the unity argument imply $S^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for any $\Omega \subset \subset \mathbb{R}^n$. That is to say
\begin{equation}
\|u\|_{L^q(\Omega)} \leq C \| Xu \|_{L^p(\Omega)}.
\end{equation}

The following lemma is useful. We refer the readers to [8] for the proof.

**Lemma 1.1** If $1 < p \leq r \leq \infty$, $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$ with $0 \leq \lambda \leq 1$ and $\mu = \left( \frac{1}{p} - \frac{1}{q} \right) \left( \frac{1}{q} - \frac{1}{r} \right)^{-1}$, then for any $\epsilon > 0$, there is
\begin{equation}
\|u\|_q \leq \varepsilon \|u\|_r + \varepsilon^{-\mu} \|u\|_p.
\end{equation}

The plan of this paper is as follows. In Section 2, we prove Theorem 1.1 with the adaption of the classical Moser’s iteration method and the choice of test functions. Section 3 is devoted to the proof of Theorem 1.2. We also give a Liouville type theorem as the consequence of the previous results.

2. The Proof of Theorem 1.1

We first prove the following lemma.

**Lemma 2.1** Let $u \in S^{1,p}(\Omega)$ be a weak subsolution of the nonhomogeneous equation (1.6) and $u \leq 0$ on $\partial \Omega$. Then there is a constant $C > 0$ such that the following inequality
\begin{equation}
\sup_\Omega u^+ \leq C(\|u^+\|_{L^p(\Omega)} + \|f\|_{L^m(\Omega)}^{-1}),
\end{equation}
where $m > 1$.

**Proof** Define
\begin{equation}
H(z) = \begin{cases} 
z^{\beta} - \kappa^{\beta}, & \text{if } \kappa \leq z \leq N; \\
\beta N^{\beta-1}(z - N) + N^{\beta} - \kappa^{\beta}, & \text{if } z \geq N,
\end{cases}
\end{equation}
where $\kappa \geq 0$ is a number that will be chosen later, $N \geq \kappa$, and $\beta > 1$. The parameter $N$ will tend to $+\infty$ at the end. We have $H(z) \in C^1([\kappa, +\infty))$ and $H'(z) \in L^\infty([\kappa, +\infty))$. Let
\begin{equation}
G(t) = \begin{cases} 
f^t_\kappa (H'(s))^{p}ds, & \text{if } t \geq \kappa; \\
(H'(\kappa))^{p} (t - \kappa), & \text{if } t < \kappa,
\end{cases}
\end{equation}
and
\begin{equation}
\varphi = G(w(x)) = \int_\kappa^{w(x)} (H'(s))^{p}ds.
\end{equation}
Then $\varphi \geq 0$, $\varphi u \geq 0$ and since $u \leq 0$ on $\partial \Omega$, it follows that $\varphi(x) = 0$ on $\partial \Omega$ and then $\varphi \in S^{1,p}(\Omega)$. Plug $\varphi(x)$ into Eq.(1.9). Since
\begin{equation}
X_j \varphi = G'(w(x)) X_j w = G'(w(x)) X_j u^+
\end{equation}
and
\[ |Xu| = \left( \sum_{j=1}^{N} |X_j u|^2 \right)^{\frac{1}{2}} \geq |X u^+|, \]

the left-hand side of (1.9) can be written as
\[
\int_{\Omega} \sum_{j=1}^{N} |Xu|^{p-2} X_j u X_j \varphi dx = \int_{\Omega} |Xu|^{p-2} \sum_{j=1}^{N} X_j u \cdot G'(w(x)) X_j u^+ dx
\]
\[
= \int_{\Omega} |Xu|^{p-2} |Xu^+|^2 G'(w) dx
\]
\[
\geq \int_{\Omega} |Xu^+|^p G'(w) dx = \int_{\Omega} |Xw|^p (H'(w))^p dx. \quad (2.3)
\]

From the monotony of \( H'(z) \) and the definition of \( G(t) \), we get \( G(t) \leq t G'(t) \). Thus (2.3) and Hölder inequality give us
\[
\int_{\Omega} |X(H(w))|^p dx = \int_{\Omega} |Xw|^p |H'(w)|^p dx
\]
\[
\leq \int_{\Omega} f G(w) dx \leq \int_{\Omega} |f| |w G'(w)| dx \leq \frac{1}{\kappa^{p-1}} \int_{\Omega} |f| |w H'(w)|^p dx
\]
\[
\leq \frac{1}{\kappa^{p-1}} \left( \int_{\Omega} |f|^m \right)^{\frac{1}{m}} \left( \int_{\Omega} |w H'(w)|^{m'p} dx \right)^{\frac{1}{m'}}, \quad (2.4)
\]

where \( m \) and \( m' \) satisfy \( \frac{1}{m} + \frac{1}{m'} = 1 \). Taking \( \kappa > 0 \) such that \( \kappa^{p-1} = \|f\|_{L^m(\Omega)} \), from the Sobolev inequality (1.14) and (2.4) we obtain
\[
\|H(w)\|_{L^q(\Omega)} \leq C(\Omega) \|w H'(w)\|_{L^{m'p}(\Omega)}. \quad (2.5)
\]

