NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATIONS OF NEUTRAL HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH DELAYS*

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(Received Sep. 24, 2004; revised Sep. 18, 2006)

Abstract This paper is concerned with the oscillations of neutral hyperbolic partial differential equations with delays. Necessary and sufficient conditions are obtained for the oscillations of all solutions of the equations, and these results are illustrated by some examples.

Key Words Partial differential equation; neutral hyperbolic type; delay; oscillation.

2000 MR Subject Classification 35L10, 35L20.

Chinese Library Classification O175.7.

1. Introduction

The oscillatory properties of partial differential equations with delays are applied widely in many fields, such as in biology, engineering, medicine, physics, chemistry and etc. In recent years, the discussion about these properties has become a hot topic. At the same time, many sufficient conditions for oscillations are obtained. We mention here [1,2,3]. However, only a few necessary and sufficient conditions for oscillations of such equations are established, in particular, about the hyperbolic differential equations. In this paper, our aim is to study the necessary and sufficient conditions for oscillations of all solutions about the hyperbolic partial differential equation

$$\frac{\partial^2}{\partial t^2}[u(t,x)] + \sum_{i=1}^{m} p_i(t)u(t - \sigma_i(t), x)] + \frac{\partial}{\partial t}[u(t,x)] + \sum_{j=1}^{n} q_j(t)u(t - \tau_j(t), x)] = a_0(t)u(t,x) + \sum_{k=1}^{l} a_k(t)\Delta u(t - \rho_k(t), x) - \sum_{s=1}^{h} r_s(t)u(t - \mu_s(t), x),$$

$$(t,x) \in \mathbb{R}_+ \times \Omega$$

*Supported by Natural Science Foundation of Hebei Province(102160) and Natural Science of Education office in Hebei Province (2004123).
with the boundary conditions as follows

\[ u(t, x) = 0, (t, x) \in R_+ \times \partial \Omega, \]  
\[ \frac{\partial u}{\partial n} = 0, (t, x) \in R_+ \times \partial \Omega, \]  

where $\Omega$ is a bounded domain in $R^n$ with a piecewise smooth boundary $\partial \Omega$, $G \equiv R_+ \times \Omega$.

Throughout this paper we assume the conditions (H) are satisfied: $p_i(t), q_j(t) \in C(R_+, R); a_0(t), a_k(t) \in C(R_+, (0, \infty)); 0 < \sigma_i(t) \leq \sigma, 0 < \tau_j(t) \leq \tau, 0 < \rho_k(t) \leq \rho, 0 < \mu_s(t) \leq \mu$, where $\sigma, \tau, \rho, \mu$ = const., $i \in \{1, 2, \ldots, m\} = I_m, j \in I_n, k \in I_l, s \in I_h$.

A solution $u(t, x)$ of the problem (1),(2)(or (1),(3)) means that $u(t, x) \in C^2(G) \cap C^1(G)$ and satisfies equation (1) in the domain $G$ and the boundary (2)(or (3)); the solution $u(t, x)$ of the problem (1),(2)(or (1),(3)) is called to be oscillatory in the domain $G$ if for any number $T > 0$, there exists a point $(t_0, x_0) \in [T, \infty) \times \Omega$ such that $u(t_0, x_0) = 0$ holds.

### 2. Main Results

Now we investigate the necessary and sufficient conditions for oscillations of problems (1),(2) and (1),(3). The following lemmas are useful in the proof of our main results.

**Lemma 1** Let $b$ be a constant, $b_0$ be the minimum eigenvalue of the problem

\[
\begin{align*}
\Delta \varphi(x) + b\varphi(x) &= 0, x \in \Omega; \\
\varphi(x) &= 0, x \in \partial \Omega,
\end{align*}
\]

and $\varphi(x)$ be its corresponding characteristic function. Then $b_0 > 0$ and $\varphi(x) > 0$ for $x \in \Omega$.

**Lemma 2** The necessary and sufficient condition for oscillations of all solutions of the neutral differential equation (4) is that its characteristic equation $F(\lambda) = 0$ has no real roots.

\[
F(\lambda) = \lambda^2(1 + \sum_{i=1}^{m} p_i e^{-\lambda \tau_i}) + \lambda(1 + \sum_{j=1}^{n} r_j e^{-\lambda \mu_j}) + \sum_{k=1}^{d} q_k e^{-\lambda \sigma_k} = 0,
\]

where $p_i, r_j, q_k > 0, \mu_j, \sigma_k \geq 0$ are all constants, $i \in I_m, j \in I_n, k \in I_d$.

The proof of Lemma 2 is similar to references [4,5]. We omit it here.
Theorem 1  The necessary and sufficient condition for oscillations of all solutions of the problem (1),(2) in domain $G$ is that all solutions of the following functional differential equation

$$[v(t) + \sum_{i=1}^{m} p_i(t)v(t - \sigma_i(t))]'' + [v(t) + \sum_{j=1}^{n} q_j(t)v(t - \tau_j(t))]'$$

$$+ b_0 a_0(t)v(t) + b_0 \sum_{k=1}^{l} a_k(t)v(t - \rho_k(t)) + \sum_{s=1}^{h} r_s(t)v(t - \mu_s(t)) = 0. \quad (5)$$

are oscillatory.

