A NOTE ON $L^2$ DECAY OF LADYZHENSKAYA MODEL

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Abstract This paper is concerned with time decay problem of Ladyzhenskaya model governed incompressible viscous fluid motion with the dissipative potential having $p$-growth ($p \geq 3$) in $\mathbb{R}^3$. With the aid of the spectral decomposition of the Stokes operator and $L^p - L^q$ estimates, it is rigorously proved that the Leray-Hopf type weak solutions decay in $L^2(\mathbb{R}^3)$ norm like $t^{-\frac{n}{2}(\frac{1}{r} - \frac{1}{2})}$ under the initial data $u_0 \in L^2(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$ for $1 \leq r < 2$. Moreover, the explicit error estimates of the difference between Ladyzhenskaya model and Navier-Stokes flow are also investigated.

Key Words Ladyzhenskaya model; $L^2$ decay; spectral decomposition.

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1. Introduction

Consider the viscous incompressible fluid motion governed by the following momentum and continuity equations

$$
\partial_t u + (u \cdot \nabla)u - \nabla \cdot \tau^v + \nabla \pi = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty),
$$

$$
\nabla \cdot u = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty)
$$

(1.1)

(1.2)

together with the boundary and initial conditions

$$
\lim_{|x| \to \infty} u(x, t) = 0 \quad \text{in } (0, \infty),
$$

$$
u(x, 0) = u_0 \quad \text{in } \mathbb{R}^3.
$$

(1.3)

(1.4)

Here, the gradient $\nabla = (\partial_{x_1}, \ldots, \partial_{x_3})$, $u = (u_1, \ldots, u_3)$ and $\pi$ denote the unknown velocity and pressure of the fluid motion at the point $(x, t) \in \mathbb{R}^3 \times (0, \infty)$, respectively, while $u_0$ is the given initial velocity vector field. For simplicity, we assume that the external force has a scalar potential and it is included into the pressure gradient. $\tau^v = (\tau^v_{ij})$ is the stress tensor specified in the following form

$$
\tau^v_{ij} = 2\left(\mu_0 + \mu_1|e(u)|^{p-2}\right) e_{ij}(u)
$$

(1.5)
for the symmetric deformation velocity tensor \( e(u) = (e_{ij}(u)) \) with

\[
e_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad |e(u)| = (e_{ij}(u)e_{ij}(u))^{\frac{1}{2}}
\] (1.6)

where the viscosities \( \mu_0 > 0 \) and \( \mu_1 \geq 0 \).

When \( \mu_1 = 0 \), the Stokes Law

\[
\tau_{ij}^v = 2\mu_0 e_{ij}(u)
\] (1.7)

holds true. The fluids, such as water and alcohol, satisfying the linear equation expressed by (1.7) are said to be Newtonian, and (1.1) turns out to be the Navier-Stokes equations (refer to [1] for details), whereas the nonlinear constitutive equation expressed by (1.5) with \( \mu_1 > 0 \) is related to other non-Newtonian fluids such as the molten plastics, dyes, adhesives, paints and greases. Equations (1.1)-(1.6) with \( \mu_1 > 0 \) were first proposed by Ladyzhenskaya [2] and have been known as the Ladyzhenskaya model which may be justified through a variety of physical and mathematical arguments. Additionally, the constitutive equation expressed by (1.5) is defined by the physical qualities of a fluid and is also called Ellis fluids model when \( p > 2 \) (refer to Chapter 2 of [3]).

There is extensive literature on the large time behavior of the viscous incompressible fluid flows. On the one hand, as for the Navier-Stokes equations, the decay problem of weak solutions was first proposed by Leray [4]. Schonbek [5] and Wiegner [6] introduced Fourier splitting methods and obtained time decay rates with respect to the whole spaces \( \mathbb{R}^n \). Kajikiya and Miyakawa [7] provided a spectral decomposition approach of the Stokes operator and also derived time decay rates in \( \mathbb{R}^n \). One may also refer to the study of He et al [8, 9] relating to the decay properties for strong solutions of Navier-Stokes equations.

On the other hand, for Ladyzhenskaya model governed incompressible viscous non-Newtonian fluid motions, the existence of weak solutions was obtained by Ladyzhenskaya [2] and J. L. Lions [10] for \( p \geq \frac{11}{5} \), and more recently, Du and Gunzburger [11] have studied the somewhat more general existence and uniqueness results in a bounded domains. Pokorny [12] investigated the Cauchy problem for this model in whole spaces. We also refer to the work of [13-15] to the nonlinear multipolar viscous fluids. Additionally, with the aid of Fourier splitting method [5], the time decay problem of Ladyzhenskaya model was recently examined by Necášová and Penel [16] for logarithmic decay in \( \mathbb{R}^2 \) and algebraic decay in \( \mathbb{R}^3 \). Guo and Zhu [17] improved the algebraic decay results in \( \mathbb{R}^n (n \geq 2) \) by the modification of Fourier splitting method [6], more precisely, when \( u_0 \in L^2(\mathbb{R}^n) \cap L^r(\mathbb{R}^n) \) for \( 1 \leq r < 2 \), they have obtained the weak solutions decay as follows

\[
\|u(t)\|_{L^2} \leq c(1 + t)^{-\frac{3}{2}(\frac{1}{r} - \frac{1}{2})}, \quad \|u(t) - e^{t\Delta}u_0\|_{L^2} \to 0, \quad t \to \infty.
\] (1.8)
By improving Schonbek’s Fourier splitting method, the optimal algebraic decay rate in $\mathbb{R}^2$ is obtained by Dong and Li [18] in the following form when $u_0 \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$

$$
\|u(t)\|_{L^2} \leq c(1 + t)^{-\frac{1}{2}}, \quad \|u(t) - e^{t\Delta}u_0\|_{L^2} \leq c(1 + t)^{-\frac{3}{4}}, \quad \forall \ t > 0.
$$