Plugging \( H(w) = w^\beta - \kappa^\beta \) and \( H'(w) = \beta w^{\beta-1} \) into (2.5) leads to
\[
\left( \int_{\Omega} |w^\beta - \kappa^\beta|^q dx \right)^{\frac{1}{q}} \leq C \beta \left( \int_{\Omega} |w^{\beta m'}|^p dx \right)^{\frac{1}{m'}}. \quad (2.6)
\]

Noting that \( |w^\beta| \leq |w^\beta - \kappa^\beta| + \kappa^\beta \) and Minkowski inequality, we have
\[
\left( \int_{\Omega} |w^\beta|^q dx \right)^{\frac{1}{q}} \leq \left( \int_{\Omega} |w^\beta - \kappa^\beta|^q dx \right)^{\frac{1}{q}} + \left( \int_{\Omega} |\kappa^\beta|^q dx \right)^{\frac{1}{q}}
\]
\[
\leq C(\Omega) \beta \left( \int_{\Omega} |w^\beta m'^p dx \right)^{\frac{1}{m'}} + C_1(\Omega) \left( \int_{\Omega} |\kappa^\beta|^m p dx \right)^{\frac{1}{m}}
\]
\[
\leq C'(\Omega) \beta \left( \int_{\Omega} |w^\beta m'^p dx \right)^{\frac{1}{m'}} + C_1(\Omega) \left( \int_{\Omega} |\kappa^\beta|^m p dx \right)^{\frac{1}{m'}}
\]
\[
= R \beta \left( \int_{\Omega} w^\beta m'^p dx \right)^{\frac{1}{m'}}, \quad (2.7)
\]
where \( R = C'(\Omega) \) is a constant only depending on \( \Omega \). It shows

\[
\|w\|_{L^{1/2}(\Omega)} \leq R^{1/2} \beta^{1/2} \|w\|_{L^{p'n'}(\Omega)}.
\] (2.8)

Let \( \theta = \frac{q}{pm'} > 1 \) and \( \beta = \theta^k \), (2.8) becomes

\[
\|w\|_{L^{p'n'\theta^{k+1}}(\Omega)} \leq R^{1/2} \theta^{1/2} \|w\|_{L^{p'n'}(\Omega)}.
\]

Therefore we have, by iterating the above inequality,

\[
\|w\|_{L^{p'n'\theta^{k+1}}(\Omega)} \leq R^{1/2} \theta^{1/2} \|w\|_{L^{p'n'}(\Omega)} \leq R^{1/2} \theta^{1/2} \|w\|_{L^{p'n'}}(\Omega).
\]

Letting \( n \to \infty \), we obtain

\[
\|w\|_{L^{\infty}(\Omega)} \leq R^{1/2} \theta^{1/2} \|w\|_{L^{p'n'}}(\Omega),
\] (2.9)

where \( \sigma = 1/(\theta - 1) \), \( \tau = \theta/(\theta - 1)^2 \). By taking \( q = pm' \), \( p = p \) and \( r = \infty \) in Lemma 1.1, we deduce

\[
\|w\|_{L^{p'n'}(\Omega)} \leq \varepsilon \|w\|_{L^{\infty}(\Omega)} + \varepsilon^{1-m'} \|w\|_{L^{p}(\Omega)}.
\]

for any \( \varepsilon > 0 \). Thus

\[
\|w\|_{L^{\infty}(\Omega)} \leq R^{\tau} \theta^{\tau} \left( \varepsilon \|w\|_{L^{\infty}(\Omega)} + \varepsilon^{1-m'} \|w\|_{L^{p}(\Omega)} \right).
\] (2.10)

Minimizing the right-hand side of inequality (2.10), we get

\[
\varepsilon = (m' - 1)^{1/m'} \|w\|_{L^{p'n'}(\Omega)} \|w\|_{L^{p}(\Omega)}^{1/m'}.
\]

Substituting this back in (2.10) shows

\[
\|w\|_{L^{\infty}(\Omega)} \leq R^{\tau} \theta^{\tau} \left[ (m' - 1)^{1/m'} + (m' - 1)^{1/m'} - 1 \right] \|w\|_{L^{p'n'}(\Omega)} \|w\|_{L^{p}(\Omega)}^{1/m'}
\]

and

\[
\|w\|_{L^{\infty}(\Omega)} \leq R^{\sigma m'} \theta^{m'm'} \left[ (m' - 1)^{1/m'} + (m' - 1)^{1-m'/m'} \right] \|w\|_{L^{p}(\Omega)}.
\]

Denote \( C' = R^{\sigma m'} \theta^{m'm'} \left[ (m' - 1)^{1/m'} + (m' - 1)^{1-m'/m'} \right] \) and let \( \kappa = \|f\|_{L^{m}(\Omega)}^{1/m} \). Keeping in mind \( w = u^+ + \kappa \), we obtain

\[
\sup_{\Omega} u^+ \leq \sup_{\Omega} u^+ + \kappa = \|w\|_{L^{\infty}(\Omega)} \leq C' \|w\|_{L^{p}(\Omega)}
\]

\[
\leq C'' \left( \|u^+\|_{L^{p}(\Omega)} + \kappa \right) = C'' \left( \|u^+\|_{L^{p}(\Omega)} + \|f\|_{L^{m}(\Omega)}^{1/m} \right).
\]
This ends the proof of (2.1).