Proof  Sufficiency. Suppose to the contrary that there is a nonoscillatory solution $u(t, x)$ of the problem (1),(2). Without loss of generality, assume that there exists a $t_0 \geq T$ such that $u(t, x) > 0, u(t - \sigma_i(t), x) > 0, u(t - \tau_j(t), x) > 0, u(t - \rho_k(t), x) > 0, u(t - \mu_s(t), x) > 0$ for $(t, x) \in [t_0, \infty) \times \Omega, i \in I_m, j \in I_n, k \in I_l, s \in I_h$. Multiplying equation (1) by $\varphi(x)$ which is the same as that in Lemma 1, and then integrating (1) with respect to $x$ over $\Omega$, we obtain

$$\frac{d^2}{dt^2} \left[ \int_{\Omega} u(t, x) \varphi(x) dx \right] + \sum_{i=1}^{m} p_i(t) \int_{\Omega} u(t - \sigma_i(t), x) \varphi(x) dx + \frac{d}{dt} \left[ \int_{\Omega} u(t, x) \varphi(x) dx \right]$$

$$+ \sum_{j=1}^{n} q_j(t) \int_{\Omega} u(t - \tau_j(t), x) \varphi(x) dx$$

$$= a_0(t) \int_{\Omega} \varphi(x) \Delta u(t, x) dx + \sum_{k=1}^{l} a_k(t) \int_{\Omega} \varphi(x) \Delta u(t - \rho_k(t), x) dx$$

$$- \sum_{s=1}^{h} r_s(t) \int_{\Omega} u(t - \mu_s(t), x) \varphi(x) dx. \quad (6)$$

From Green’s formula and boundary condition (2), we have

$$\int_{\Omega} \varphi(x) \Delta u(t, x) dx = \int_{\Omega} u(t, x) \Delta \varphi(x) dx = -b_0 \int_{\Omega} u(t, x) \varphi(x) dx,$$

$$\int_{\Omega} \varphi(x) \Delta u(t - \rho_k(t), x) dx = -b_0 \int_{\Omega} u(t - \rho_k(t), x) \varphi(x) dx.$$

Substitute the above two formulas into (6), and let $v(t) = \int_{\Omega} u(t, x) \varphi(x) dx$, then $v(t) > 0$. It follows that

$$[v(t) + \sum_{i=1}^{m} p_i(t)v(t - \sigma_i(t))]'' + [v(t) + \sum_{j=1}^{n} q_j(t)v(t - \tau_j(t))]'$$

$$+ b_0 a_0(t)v(t) + b_0 \sum_{k=1}^{l} a_k(t)v(t - \rho_k(t)) + \sum_{s=1}^{h} r_s(t)v(t - \mu_s(t)) = 0.$$
This indicates that \( v(t) \) is a nonoscillatory solution of equation (5), which contradicts that all solutions of (5) oscillate. This ends the proof of the sufficiency.

Necessity. Still we suppose to the contrary that \( v(t) \) is a nonoscillatory solution of delay differential equation (5). Without loss of generality, we assume that \( v(t) > 0 \) and \( v(t - \sigma_i(t)) > 0, v(t - \tau_j(t)) > 0, v(t - \rho_k(t)) > 0, v(t - \mu_s(t)) > 0, \ i \in I_m, j \in I_n, k \in I_l, s \in I_h \). Multiply both sides of equation (5) by \( \varphi(x) \), and we’ll get

\[
\frac{\partial^2}{\partial t^2}[v(t)\varphi(x) + \sum_{i=1}^{m} p_i(t)v(t - \sigma_i(t))\varphi(x)] + \frac{\partial}{\partial t}[v(t, x)\varphi(x) + \sum_{j=1}^{n} q_j(t)v(t - \tau_j(t))\varphi(x)] \\
+ b_0a_0(t)v(t)\varphi(x) + b_0\sum_{k=1}^{l} a_k(t)v(t - \rho_k(t))\varphi(x) + \sum_{s=1}^{h} r_s(t)v(t - \mu_s(t))\varphi(x) = 0.
\]

Set \( u(t, x) = v(t)\varphi(x) \), then \( u(t, x) > 0 \) and we can easily obtain that

\[
\Delta u(t, x) = \Delta[v(t)\varphi(x)] = v(t)\Delta\varphi(x) = -b_0v(t)\varphi(x),
\]

\[
\Delta u(t - \rho_k(t), x) = \Delta[v(t - \rho_k(t))\varphi(x)] = -b_0v(t - \rho_k(t))\varphi(x).
\]

Combining the above three formulas, we know that \( u(t, x) \) satisfies equation (1). Because \( u(t, x) = v(t)\varphi(x) = 0 \) for \( x \in \partial\Omega \). That is, it also satisfies boundary condition (2). This means problem (1), (2) has a nonoscillatory solution \( u(t, x) \). This is a contrary. The proof of necessity is completed.