(1.9)

However, it is to be mentioned that the decay results of [17] and [16] are stated for the arbitrary weak solution. But their proofs obviously require some additional smoothness of such solution, especially in the high dimensional case (in the two-dimensional case, the difficulty can be easily overcome by the fact that weak solutions are indeed classic solutions). In fact, in the original paper of Schonbek [5], the mentioned difficulty was overcome by using the smooth approximate solution constructed by Caffarelli, Kohn and Nirenberg [19] and derivation of the number of estimates independent of the parameter of regularization. Hereby, the decay results of [5] hold not for any weak solution but only for the solutions which can be obtained by the smooth appropriate approximations. As this sort Ladyzhenskaya model (unlike the Navier-Stokes equations) has the nonlinear principal part, the possibility of such a regularization seems not to be quite evident and (if it is true) it should be explained.

The present paper focuses on the rigorous analysis to the time decay problem of Ladyzhenskaya model with the dissipative potential having $p$-growth with $p \geq 3$ in $\mathbb{R}^3$. By using the $L^p - L^q$ estimates of the Stokes flows and the spectral decomposition and fractional powers of the Stokes operator (see Kajikiya and Miyakawa [7]), we obtain that this sort non-Newtonian flow has the same time decay rates as Navier-Stokes flow. That is, the approximate solution $u_k(t)$ of (1.1)-(1.6) which is constructed like Caffarelli, Kohn and Nirenberg [19] enjoys the optimal algebraic decay estimates,

$$
\|u_k(t)\|_{L^2} \to 0, \quad t \to \infty, \text{ when } u_0 \in L^2,
$$

$$
\|u_k(t)\|_{L^2} \leq ct^{-\frac{n}{2} \left(\frac{2}{r} - \frac{1}{2}\right)}, \quad \text{when } u_0 \in L^2 \cap L^r, \quad 1 \leq r < 2.
$$

(1.10)

and since approximate solutions $u_k$ converge strongly in $L^2_{loc}(\mathbb{R}^3 \times (0, \infty))$ to the Leray-Hopf type weak solutions $u(t)$ and $\|u(t)\|_2$ is lower semi-continuous, therefore, the time decay of $u_k$ implies the same decay of $u$. Moreover, to understand the asymptotic behavior relation between the Ladyzhenskaya fluid model $u(t)$ and the Newtonian flow $\tilde{u}(t)$ ( or the Navier-Stokes flow expressed by (1.1)-(1.6) with $p = 2$), we investigated the optimal algebraic decay rates of the difference $u(t) - \tilde{u}(t)$. The main difficulty in the study of the sort non-Newtonian fluid motion system is to control nonlinear viscous term $\nabla \cdot (|e(u)|^{p-2}e(u))$. When $p \geq 3$, this nonlinear term can be bounded by the energy estimate of (1.1)-(1.6) with the aid of the Gagliardo-Nirenberg inequality (refer to [20]). However, for the case $2 < p < 3$, this approach is failed and time decay problem of (1.1)-(1.6) remains unsolved with the large initial data.

The remains of this paper are organized as follows. In Sections 2 we study convergence of approximate solutions. Decay estimates of the non-Newtonian flow are described in Section 3, whereas the error estimates of the difference between non-Newtonian and Newtonian flows $u(t) - \tilde{u}(t)$ are derived in Section 4.
2. Convergence of Smooth Approximate Solutions

Let \( 1 \leq q \leq \infty, \| \cdot \|_q = \| \cdot \|_{L^q} \) be the norm of the usual scalar and vector Lebesgue space \( L^q(\mathbb{R}^3) \) and \( \| \cdot \|_{m,q} = \| \cdot \|_{W^{m,q}(\mathbb{R}^3)} \) be the norm of the Sobolev space \( W^{m,q}(\mathbb{R}^3) \). The space \( L^q_\sigma(\mathbb{R}^3) \) denotes the \( L^q \)-closure of \( C_0^\infty(\mathbb{R}^3) \), which is the set of smooth divergence-free vector fields with compact supports in \( \mathbb{R}^3 \), in particular, let \( H = L^2_\sigma(\mathbb{R}^3), V = W^{1,2}_\sigma(\mathbb{R}^3), V_q = W^{1,q}_\sigma(\mathbb{R}^3) \) and \( V', V'_q \) be the dual spaces of \( V, V_q \).

Without loss of generality, we assume that \( \mu_0 = \mu_1 = 1 \) in (1.5). Substitution of (1.5) into (1.1) produces

\[
\begin{align*}
\partial_t u - \Delta u + (u \cdot \nabla)u - \nabla \cdot (|e(u)|^{p-2}e(u)) + \nabla \pi &= 0, \\
(2.1)
\end{align*}
\]

By a Leray-Hopf type weak solution of the Cauchy problem (1.2)-(1.4) and (2.1) with \( u_0 \in H \) and \( p \geq 3 \), we mean an \( \mathbb{R}^3 \)-value vector field (refer to Ladyzhenskaya [2])

\[
\begin{align*}
u \in L^p(0; T; V_p) \cap L^\infty(0; T; H) \cap L^2(0; T; V), \quad \forall T > 0
\end{align*}
\]

satisfying

\[
\begin{align*}
- \int_0^\infty \int_{\mathbb{R}^3} u \cdot \frac{\partial \varphi}{\partial s} dxds + \int_0^\infty \int_{\mathbb{R}^3} (1 + |e(u)|^{p-2})e_{ij}(u) \cdot e_{ij}(\varphi) dxds \\
+ \int_0^\infty \int_{\mathbb{R}^3} u_k \frac{\partial u_k}{\partial x_j} \varphi_i dxds = \int_{\mathbb{R}^3} u_0 \cdot \varphi(0) dx
\end{align*}
\]

for every \( \varphi \in C^1([0, \infty); L^3 \cap V \cap V_p) \).