**Proof of Theorem 1.1**

Let \( l = \sup_{\partial \Omega} u^+ \). If \( l \) is infinite, then the result is true. It suffices to consider that \( l \) is finite. We may assume, without loss of generality, that \( l \) equals zero. Take the test function

\[
\varphi(x) = \frac{1}{p-1} \left[ \frac{1}{(M + \kappa - u^+)^{p-1}} - \frac{1}{(M + \kappa)^{p-1}} \right],
\]

where \( M = \sup_{\Omega} u^+ \) and \( \kappa > 0 \) to be determined later. When \( u^+ = 0 \), we have \( \varphi = 0 \). It shows \( \varphi \in \dot{S}^{1,p}(\Omega) \) from \( u^+ \in \dot{S}^{1,p}(\Omega) \). So \( \varphi \) can be taken as a test function. Noting that

\[
X\varphi = \frac{Xu^+}{(M + \kappa - u^+)^p},
\]

and plugging it into the left-hand side of (1.9), we obtain

\[
\int_{\Omega} \sum_{j=1}^{N} |Xu|^{p-2} X_j u X_j \varphi \, dx = \int_{\Omega} |Xu|^{p-2} |Xu^+|^2 \, dx \geq \int_{\Omega} \frac{|Xu^+|^p}{(M + \kappa - u^+)^p} \, dx.
\]

Since \( M \geq u^+ \), the right-hand side of (1.9) has the estimate

\[
\int_{\Omega} |f| \varphi \, dx \leq \int_{\Omega} \frac{1}{p-1} |f| \frac{1}{(M + \kappa - u^+)^{p-1}} \, dx \leq \frac{1}{p-1} \int_{\Omega} \frac{|f|}{\kappa^{p-1}} \, dx \quad (p > 1).
\]

Thus

\[
\int_{\Omega} \frac{|Xu^+|^p}{(M + \kappa - u^+)^p} \, dx \leq \frac{1}{p-1} \int_{\Omega} \frac{|f|}{\kappa^{p-1}} \, dx. \tag{2.11}
\]

Letting \( w = \ln \frac{M + \kappa}{M + \kappa - u^+} \) and noting \( Xw = \frac{Xu^+}{M + \kappa - u^+} \), we obtain from (2.11)

\[
\int_{\Omega} |Xw|^p \, dx \leq \frac{1}{p-1} \int_{\Omega} \frac{|f|}{\kappa^{p-1}} \, dx \leq \frac{1}{p-1} \int_{\Omega} \frac{|f|^m}{\kappa^{p-1}} \, dx \quad \text{for } m > 1.
\]

where we have let \( \kappa = \|f\|_{L^m(\Omega)}^{\frac{p-1}{p}} \) as before. Thus

\[
\|Xw\|_{L^p(\Omega)} \leq C, \tag{2.12}
\]

where \( C \) is a constant only depended on \( p, \Omega \) and \( m > 1 \).

We assert that \( w \) is a weak subsolution of equation

\[
- \sum_{j=1}^{N} X_j \left( |Xw|^{p-2} X_j w \right) = \frac{|f|}{\kappa^{p-1}}. \tag{2.13}
\]
In fact, for all \( \eta \in \dot{S}^{1,p}(\Omega) \) satisfying \( \eta \geq 0 \) and \( \eta w \geq 0 \) in \( \Omega \); or equivalently, for all \( \eta \in \dot{S}^{1,p}(\Omega) \) such that \( \text{supp} \eta \subset \text{supp} u^+ \), let \( \psi = \eta(M + \kappa - u^+)^{p+1} \). Note explicitly that

\[
X_j \psi = X_j \eta(M + \kappa - u^+)^{p+1} + (p - 1) \eta X_j u^+ (M + \kappa - u^+)^{-p},
\]

\( \eta u \geq 0 \), and \( \eta X_j u^+ = \eta X_j u, \forall j = 1, \ldots, N \). Since \( u \) is the subsolution of function \( L_p u = f \) on \( \Omega \), we get

\[
\int_{\Omega} |Xu|^{p-2} \sum_{j=1}^{N} X_j u X_j \psi \mathrm{d}x \leq \int_{\Omega} f \frac{\eta}{(M + \kappa - u^+)^{p-1}} \mathrm{d}x. \tag{2.14}
\]

The left-hand side of the above equation satisfies

\[
\int_{\Omega} |Xu|^{p-2} \sum_{j=1}^{N} X_j u \left[ \frac{X_j \eta}{(M + \kappa - u^+)^{p-1}} + (p - 1) \frac{\eta X_j u^+}{(M + \kappa - u^+)^p} \right] \mathrm{d}x \geq \int_{\Omega} |Xu|^{p-2} \sum_{j=1}^{N} \frac{X_j u X_j \eta}{(M + \kappa - u^+)^{p-1}} \mathrm{d}x \geq \int_{\Omega} \frac{|Xu^+|^{p-2}}{(M + \kappa - u^+)^{p-1}} \sum_{j=1}^{N} X_j u^+ X_j \eta \mathrm{d}x = \int_{\Omega} |Xw|^{p-2} \sum_{j=1}^{N} X_j w X_j \eta \mathrm{d}x.
\]

As for the right-hand side of (2.14), we have

\[
\int_{\Omega} f \frac{\eta}{(M + \kappa - u^+)^{p-1}} \mathrm{d}x \leq \int_{\Omega} |f| \frac{\eta}{\kappa^{p-1}} \mathrm{d}x.
\]

Thus joining two inequalities yields

\[
\int_{\Omega} \sum_{j=1}^{N} |Xw|^{p-2} X_j w X_j \eta \mathrm{d}x \leq \int_{\Omega} |f| \frac{\eta}{\kappa^{p-1}} \mathrm{d}x,
\]

and \( w \) is a weak subsolution of the equation (2.13).

Applying Lemma 2.1 to the function \( w \), we obtain

\[
\sup_{\Omega} w \leq C \left( \|w\|_{L^p(\Omega)} + \frac{1}{\kappa} \|f\|_{L^m(\Omega)}^{\frac{1}{m-1}} \right).
\]

Combining (2.12) with the Sobolev inequality we get

\[
\sup_{\Omega} w^+ \leq C(\Omega, X, m, p, q). \tag{2.15}
\]
From the definition of $w$ we get
\[ M + \kappa \leq e^C (M + \kappa - u^+). \]

Minimizing the right-hand side with respect to $x$ on $\Omega$, we get $M \leq \kappa (e^C - 1)$. That is
\[ \sup_{\Omega} u^+ \leq (e^C - 1) \| f \|_{L^\infty(\Omega)}. \]

The assertion follows.

We want to prove next that the assumption $m > \frac{Q}{p}$ in Theorem 1.1 is sharp. Let $X = \{ X_1, \ldots, X_N \}$ be a family of smooth vector fields satisfying Hörmander’s condition (1.1). Let $\alpha_1, \ldots, \alpha_n$ be positive integers, and introduce $\delta_R x = (R^{\alpha_1} x_1, \ldots, R^{\alpha_n} x_n)$, for $R > 0$, $x \in \mathbb{R}^n$ and $Q = \alpha_1 + \cdots + \alpha_n$ the homogeneous dimension attached to $\{ \delta_R \}$. We say that the vector fields $\{ X_j \}_{j=1}^N$ are dilation invariant with respect to $\{ \delta_R \}$ if the following homogeneity property is satisfied:
\[ X_j(\delta_R u)(x) = R(X_j u)(\delta_R x), \]
for each smooth function $u$, where $\delta_R u(x) = u(\delta_R x)$. See [9] and [10].

Suppose $p > \frac{Q}{Q - 1}$ and $X_j^* = -X_j$. Let $L_p$ denote the formally self-adjoint operator
\[ L_p u = \sum_{j=1}^N X_j(|X u|^{p-2} X_j u). \]

We use an adaptation of the classical log $|\log|$ example to our setting and seek an unbounded function $u \in \dot{S}^{1,p}(\Omega, X)$, weak solution to $L_p u = h$, with $h \in L^{\frac{Q}{p}}(\Omega)$, and then the optimality of condition $m > \frac{Q}{p}$ in Theorem 1.1 follows.

Let $\rho: \mathbb{R}^n \setminus \{ 0 \} \rightarrow \mathbb{R}$ be a positive and $C^\infty$ function homogeneous of degree one with respect to dilations $\delta_R x$. As in [4], we define
\[ \rho(x) = \left[ (x_1^2)^{\frac{\alpha_1}{p}} + \cdots + (x_n^2)^{\frac{\alpha_n}{p}} \right]^{\frac{1}{2p}}, \]
where $s$ is the least common multiple of $\alpha_1, \ldots, \alpha_n$.

Let $\Omega$ be the open set $\Omega := \{ x \in \mathbb{R}^n | \rho(x) < \frac{1}{2} \}$ and choose for $x \in \overline{\Omega} \setminus \{ 0 \}$, $u(x) = f(\rho(x)) - f(\frac{1}{2})$, with $f(s) = \ln |\ln s|$. Consequently, $u \in C^\infty(\overline{\Omega} \setminus \{ 0 \})$, $u$ is unbounded in $\Omega$, $u = 0$ on $\partial \Omega$, $X_j u = f'(\rho) X_j \rho$, and
\[ L_p u = (-1)^p \{ (p - 1)f''(\rho)[f'(\rho)]^{p-2}|X \rho|^p + [f'(\rho)]^{p-1}L_p \rho \} := h. \]

In order to get the result, we have to check that
\begin{enumerate}
  \item[(I)] $u \in L^p(\Omega)$;
  \item[(II)] $X_j u \in L^p(\Omega)$, for $j = 1, \ldots, N$; and
  \item[(III)] $h \in L^{\frac{Q}{p}}(\Omega)$
\end{enumerate}

The following known proposition that follows from the homogeneity of $\rho$, is needed.
Proposition 2.1 Let \( g : (0, \frac{1}{2}] \to \mathbb{R} \) be a continuous function such that \( r^{Q-1}g(r) \) is in \( L^1((0, \frac{1}{2}]) \). Then \( g(\rho(x)) \) belongs to \( L^1(\Omega) \), and
\[
\int_{\Omega} g(\rho(x)) \, dx = \omega_Q \int_0^{1/2} r^{Q-1}g(r) \, dr,
\]
where \( \omega_Q \) is a positive constant.