The global existence of the Leray-Hopf type weak solutions of Ladyzhenskaya equations has been studied by Ladyzhenskaya [2] and J. L. Lions [10]. In order to investigate the large time behavior of Ladyzhenskaya equations (1.2)-(1.4) and (2.1), we give one sort of their smooth approximate solutions constructed like Caffarelli, Kohn and Nirenberg [19] as those functions satisfy

\[
\begin{align*}
\frac{\partial u_k}{\partial t} + (w_k \cdot \nabla)u_k - \Delta u_k - \nabla \cdot (|e(w_k)|^{p-2}e(u_k)) + \nabla \pi &= 0, \\
\nabla \cdot u_k &= 0, \\
u_k(x, 0) &= u_0.
\end{align*}
\]

Here the retarded modification \( w_k \) of \( u_k \) is defined by

\[
w_k(x, t) = \delta^{-4} \int \int \psi(x-y, t-s) \tilde{u}_k(y, t-s) dyds, \quad k = \frac{T}{\delta}
\]

where \( \psi \in C^\infty_0(\mathbb{R}^3 \times (0, \infty)) \) and satisfies

\[
\psi \geq 0, \quad \int \int \psi dx dt = 1 \quad \text{and} \quad \text{supp } \psi \subset \{(x, t) : |x|^2 < t, 1 < t < 2\},
\]

\[
\]
$\bar{u}_k$ is the zero extension of the function $u_k$ which is originally defined for $t \geq 0$. It is to be noted that $w_k$ at time $t$ clearly depends only on the values of $u_k$ at times $\tau \in (t - 2\delta, t - \delta)$, so for a fixed $\delta$, solving (2.3) amount to solving a linear equation on each strip $\mathbb{R}^3 \times (m\delta, (m + 1)\delta)$, $0 \leq m \leq k - 1$.

In order to prove the convergence of smooth approximate solution $u_k$, we need to some preliminary lemmas. Firstly, by the definition of $w_k$ and $\nabla \cdot u_k = 0$, we can immediately obtain the follow results with the similar arguments in the appendix of [19].

**Lemma 2.1** Assume $u_k \in L^p(0, T; V_p) \cap L^\infty(0, T; H) \cap L^2(0, T; V) \ (T > 0)$, we have

\[
\nabla \cdot w_k = 0; \\
\|u_k(t)\| \leq \text{esssup}_{0 < s < t}\|u_k(s)\| \leq \|u_0\|, \ \forall \ t \in (0, T); \\
\|w_k\|_{L^2(\mathbb{R}^3 \times (0, T))} \leq c\|u_k\|_{L^2(\mathbb{R}^3 \times (0, T))}; \\
\|\nabla w_k\|_{L^2(\mathbb{R}^3 \times (0, T))} \leq c\|\nabla u_k\|_{L^2(\mathbb{R}^3 \times (0, T))}; \\
\|\nabla w_k\|_{L^p(\mathbb{R}^3 \times (0, T))} \leq c\|\nabla u_k\|_{L^p(\mathbb{R}^3 \times (0, T))}.
\] (2.4)

**Lemma 2.2** For any $u_0 \in H$, there exists a unique solution $u_k$ of (2.3) which satisfies

\[u_k \in L^p(0, T; V_p) \cap L^\infty(0, T; H) \cap L^2(0, T; V), \ \forall \ T > 0 \] (2.5)

and

\[-\int_0^\infty \int_{\mathbb{R}^3} u_k \cdot \frac{\partial \phi}{\partial s} \, dx \, ds + \int_0^\infty \int_{\mathbb{R}^3} (1 + |e(w_k)|^{p-2})e_{ij}(u_k) \cdot e_{ij}(\phi) \, dx \, ds \\
+ \int_0^\infty \int_{\mathbb{R}^3} (w_k \cdot \nabla u_k) \cdot \phi \, dx \, ds = \int_{\mathbb{R}^3} u_0 \cdot \phi(0) \, dx \] (2.6)

for every $\phi \in C^1([0, \infty) ; L^3 \cap V \cap V_p)$. Moreover, the approximate solution $u_k$ possesses the smooth properties $u_k \in C^1((0, T); H) \cap C([0, T); H)$ and therefore satisfies

\[
\frac{d}{dt}\|u_k(t)\|^2 = 2(\dot{u}_k(t), u_k(t)). \] (2.7)

**Proof** The existence of the unique approximate $u_k$ in the sense of (2.5)-(2.6) may be proved by the standard Faedo-Galerkin method, the argument is somewhat parallel to the proof of Theorem 1.1 in Chapter 3 of [21] and here we omit the details. Furthermore, take the divergence of (2.3) to obtain the expression of the pressure $p_k$

\[-\Delta p_k = \frac{\partial^2}{\partial x_i \partial x_j} \left( w^i_k u^j_k - |e(w_k)|^{p-2}e_{ij}(u_k) \right) \]
and so
\[ p_k = R_i R_j \left( w_k^i w_k^j - |e(w_k)|^{p-2} e_{ij}(u_k) \right); \quad (2.8) \]
\[ \frac{\partial p_k}{\partial x_i} = R_i R_j \left( (w_k \cdot \nabla) u_k - \nabla \cdot ([|e(w_k)|^{p-2} e(u_k))] \right). \quad (2.9) \]

where \( R_i, i = 1, \ldots, n \) are the Riesz transforms and the strong \((q,q)\) type operators for \( 1 < q < \infty \) \([22]\). We denote by \( P \) the projection operator from \( L^q \) onto \( L^q_\sigma \) and apply \( P \) to (2.3) to obtain formally the abstract Cauchy problem as follow
\[
\frac{\partial u_k}{\partial t} + A u_k + B(u_k) = 0, \quad u_k(0) = u_0. \quad (2.10)
\]

where \( A \) is the stokes operator defined as
\[
A u = -P \Delta u, \quad \text{for } u \in D(A) \equiv W^{2,2} \cap V \cap V_p
\]

and the nonlinear term
\[
B(u_k) = P(w_k \cdot \nabla) u_k - P \nabla \cdot ([|e(w_k)|^{p-2} e(u_k)]). \quad (2.11)
\]