By using this proposition, the claim (I) immediately deduces by treating \( g(r) = f^p(r) = (\ln |\ln r|)^p \).

To prove (II), let us notice that the functions \( X_j \rho \), \( j = 1, \ldots, N \) are bounded since they are smooth away from the origin and homogeneous of degree zero with respect to \( \delta_R \). Then
\[
|X_j u| = |f'(\rho)||X_j \rho| \leq C |f'(\rho)|
\]
and again from Proposition 2.1
\[
\int_{\Omega} |X_j u|^p \, dx \leq C \omega_Q \int_0^{1/2} r^{Q-1}|f'(\rho)|^p \, dr < \infty
\]
by considering \( |f'(\rho)|^p = |r \ln r|^{-p} \).

Finally, we investigate (III). We see that the functions \( |X \rho|^2 \) and \( \rho L_p \rho \) are both smooth away from the origin and homogeneous of degree zero with respect to \( \delta_R \). Then, for suitable positive constant \( C \) and \( C_1 \),
\[
|h| \leq C \left( |f''(\rho)||f'(\rho)|^{p-2} + \frac{|f'(\rho)|^{p-1}}{\rho} \rho L_p \rho \right)
\]
\[
\leq C \left( \frac{1}{\rho^2 |\ln \rho|} \frac{1}{(\rho |\ln \rho|)^{p-2}} + \frac{1}{\rho} \frac{1}{|\rho \ln \rho|^{p-1}} \right)
\]
\[
\leq C_1 \left( \frac{1}{\rho^2 |\ln \rho|^{p-2}} \right).
\]
Since \( \frac{Q}{p} > 1 \), we have
\[
\int_{\Omega} |h|^\frac{Q}{p} \, dx \leq C \int_0^{1/2} \frac{1}{r |\ln r|^{\frac{Q}{p}(p-1)}} < \infty.
\]

3. A Harnack Inequality for Homogeneous Equations

In this section we consider the Harnack inequality for the weak solution of homogeneous equations (1.2). The following lemma is important in our proof.

**Lemma 3.1** There is a real number \( R_0 > 0 \) so that d-ball \( B(x,t) \subset \subset \Omega \) (\( t \leq R_0 \) and \( 0 < s < t \)) and there is a function \( \psi \in C_0^\infty(B(x,t)) \) that has the properties: \( 0 \leq \psi \leq 1 \) and \( |X \psi| \leq C \) on \( B(x,t) \subset \subset \Omega \); \( \psi \equiv 1 \) on \( B(x,s) \) which is in d-ball \( B(x,t) \). Here \( C > 0 \) is a constant independent of \( s \) and \( t \).
We refer to [7] for the proof.

**Proof of Theorem 1.2**

If there is $R_0 > 0$ such that $u \equiv 0$ in $B(x, R_0)$, the equation (1.11) comes into existence directly. So we just need to prove the theorem under the assumption that $u \not\equiv 0$ in $B(x, R_0)$ for any $R_0 > 0$. In this case, the measure of $\{u = 0\}$ is vanishing.

Let $\eta \in \dot{S}^{1,p}(\Omega)$ be a cut-off function and choose $\varphi = \eta^p u^\beta$, where $\beta \in \mathbb{R}$, $\beta \neq 0$ and $\varphi \in \dot{S}^{1,p}(\Omega)$. Then

$$X_j(\eta^p u^\beta) = p\eta^{p-1} u^\beta X_j \eta + \beta \eta X_j u^\beta X_j u.$$

Using (1.9) we get

$$\int_{\Omega} \beta \eta^{p-1} u^\beta |Xu|^p dx = -\int_{\Omega} p\eta^{p-1} u^\beta |Xu|^{p-2} \sum_{j=1}^{N} X_j u X_j \eta dx.$$  

Noticing that $\beta \neq 0$, by applying Hölder inequality we have

$$\int_{\Omega} |\eta u^\beta X u|^p dx \leq |\beta| \int_{\Omega} \eta^{p-1} u^\beta |Xu|^{p-1} |\eta| dx$$

$$= \frac{|\beta|}{p} \int_{\Omega} |\eta u^\beta X u|^{p-1} |u^\frac{1}{p} - 1| \eta X \eta dx. \tag{3.1}$$

If $\beta \neq -p + 1$, let $w = u^\frac{1}{p} - 1$, then $X w = \frac{\beta + p - 1}{p} u^\frac{1}{p} - 1 X u$ and (3.1) becomes

$$\int_{\Omega} |\eta X w|^p dx \leq \frac{|\beta + p - 1|}{|\beta|} \int_{\Omega} |\eta X w|^p |w X \eta| dx \tag{3.2}$$

and therefore

$$\|\eta X w\|_{L^p(\Omega)}^p \leq \frac{|\beta + p - 1|}{|\beta|} \|\eta X w\|_{L^p(\Omega)}^{p-1} \|w X \eta\|_{L^p(\Omega)}.$$  

It shows

$$\|\eta X w\|_{L^p(\Omega)} \leq C_0 \|w X \eta\|_{L^p(\Omega)}, \tag{3.3}$$

where $C_0$ only depends on $\beta$ and $p$.

If $\beta = -p + 1$, we can choose $w = \ln u$ when $u > 0$ and $w = 0$ when $u = 0$, then $X w = u^{-1} X u$ a.e. $\Omega$. From (3.1) we get

$$\int_{\Omega} |\eta X w|^p dx \leq \frac{p}{|\beta|} \int_{\Omega} |\eta X w|^{p-1} |X \eta| dx. \tag{3.4}$$

Analogously, we have

$$\|\eta X w\|_{L^p(\Omega)} \leq \frac{p}{|\beta|} \|X \eta\|_{L^p(\Omega)}. \tag{3.5}$$

Especially, (3.3) and (3.5) still hold for $B_{4r}$. 
For (3.2), applying the inequality (1.12) to the function \( \eta w \), we get
\[
\| \eta w \|_{L^p_t(B_r)} \leq C r \| X(\eta w) \|_{L_t^p(B_{r+})} \leq C r \left( \| X^* \|_{L^p_t(B_{r+})} + \| \eta X w \|_{L^p_t(B_{r+})} \right)
\]
\[
\leq C r \left( 1 + \frac{\beta + p - 1}{|\beta|} \right) \| X^* \|_{L^p_t(B_{r+})},
\]
where \( \| f \|_{L^p_t(B_r)} = \left( \frac{1}{|B_r|} \int_{B_r} |f(s)|^p dx \right)^{\frac{1}{p}} \). By assuming \( \| f \|_{L^p_t(B_{r+})} = 1 \), we have chosen \( r \leq r_2 < r_1 \leq 2r \). The following proof is analogous to that in [8]p197. By assuming \( |\beta| \geq 1 \), we have \( C_0 = \frac{|\beta + p - 1|}{|\beta|} \leq |\beta + p - 1| \). (3.7) can be written as
\[
\| w \|_{L^p_t(B_{r+})} \leq C (1 + |\beta + p - 1|) \frac{r}{r_1 - r_2} \| w \|_{L^p_t(B_{r+})}.
\]
The following iteration progress is based on this inequality.

First applying \( w = u \frac{2^p - 1}{2^p} \) to (3.8), we get
\[
\left[ \frac{1}{|B_r|} \int_{B_r} \left( u \frac{2^p - 1}{2^p} \right)^q dx \right]^{\frac{1}{q}} \leq C (1 + |\beta + p - 1|) \frac{r}{r_1 - r_2} \left[ \frac{1}{|B_r|} \int_{B_r} \left( u \frac{2^p - 1}{2^p} \right)^p dx \right]^{\frac{1}{p}}
\]
and then
\[
\| u \|_{L^p_t(\beta + p - 1)(B_{r+})} \leq C \frac{2^p - 1}{2^p} (1 + |\beta + p - 1|) \frac{r}{r_1 - r_2} \left( \frac{r}{r_1 - r_2} \right)^{\frac{p}{\beta + p - 1}} \| u \|_{L^p_t(\beta + p - 1)(B_{r+})}.
\]
Letting \( \frac{2^p - 1}{2^p} = \theta \) with \( \theta \geq 1 \) and picking \( \beta > 0 \) such that \( \beta \geq 1 \) and \( \beta + p - 1 = \theta^k s \) \( (s > 1) \) in (3.9), we obtain
\[
\| u \|_{L^p_t(\beta + p - 1)(B_{r+})} \leq C \frac{2^p - 1}{2^p} (1 + \theta^k s)^{\frac{p}{\beta}} \left( \frac{r}{r_1 - r_2} \right)^{\frac{p}{\beta}} \| u \|_{L^p_t(\beta + p - 1)(B_{r+})}.
\]
Now we let \( r_k = r(1 + \frac{1}{2^k}) \) be a monotone decreasing sequence such that \( r_\infty = r \), \( r_0 = 2r \) and \( r_{k-1} - r_k = r(\frac{1}{2^{k-1}} - \frac{1}{2^k}) = \frac{1}{2^k} r \). Thus for every \( k \in \mathbb{N} \), we have
\[
\| u \|_{L^p_t(\beta + p - 1)(B_{r+k+})} \leq C \frac{2^p - 1}{2^p} (1 + \theta^k s)^{\frac{p}{\beta}} \left( \frac{r}{r_k - r_{k+1}} \right)^{\frac{p}{\beta}} \| u \|_{L^p_t(\beta + p - 1)(B_{r+k})}
\]
\[
\leq C \frac{2^p - 1}{2^p} \left( \frac{r}{r_k - r_{k+1}} \right)^{\frac{p}{\beta}} \left[ (1 + \theta^k s)^{\frac{p}{\beta}} (1 + \theta^{k-1} s)^{\frac{p}{\beta}} \cdots (1 + s)^{\frac{p}{\beta}} \right]^{\frac{1}{\beta}} \| u \|_{L^p_t(\beta + p - 1)(B_{r+k})}
\]
\[
\leq C \frac{2^p - 1}{2^p} \left( \sum_{n=0}^{k} (1 + \theta^n s)^{\frac{p}{\beta}} \right)^{\frac{1}{\beta}} \| u \|_{L^p_t(\beta + p - 1)(B_{r+k})}
\]
\[
\leq C \frac{2^p - 1}{2^p} \left( \sum_{n=0}^{k} (1 + \theta^n s)^{\frac{p}{\beta}} \right)^{\frac{1}{\beta}} \| u \|_{L^p_t(\beta + p - 1)(B_{r+k})}.
\]
Letting $k \to +\infty$ and noting $\sum_{n=0}^{+\infty} \frac{1}{\theta^{n}} = \frac{\theta}{\theta - 1}$, $\sum_{n=0}^{+\infty} \frac{n}{\theta^{n}} = \frac{\theta}{(\theta - 1)^{2}}$, 