Because the stokes operator \(-A\) generates an analytic semigroup \( \{e^{-At}; t \geq 0\} \) in \( L^q_\sigma \) and by the smoothness and bound of \( w_k \), we can easily obtain that \( u_k \in C^1((0,T); H) \cap C([0,T]; H) \) and \( Au_k \in C((0,T); H) \). In particular, we have
\[
\frac{d}{dt} \|u_k(t)\|^2 = 2(u'_k(t), u_k(t)). \quad (2.12)
\]

The proof of the lemma is completed.

**Theorem 2.1** For any \( u_0 \in H \), and \( p \geq 3 \), the smooth approximate solution \( u_k \) converges to the Leray-Hopf type weak solution \( u \) of Ladyzhenskaya fluid model (1.2)-(1.4) and (2.1) in the sense of (2.2).

**Proof** Because \( u_k \in L^p(0,T; V_p) \cap L^\infty(0,T; H) \cap L^2(0,T; V) \) and \( u_k \in C^1((0,T); H) \cap C([0,T]; H) \),
\[
u_k \to u, \quad \text{weakly in } L^p(0,T; V_p);
u_k \to u, \quad \text{weakly star in } L^\infty(0,T; H);
u_k \to u, \quad \text{weakly in } L^2(0,T; V). \quad (2.13)
\]

As \( k \to \infty \), the proof of (2.6) convergence to (2.2) is obvious besides the following limits for every \( \varphi \in C^1([0,\infty); L^3 \cap V \cap V_p)\)
\[
\int_0^\infty \int_{\mathbb{R}^3} (w_k \cdot \nabla u_k) \cdot \varphi \, dx ds \to \int_0^\infty \int_{\mathbb{R}^3} u_j \frac{\partial u_i}{\partial x_j} \varphi_i \, dx ds; \quad (2.14)
\]
\[
\int_0^\infty \int_{\mathbb{R}^3} |e(w_k)|^{p-2} e_{ij}(u_k) \cdot e_{ij}(\varphi) \, dx ds \to \int_0^\infty \int_{\mathbb{R}^3} |e(u)|^{p-2} e_{ij}(u) \cdot e_{ij}(\varphi) \, dx ds. \quad (2.15)
\]
The proof of (2.15) is parallel to the proof of Theorem 4.2 of [11]. In order to prove (2.14), we need $u_k$ converges strongly in $L^2_{\text{loc}}(\mathbb{R}^3 \times (0, \infty))$. In fact $u_k$ stay bounded in $L^{\infty}(0, T; H) \cap L^{2+\frac{1}{2}}(\mathbb{R}^3 \times (0, T))$ by the interpolation and Lemma 2.1. Furthermore, (2.8) implies $p_k \in L^{1+\frac{2}{3}}(\mathbb{R}^3 \times (0, T; H^{-m}))$ for $m \geq \frac{3}{2}$ and $\Delta u_k \in L^2(0, T; H^{-m}) \subset L^{1+\frac{2}{3}}(0, T; H^{-m})$. Therefore, (2.3) implies $\frac{\partial u_k}{\partial t} \in L^{1+\frac{2}{3}}(0, T; H^{-m})$, thus by the compactness theorem stated in Theorem 2.1, Chapter 3 of [21], we conclude that $u_k$ is precompact in $L^2_{\text{loc}}(\mathbb{R}^3 \times (0, \infty))$, i.e. there is a subsequence denoted also by $u_k$ satisfying

$$u_k \rightarrow u, \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^3 \times (0, \infty)).$$

Moreover, because $w_k$ is also precompact in $L^2_{\text{loc}}(\mathbb{R}^3 \times (0, \infty))$ and converges strongly to $u$ in $L^2_{\text{loc}}(\mathbb{R}^3 \times (0, \infty))$ and together with $u_k$ weakly converges to $u$ in $L^2(0, \infty; V)$, we can obtain (2.14) and the proof is completed.

3. Decay Estimates of Ladyzhenskaya Model

It is to be noted that if $u_k$ have some decay properties $\|u_k(t)\| \leq g(t)$, then $\|u(t)\| \leq g(t)$ except possible points in a measure zero set. We know that the weak solution $u(t)$ obtained as the limit of $\{u_k\}$ is in $C_w([0, \infty), H)$ in the sense of weak topology and $\|u(t)\|$ is lower semi-continuous, i.e.

$$\|u(t)\| \leq \liminf_{k \rightarrow \infty} \|u_k(t)\|.$$  

Therefore, our proofs for time decay of Ladyzhenskaya model are based on the estimate of smooth approximate solution $u_k$. More precisely, we only need to investigate the decay properties of smooth approximate solution $u_k$ of (2.3).