$$\prod_{n=0}^{\infty} (1 + \theta^{n}s)^{\frac{1}{\theta^{n}}} = e^{\frac{1}{\theta} \sum_{n=0}^{\infty} \frac{1}{\theta^{n}} \ln(1 + \theta^{n}s) \leq e^{\frac{1}{\theta} \sum_{n=0}^{\infty} \frac{1}{\theta^{n}} \ln(\theta^{n}s)} = e^{\frac{1}{\theta} \left[ \frac{\theta \ln(\theta^{2}s)}{\theta - 1} + \frac{\theta \ln \theta}{(\theta - 1)^{2}} \right]}$$

we obtain 

$$\sup_{B_{r}} u \leq C\|u\|_{L^{\theta}(B_{2r})}, \quad (3.11)$$

where $C$ only depends on $p, q, s$.

Notice that $f(x) = x^{\alpha}$ is monotone decreasing on $\mathbb{R}^{+}$ if $\alpha < 0$ and select $\beta < -1$ such that $\beta + p - 1 < 0$. The equation (3.9) gives 

$$\|u\|_{L^{\beta+p-1}(\Omega)} \geq C_{\beta+p-1}^{\beta+p-1} \frac{p}{p-1} \left( \frac{r}{r_{1} - r_{2}} \right)^{\frac{p}{p-1}} \|u\|_{L^{\beta+p-1}(\Omega_{1})}. \quad (3.12)$$

Let $\theta = \frac{q}{p} > 1$ and $s' > 0$ such that $\forall k \in \mathbb{Z}^{+}, \beta + p - 1 = -\theta k s' < 0$. Plugging it into the above inequality, we obtain 

$$\|u\|_{L^{\beta+1,s'}(B_{r})} \geq C_{\beta+1,s'}^{\beta+1,s'} \left( 1 + \theta k s' \right)^{\frac{p}{\theta k s'}} \frac{p}{\theta k s'} \left( \frac{r}{r_{1} - r_{2}} \right)^{\frac{p}{\theta k s'}} \|u\|_{L^{\beta+1,s'}(B_{r})} \quad (3.13)$$

or 

$$C_{\beta+1,s'}^{\beta+1,s'} \left( 1 + \theta k s' \right)^{\frac{p}{\theta k s'}} \frac{p}{\theta k s'} \|u\|_{L^{\beta+1,s'}(B_{r})} \geq \|u\|_{L^{\beta+1,s'}(B_{r})}. \quad (3.14)$$

Letting $r_{k} = r(1 + \frac{2}{k})$, then using successively the above inequality and letting $k \to \infty$, we have 

$$C\|u\|_{L^{\beta}(\Omega)} \geq \|u\|_{L^{\beta}(\Omega)}, \quad (3.15)$$

and then 

$$\|u\|_{L^{\beta}(\Omega)} \leq C \inf_{B_{r}} u. \quad (3.16)$$

To finish the proof, we also need the following two inequalities for some nonnegative real number $s'$ sufficiently small,

$$\left( \frac{1}{|B_{3r}|} \int_{B_{3r}} u^{s'} dx \right)^{\frac{1}{s'}} \leq C \left( \frac{1}{|B_{3r}|} \int_{B_{3r}} u^{-s} dx \right)^{\frac{1}{s}}, \quad (3.17)$$

$$\left( \frac{1}{|B_{2r}|} \int_{B_{2r}} u^{s'} dx \right)^{\frac{1}{s'}} \leq C \left( \frac{1}{|B_{3r}|} \int_{B_{3r}} u^{s} dx \right)^{\frac{1}{s'}}. \quad (3.18)$$