Firstly we use the spectral decomposition of the Stokes operator which was studied by Kajikiya and Miyakawa [7],

$$A = \int_{0}^{\infty} \lambda \, dE(\lambda),$$

and it is easy checked that

$$\|E(\lambda)u\|^2 = \int_{|\xi|^2 \leq \lambda} |\hat{u}(\xi, t)|^2 d\xi.$$  

(3.1)

Since the stokes operator $-A$ generates an analytic semigroup $\{e^{-At}; t \geq 0\}$ in $L^q_H$, the fractional powers of the Stokes operator can be defined as follows

$$A^\alpha = \int_{0}^{\infty} \lambda^\alpha \, dE(\lambda), \quad \alpha > 0.$$  

(3.2)

Let us now recall some preliminary lemmas.
Lemma 3.1 Let either $1 < r \leq q < \infty$ or $1 \leq r < q \leq \infty$ and $u_0 \in L^r(\mathbb{R}^3)$. Then the $L^r - L^q$ estimate satisfies

$$\|D^\alpha e^{-A_t}u_0\|_q \leq ct^{-\frac{\alpha}{2} - \frac{3}{2}(\frac{1}{r} - \frac{1}{q})}\|u_0\|_r, \quad \alpha \geq 0. \quad (3.3)$$

Now using the spectral decomposition approach we obtain an estimate of the nonlinear operator $B(u_k)$.

Lemma 3.2 Suppose $u_k$ is a smooth approximate solution of (2.3), then

$$\|E(\lambda)B(u_k)\| \leq c\|w_k\|\|u_k\|^\frac{5}{2} + c\|\nabla w_k\|_{p-1}^2\|\nabla u_k\|_{p-1}\lambda^\frac{5}{2}, \quad \lambda > 0.$$

**Proof** Since the projection P on $L^q(\mathbb{R}^3)\ (1 < q < \infty)$ can commute with $E(\lambda)$, together with (3.1) and Lemma 2.1, we have

$$\|E(\lambda)B(u_k)\|^2 = \|PE(\lambda)((w_k \cdot \nabla)u_k - \nabla \cdot (|e(w_k)|^{p-2}e(u_k)))\|^2$$

$$\leq 2 \int_{|\xi|^2 \leq \lambda} |F[(w_k \cdot \nabla)u_k]|^2 d\xi + 2 \int_{|\xi|^2 \leq \lambda} |F[\nabla \cdot (|e(w_k)|^{p-2}e(u_k))]|^2 d\xi$$

$$\leq 2 \int_{|\xi|^2 \leq \lambda} |\xi|^2 |F[(w_k \otimes u_k)]|^2 d\xi + 2 \int_{|\xi|^2 \leq \lambda} |\xi|^2 |F[w_k]|^{p-2}d\xi$$

$$\leq c \int_{|\xi|^2 \leq \lambda} |\xi|^2 \|w_k\|^2 \|u_k\|^2 d\xi + c \int_{|\xi|^2 \leq \lambda} |\xi|^2 \|\nabla w_k\|_{p-1}^2 \|\nabla u_k\|_{p-1}^2 d\xi$$

$$\leq c\|w_k\|^2\|u_k\|^2\lambda^\frac{5}{2} + c\|\nabla u_k\|_{p-1}^2\|\nabla u_k\|_{p-1}^2\lambda^\frac{5}{2}.$$

We thus have the desired assertion and complete the proof.

As is well known that the weak solutions of Navier-Stokes equations have the optimal decay estimates in $\mathbb{R}^3$ (refer to [5-7]). In this section, we show that the approximate solution $u_k$ has the same optimal decay estimates with the Newtonian flow. Now the results read as follows.

**Theorem 3.1** Suppose $u_k$ is the smooth approximate solution of (2.3),

(i) If $u_0 \in H$, then $\|u_k(t)\| \to 0$ as $t \to \infty$.

(ii) If $u_0 \in H \cap L^r$, for some $1 \leq r < 2$, then

$$\|u_k(t)\| \leq ct^{-\frac{3}{2}(\frac{1}{r} - \frac{1}{2})}, \quad \forall \ t \geq 1.$$

We prove these results by borrowing the spectral decomposition approach due to Kajikiya and Miyakawa [7].

**Proof of Theorem 3.1** We multiply (2.10) by $u_k$ and use the property (2.7) together with $(P(w_k \cdot \nabla)u_k, u_k) = 0$ and $-P\nabla \cdot (|e(w_k)|^{p-2}e(u_k)), u_k) \geq 0$ to get

$$\frac{d}{dt}\|u_k(t)\|^2 + 2\|A^\frac{1}{2}u_k(t)\|^2 \leq 0. \quad (3.4)$$
To derive the lower bound of the second term on the left-hand side, we use (3.2) to deduce, for \( \rho > 0 \),

\[
\|A^{\frac{1}{2}} u_k(t)\|^2 = \int_0^\infty \lambda d\|E(\lambda)u_k(t)\|^2 \\
\geq \int_\rho^\infty \lambda d\|E(\lambda)u_k(t)\|^2 \\
\geq \rho \int_\rho^\infty d\|E(\lambda)u_k(t)\|^2 \\
\geq \frac{\rho}{2} \left( \|u_k(t)\|^2 - \|E(\rho)u_k(t)\|^2 \right)
\]

(3.5)

Hence we have

\[
\frac{d}{dt}\|u_k(t)\|^2 + \rho \|u_k(t)\|^2 \leq \rho \|E(\rho)u_k(t)\|^2.
\]

(3.6)

To estimate the right-hand side of (3.6), because the smooth approximate solution \( u_k \) is also the mild solution of (2.10), we consider the integral form of (2.10)

\[
u_k(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}B(u_k)ds,
\]

which is derived from (2.2) by choosing the test function \( \varphi(t) = e^{-At}u_0 \). Applying the operator \( E(\rho) \) to both sides and integrating by parts, we obtain

\[
E(\rho)u_k(t) = E(\rho)e^{-tA}u_0 + \int_0^t \int_0^\rho e^{-\lambda(t-s)}d(E(\lambda)B(u_k))ds \\
= E(\rho)e^{-tA}u_0 + \int_0^t \int_0^\rho e^{-\rho(t-s)}(E(\rho)B(u_k))ds \\
+ \int_0^t \int_0^\rho (t-s) e^{-\lambda(t-s)}E(\lambda)B(u_k)d\lambda ds.
\]

(3.7)

This together with Lemma 3.2 and Lemma 2.1 implies

\[
\|E(\rho)u_k(t) - E(\rho)e^{-tA}u_0\| \\
\leq \int_0^t \int_0^{\rho(t-s)}\|E(\rho)B(u_k)\||d\lambda ds + \int_0^t \int_0^\rho (t-s) e^{-\lambda(t-s)}\|E(\lambda)B(u_k)\||d\lambda ds \\
\leq c \int_0^t \int_0^{\rho(t-s)}\rho^{\frac{3}{2}} \left( \|w_k\|\|u_k\| + \|\nabla w_k\|_{p-1}^{p-2}\|\nabla u_k\|_{p-1} \right) ds \\
+ c \int_0^t \int_0^\rho (t-s) e^{-\lambda(t-s)}\left( \|w_k\|\|u_k\| + \|\nabla w_k\|_{p-1}^{p-2}\|\nabla u_k\|_{p-1} \right) d\lambda ds \\
\leq c \rho^{\frac{3}{2}} \int_0^t \left( \|w_k\|\|u_k\| + \|\nabla w_k\|_{p-1}^{p-2}\|\nabla u_k\|_{p-1} \right) ds \\
+ c \int_0^t \rho^{\frac{3}{2}} \left( \int_0^\rho (t-s) e^{-\lambda(s)}d\lambda \right) \left( \|w_k\|\|u_k\| + \|\nabla w_k\|_{p-1}^{p-2}\|\nabla u_k\|_{p-1} \right) ds \\
\leq c \rho^{\frac{3}{2}} \int_0^t \left( \|w_k\|\|u_k\| + \|\nabla w_k\|_{p-1}^{p-2}\|\nabla u_k\|_{p-1} \right) ds
\]
follows from (3.12) with
as
\[ t \]
Choosing \( \alpha \) for all \( n \), we have
Insert (3.10) to (3.6) to get, note that (3.11) by
Now we carry out the proof of Theorem 3.1 (ii) for
Since \( n \) is large and \( \alpha \) is chosen afterwards and multiply both sides of (3.11) by \( t^n \) to obtain
Choosing \( \alpha \) so that \( \alpha - \frac{9}{2} > 0 \) and integrating with respect to \( t \), we have
\[ \| u_k(t) \|^2 \leq c \alpha t^{-\alpha} \int_0^t s^{\alpha-1} \| e^{-sA} u_0 \|^2 ds + c(\alpha + 1 - \frac{3}{2})^{-1} \alpha^\frac{7}{2} t^{-\frac{5}{2}} + c(\alpha - 3 - \frac{3}{2})^{-1} \alpha^\frac{7}{2} t^{-\frac{5}{2}}. \]
Since \( n \geq 3 \) and \( \| e^{-tA} u_0 \|^2 \to 0 \) as \( t \to \infty \) for \( u_0 \in H \), we conclude that \( \| u_k(t) \| \to 0 \) as \( t \to \infty \) and complete the proof of Theorem 3.1 (i).
Now we carry out the proof of Theorem 3.1 (ii) for \( u_0 \in H \cap L^r (1 \leq r < 2) \). It follows from (3.12) with \( \alpha = 3 \) and Lemma 3.1 that
\[ \| u_k(t) \|^2 \leq c \left( t^{-(\frac{7}{2} - \frac{3}{2})} + t^{-\frac{5}{2}} + t^{-\frac{5}{2}} \right). \]
Since \(\frac{3}{r} - \frac{3}{2} < \frac{3}{2}\) \(< \frac{5}{2}\), the desired assertion is obtained if \(\frac{3}{r} - \frac{3}{2} \leq \frac{1}{2}\). It remains to consider the case \(\frac{3}{r} - \frac{3}{2} > \frac{1}{2} \geq \frac{1}{2}\). Hence (3.13) implies that
\[
\|u_k(t)\|^2 \leq ct^{-\frac{1}{2}}. 
\] (3.14)
Substituting (3.14) into (3.10), then substituting the resulting inequality into (3.6), we get
\[
\frac{d}{dt} \|u_k(t)\|^2 \leq c\left(\rho t^{-\left(\frac{3}{r} - \frac{3}{2}\right)} + \rho \frac{7}{2} t + \rho \frac{3}{2}\right).
\]
Let \(\rho = 3t^{-1}\) and then multiply the both sides of this equation by \(t^3\) to get
\[
\frac{d}{dt} (t^3 \|u_k(t)\|^2) \leq c\left(t^2 - \left(\frac{3}{r} - \frac{3}{2}\right) + t^2 + t^{-\frac{1}{2}}\right).
\]
Since \(1 \leq r < 2\) implies \(\frac{3}{r} - \frac{3}{2} \leq \frac{3}{2}\), we have
\[
\|u_k(t)\|^2 \leq c\left(t^{-\left(\frac{3}{r} - \frac{3}{2}\right)} + t^{-\frac{3}{2}} + t^{-\frac{1}{2}}\right) \leq ct^{-\left(\frac{3}{r} - \frac{3}{2}\right)}, \quad t \geq 1.
\]
and the desired estimate is obtained.

4. Error Estimates of Ladyzhenskaya Model and Navier-Stokes Equations

Obviously Ladyzhenskaya fluid model is modified from the Navier-Stokes equations. It is interest of showing the \(L^2\) decay estimates of the difference between \(u(t)\) and \(\tilde{u}(t)\), here \(u(t)\) denotes the Leray-Hopf type weak solution of the Ladyzhenskaya fluid model (1.2)-(1.4) and (2.1), whereas \(\tilde{u}\) denotes the weak solution of Navier-Stokes equations. As the arguments in the former section, we also only need to investigate the \(L^2\) decay estimates of \(u_k(t) - \tilde{u}(t)\).