From now on we denote $w = \ln u$. For any metric ball $B_{\rho}$ such that $B_{2\rho} \subseteq \Omega$, by Lemma 3.1, we choose a cut-function $\eta \in S^{1,p}(\Omega)$ satisfying the following properties : $\eta \equiv 1$ on $B_{\rho}$ and $\eta \equiv 0$ in $\Omega \setminus B_{2\rho}$; $|X\eta| \leq \frac{C_{\rho}}{\rho}$ on $B_{2\rho}$. Putting this $\eta$ into the inequality (3.5) and noting the doubling condition 

$$0 < |B_{2\rho}| \leq A|B_{\rho}|,$$
where $A$ is a constant independent of $\rho$, we get
\[ \frac{1}{|B_\rho|} \int_{B_\rho} |Xw|^p dx \leq \left( \frac{p}{|\beta|} \right)^p \frac{1}{|B_\rho|} \int_{B_{2\rho}} \left( \frac{C_1}{\rho} \right)^p dx \leq A \rho^{-p} \left( \frac{pC_1}{|\beta|} \right)^p. \] (3.19)

Using the following Poincaré inequality [11]
\[ \int_{B_\rho} |w - w_\rho|^p dx \leq C_{X,p} \rho^p \int_{B_\rho} |Xw|^p dx, \] (3.20)
we obtain
\[ \frac{1}{|B_\rho|} \int_{B_\rho} |w - w_\rho|^p dx \leq C_{X,p} \rho^p \frac{1}{|B_\rho|} \int_{B_\rho} |Xw|^p dx \leq C = C_{X,p} A \left( \frac{pC_1}{|\beta|} \right)^p, \] (3.21)
where $C$ is a constant only depending on $\{X_j\}, p, A$. By Theorem 0.3 and Theorem 0.4 of [12] (also see [13]) we know that for any ball $B_\rho$ such that $B_{4\rho} \subseteq \Omega$, the following John-Nirenberg type inequality holds
\[ \frac{1}{|B_{3\rho}|} \int_{B_{3\rho}} \exp(s'|w - w_\rho|) dx \leq M, \] (3.22)
where $w_\rho = \frac{1}{|B_{3\rho}|} \int_{B_{3\rho}} w dx$ and $M$ is a constant. Since
\[ \frac{1}{|B_{3\rho}|} \int_{B_{3\rho}} \exp(s'|w - w_\rho|) dx \geq \frac{1}{|B_{3\rho}|} \int_{B_{3\rho}} \exp(s'w - s'w_\rho) dx, \]
\[ \frac{1}{|B_{3\rho}|} \int_{B_{3\rho}} \exp(s'|w - w_\rho|) dx \geq \frac{1}{|B_{3\rho}|} \int_{B_{3\rho}} \exp(s'w_\rho - s'w) dx, \]
we have
\[ \int_{B_{3\rho}} u^{-s'} dx \leq \frac{1}{|B_{3\rho}|} \int_{B_{3\rho}} \exp(-s'w) dx \leq \frac{1}{|B_{3\rho}|} \int_{B_{3\rho}} \exp(s'w) dx \]
\[ = \frac{1}{|B_{3\rho}|} \int_{B_{3\rho}} \exp(s'w - s'w_\rho) dx \frac{1}{|B_{3\rho}|} \int_{B_{3\rho}} \exp(s'w_\rho - s'w) dx \]
\[ \leq \left( \frac{1}{|B_{3\rho}|} \int_{B_{3\rho}} \exp(s'|w - w_\rho|) dx \right)^2 \leq M^2. \]
This proves (3.17).

The proof of (3.18) is analogous to that of (3.11). In fact, we take $r_n = r \left( 3 - \frac{n}{k+1} \right)$ for any fixed $k$. Thus $r_0 = 3r$, $r_{k+1} = 2r$, and $r_n - r_{n+1} = \frac{r}{k+1}$. By iterating (3.10), we get
\[ \| u \|_{L^{p_{k+1}}(B_{r_{k+1}})} \leq C_{\rho} \left( 1 + \theta^k \tau \right)^{\frac{p}{|\beta|} \tau} (k + 1)^{\frac{p}{|\beta|} \tau} \| u \|_{L^{p_{k+1}}(B_{r_k})} \]
\[ \leq C \left( 1 + \theta^k \tau \right)^{\frac{p}{|\beta|} \tau} (k + 1)^{\frac{p}{|\beta|} \tau} \left( 1 + \theta^{k-1} \tau \right)^{\frac{p}{|\beta|} \tau} \cdots (1 + \tau)^{\frac{p}{|\beta|} \tau} \| u \|_{L^2(B_{r_0})}. \]
Particularly, taking $\tau = s' = \frac{1}{\theta k+1} s$ and letting $k$ sufficiently large, we have that (3.18) holds.

Thus the statement of Theorem 1.2 follows from (3.11), (3.17), (3.18) and (3.16).

**Remark**  As a consequence, we have the following Liouville type theorem:

If $u$ is a nonnegative global solution of (1.2), then $u$ is a constant.

In fact, we can assume $\inf u = 0$, otherwise it suffices to consider the function $v = u - \inf u$ which is still a weak solution of (1.2). Hence, for any $\varepsilon > 0$, $u(x_0) < \varepsilon$ for some $x_0$. We have from (1.11) that $u < C\varepsilon$ for all $x \in B_R(x_0)$. Since $C$ is a constant independent of $R$, it follows that $u(x) < C\varepsilon$ for all $x \in \mathbb{R}^n$, and the conclusion is immediately obtained by letting $\varepsilon \to 0$.

**References**