The results read as follows.

**Theorem 4.1** (i) If \(u_0 \in H\), then
\[
\|u_k(t) - \tilde{u}(t)\| \leq ct^{-\frac{1}{2}}, \quad t \geq 1.
\]
(ii) If \(u_0 \in H \cap L^r(\mathbb{R}^3)\) with \(1 \leq r < 2\), then
\[
\|u_k(t) - \tilde{u}(t)\| \leq t^{-\frac{3}{r} - \frac{1}{2}}, \quad t \geq 1.
\]

**Remark 4.2** From the assertion (ii) of Theorem 4.1, it is easy to deduce that when \(u_0 \in H \cap L^r\) for some \(1 \leq r < 2\),
\[
\|u_k(t) - \tilde{u}(t)\| = o\left(t^{-\left(\frac{3}{r} - \frac{1}{2}\right)}\right), \quad as \ t \to \infty.
\]
Proof of Theorem 4.1  It is to be noted that the estimates of Theorem 4.1 with $u_k(t) - \tilde{u}(t)$ replaced by $e^{-tA}u_0 - \tilde{u}(t)$ also hold true (see Kajikiya and Miyakawa [7]). Thus from the inequality

$$\|u_k(t) - \tilde{u}(t)\| \leq \|u_k(t) - e^{-tA}u_0\| + \|e^{-tA}u_0 - \tilde{u}(t)\|,$$

we only need to prove the validity of the estimates of Theorem 4.1 with $u_k(t) - \tilde{u}(t)$ replaced by $u_k(t) - e^{-tA}u_0$.

Indeed, let $U(t) = u_k(t) - v(t)$ with $v(t) = e^{-tA}u_0$, because $u_0 \in H$, $v(t)$ is a smooth solution of heat equation, $U(t)$ has the same smooth properties of $u_k$ and $\nabla \cdot U = 0$, by (2.10), $U(t)$ satisfies the following abstract Cauchy problem

$$U_t + AU + B(u_k) = 0, \quad U(x, 0) = 0. \quad (4.1)$$

Similar to the estimation of (3.4), taking the scaler product of (4.1) with $U$ yields

$$\frac{d}{dt}\|U(t)\|^2 + 2\|A^\frac{1}{2} U(t)\|^2 + 2B(u_k, v) = 0 \quad (4.2)$$

where

$$B(u_k, v) = (P(w_k \cdot \nabla)u_k, U) - \left( P\nabla \cdot (|e(w_k)|^{p-2}e(u_k)), U \right)$$

$$= - ((w_k \cdot \nabla)U, u_k) + \left( |e(w_k)|^{p-2}e(u_k), e(U) \right)$$

$$= - ((w_k \cdot \nabla)U, v) - \left( |e(w_k)|^{p-2}e(u_k), e(v) \right) + \left( |e(w_k)|^{p-2}e(u_k), e(u_k) \right)$$

$$= - ((w_k \cdot \nabla)U, v) - \left( |e(w_k)|^{p-2}e(u_k), e(v) \right) + \|\nabla w_k\|_{p}^{2-2}\|\nabla u_k\|^2_{p}.$$

Here we have used $((w_k \cdot \nabla)U, U) = 0$. Equation (4.2) makes sense whenever the non-linear term $B(u_k, v)$ is integrable with respect to $t > 0$. The integrability is implied from (2.5) and the process of the following derivation

$$2 \left| ((w_k \cdot \nabla)U, v) + \left( |e(w_k)|^{p-2}e(u_k), e(v) \right) \right|$$

$$\leq 2\|w_k\| \|\|\nabla U\|\|v\|_\infty + 2\|\nabla w_k\|_{p-1}^{p-2}\|\nabla u_k\|_{p-1}^{p-2}\|\nabla v\|_\infty$$

$$= 2\|w_k\| \|A^\frac{1}{2} U\| \|v\|_\infty + 2\|\nabla w_k\|_{p-1}^{p-2}\|\nabla u_k\|_{p-1}^{p-2}\|\nabla v\|_\infty$$

$$\leq \|A^\frac{1}{2} U\|^2 + \|w_k\|^2 \|v\|_\infty^2 + 2\|\nabla w_k\|_{p-1}^{p-2}\|\nabla u_k\|_{p-1}^{p-2}\|\nabla v\|_\infty.$$
Thus from (4.3) we have
\[ \frac{d}{dt} \|U\|^2 + \|A^{\frac{1}{2}}U\|^2 \leq ct^{-\frac{3}{2}} \|u_0\|^2 \|w_k\|^2 + ct^{-\frac{3}{2} - \frac{1}{2}} \|u_0\|_r \|\nabla w_k\|_{p-2} \|\nabla u_k\|_{p-1}. \quad (4.4) \]

By the same proof of (3.5), we have
\[ \|A^{\frac{1}{2}}U\|^2 \geq \rho \left( \|U\|^2 - \|E(\rho)U\|^2 \right), \quad \forall \rho > 0, \]
which implies, by (4.4),
\[ \frac{d}{dt} \|U\|^2 + \rho \|U\|^2 \leq \rho \|E(\rho)U\|^2 + ct^{-\frac{3}{2}} \|w_k\|^2 + ct^{-\frac{3}{2} - \frac{1}{2}} \|\nabla w_k\|_{p-2} \|\nabla u_k\|_{p-1}. \quad (4.5) \]

In order to estimate the first term on the right-hand side of (4.5), we consider the integral equation of (4.1)
\[ U(t) = -\int_0^t e^{-(t-s)A}B(u_k)ds. \]

Applying the operator $E(\rho)$ to the both sides of the above formula and using Lemma 3.2 yield
\[ \|E(\rho)U(t)\| \leq c\rho^{\frac{7}{4}} \int_0^t \|u_k(s)\|^2 ds + c\rho^{\frac{5}{2}}. \quad (4.6) \]

The substitution of (4.6) into (4.5) gives
\[ \frac{d}{dt} \|U\|^2 + \rho \|U\|^2 \leq c\rho^{\frac{7}{4}} \left( \int_0^t \|u_k(s)\|^2 ds \right)^2 + c\rho^{\frac{5}{2}} + \rho \|E(\rho)U\|^2 + ct^{-\frac{3}{2}} \|w_k\|^2 + ct^{-\frac{3}{2} - \frac{1}{2}} \|\nabla w_k\|_{p-2} \|\nabla u_k\|_{p-1}. \quad (4.7) \]

Multiplying the both sides of (4.7) by $t^\alpha$ and letting $\rho = \alpha t^{-1}$ for $\alpha > \frac{7}{4}$, we have for $r = 2$, noting that $\|u_k(t)\| \leq \text{esssup}_{0 < s < t} \|u_k(s)\| \leq \|u_0\|$ from Lemma 2.1
\[ \frac{d}{dt} (t^\alpha \|U\|^2) \leq ct^{\alpha - \frac{7}{2}} \left( \int_0^t \|u_k(s)\|^2 ds \right)^2 + ct^{\alpha - \frac{7}{2}} + ct^{\alpha - \frac{5}{2}} \|w_k\|^2 + ct^{\alpha - \frac{5}{2}} \|\nabla w_k\|_{p-2} \|\nabla u_k\|_{p-1} \leq ct^{\alpha - \frac{5}{2}} + ct^{\alpha - \frac{5}{2}} \|\nabla w_k\|_{p-2} \|\nabla u_k\|_{p-1}, \quad t \geq 1. \]

This implies, together with (3.9)
\[ \|U(t)\|^2 \leq ct^{\frac{3}{2}} + c \int_0^t s^{\alpha - \frac{3}{2}} \|\nabla w_k\|_{p-2} \|\nabla u_k\|_{p-1} ds \leq ct^{\frac{3}{2}} + ct^{\alpha - \frac{3}{2}} \int_0^t \|\nabla u_k\|_{p-1} ds \leq ct^{\frac{3}{2}}, \quad t \geq 1. \]
Thus we obtain Theorem 4.1 (i). To prove Theorem 4.1 (ii), we begin with restriction
\[ \frac{3}{r} - \frac{3}{2} < 1. \]
By Theorem 3.1, we have
\[ \|u_k(t)\| \leq ct^{-\frac{3}{2}(\frac{1}{r} - \frac{1}{2})}, \quad 1 \leq r < 2, \]
furthermore, by Lemma 2.1, \( w_k \) has the same decay rate in \( L^2 \). Hence (4.7) gives
\[
\frac{d}{dt}\|U\|^2 + \rho\|U\|^2 \leq c\rho^2 t^{2-2(\frac{3}{2} - \frac{3}{2})} + c\rho^2 \\
+ ct^{-\frac{3}{2}(\frac{1}{r} - \frac{1}{2})} + ct^{-\frac{3}{2} - \frac{1}{2}}\|\nabla w_k\|_{p-1}\|
\]
In the same way, we let \( \rho = \alpha t^{-1} \) for a suitable large constant \( \alpha > 0 \), and multiply (4.8) by \( t^\alpha \) to obtain, by (3.9) and the condition \( \frac{3}{r} - \frac{3}{2} < 1 \),
\[
\|U(t)\|^2 \leq ct^{\frac{1}{2} - \frac{r}{2}} + ct^{-\frac{3}{2}} + ct^{1-(\frac{3}{2} - \frac{3}{2})} + ct^{-\frac{3}{2} - \frac{1}{2}} \\
\leq ct^{-\frac{3}{2} - \frac{1}{2}}, \quad t \geq 1. \tag{4.9}
\]
Next, we consider the case \( \frac{3}{r} - \frac{3}{2} = 1 \). The inequality \( \|u_k(t)\| \leq \|u_0\| \) and Theorem 3.1(ii) imply that
\[
\|u_k(t)\|^2 \leq c(1 + t)^{-1}. \tag{4.10}
\]
and the same decay rate to \( w_k \). Similar to the derivation of (4.9), we have from (4.7)
\[
\|U(t)\|^2 \leq ct^{\frac{1}{2} - \frac{r}{2}(\ln(1 + t))^2} + ct^{-\frac{3}{2}} + ct^{-\frac{3}{2} + 2t^{-\frac{3}{2} - \frac{1}{2}}} \\
\leq ct^{-\frac{3}{2} - \frac{1}{2}}, \quad t \geq 1.
\]
Finally, if \( \frac{3}{r} - \frac{3}{2} > 1 \)
\[
\|u_k(t)\|^2 \leq c(1 + t)^{-(\frac{3}{2} - \frac{3}{2})} \text{ and } \int_0^t \|u_k(s)\|^2 \, ds \leq c,
\]
after similar manipulation, we obtain from (4.7) that
\[
\|U(t)\|^2 \leq ct^{\frac{3}{2} - \frac{3}{2}} + ct^{-\frac{3}{2} - \frac{1}{2}} + ct^{-\frac{3}{2} - \frac{1}{2}} + 2t^{-\frac{3}{2} - \frac{1}{2}} \\
\leq ct^{-\frac{3}{2} - \frac{1}{2}}, \quad t \geq 1.
\]
Hence the proof of Theorem 4.1 is completed.

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References