**Theorem 2** The necessary and sufficient condition for oscillations of all solutions of the problem (1), (3) in domain \( G \) is that all solutions of the following functional differential equation

\[
[v(t) + \sum_{i=1}^{m} p_i(t)v(t - \sigma_i(t))]' + [v(t) + \sum_{j=1}^{n} q_j(t)v(t - \tau_j(t))]' + \sum_{s=1}^{h} r_s(t)v(t - \mu_s(t)) = 0.
\]

are oscillatory.

**Proof** sufficiency. Suppose to the contrary that there is a nonoscillatory solution \( u(t, x) \) of the problem (1), (3). Then similar to the proof of Theorem 1, we have \( u(t, x) > 0, u(t - \sigma_i(t), x) > 0, u(t - \tau_j(t), x) > 0, u(t - \rho_k(t), x) > 0, u(t - \mu_s(t), x) > 0, \ i \in I_m, j \in I_n, k \in I_l, s \in I_h \). Integrating equation (1) with respect to \( x \) over \( \Omega \), we get

\[
\frac{d^2}{dt^2} \int_{\Omega} u(t, x) dx + \sum_{i=1}^{m} p_i(t) \int_{\Omega} u(t - \sigma_i(t), x) dx \\
- \frac{d}{dt} \int_{\Omega} u(t, x) dx + \sum_{j=1}^{n} q_j(t) \int_{\Omega} u(t - \tau_j(t), x) dx \\
= a_0(t) \int_{\Omega} \Delta u(t, x) dx + \sum_{k=1}^{l} a_k(t) \int_{\Omega} \Delta u(t - \rho_k(t), x) dx \\
- \sum_{s=1}^{h} r_s(t) \int_{\Omega} u(t - \mu_s(t), x) dx.
\]
From Green’s formula and boundary condition (3), we have

$$\int_{\Omega} \Delta u(t, x) dx = \int_{\partial \Omega} \frac{\partial u}{\partial n} ds, \quad \int_{\Omega} \Delta u(t - \rho_k(t), x) dx = 0.$$ 

Substituting the above two formulas into (8), and let \(v(t) = \int_{\Omega} u(t, x) dx\), then \(v(t) > 0\). It follows that

$$[v(t) + \sum_{i=1}^{m} p_i(t) v(t - \sigma_i(t))]'' + [v(t) + \sum_{j=1}^{n} q_j(t) v(t - \tau_j(t))]' + \sum_{s=1}^{h} r_s(t) v(t - \mu_s(t)) = 0.$$ 

This indicates that \(v(t)\) is a nonoscillatory solution of delay differential equation (7), which contradicts that all solutions of (7) oscillate. This ends the proof of the sufficiency.

Necessity. We still suppose to the contrary that \(v(t)\) is a nonoscillatory solution of delay differential equation (7). Without loss of generality, assume that \(v(t) > 0\) and \(v(t - \sigma_i(t)) > 0, v(t - \tau_j(t)) > 0, v(t - \rho_k(t)) > 0, v(t - \mu_s(t)) > 0, i \in I_m, j \in I_n, k \in I_l, s \in I_k\). Let \(u(t, x) = v(t)\), then \(u(t, x) > 0\) and we know

$$\Delta u(t, x) = \Delta v(t) = 0, \quad \Delta u(t - \rho_k(t), x) = \Delta v(t - \rho_k(t)) = 0.$$ 

In view of the above two formulas and (7), we know that \(u(t, x)\) satisfies equation (1). As well as \(\partial u/\partial n = \partial v(t)/\partial n = 0, (t, x) \in (0, \infty) \times \partial \Omega\). That is \(u(t, x)\) also satisfies boundary condition (3). Therefore \(u(t, x) = v(t)\) is a nonoscillatory solution of problem (1),(3). A contradiction. The proof of necessity is completed.

Using Lemma 2, we can obtain the following results.

**Theorem 3** The necessary and sufficient condition for oscillations of all solutions of the problem (1),(2) in domain \(G\) is that its characteristic equation

$$D_1(\lambda) = \lambda^2(1 + \sum_{i=1}^{m} p_i e^{-\lambda \sigma_i}) + \lambda(1 + \sum_{j=1}^{n} q_j e^{-\lambda \tau_j}) + \sum_{s=1}^{h} r_s e^{-\lambda \mu_s} + a_0 b_0 + b_0 \sum_{k=1}^{l} a_k e^{-\lambda \rho_k},$$

has no real roots, where \(p_i, q_j \in R, r_s, a_0, a_k > 0, \sigma_i, \tau_j, \mu_s, \rho_k \geq 0\) are constants, \(i \in I_m, j \in I_n, s \in I_k, k \in I_l\).

**Theorem 4** The necessary and sufficient condition for oscillations of all solutions of the problem (1),(3) in domain \(G\) is that its characteristic equation

$$D_2(\lambda) = \lambda^2(1 + \sum_{i=1}^{m} p_i e^{-\lambda \sigma_i}) + \lambda(1 + \sum_{j=1}^{n} q_j e^{-\lambda \tau_j}) + \sum_{s=1}^{h} r_s e^{-\lambda \mu_s},$$

has no real roots, where \(p_i, q_j \in R, r_s > 0, \sigma_i, \tau_j, \mu_s \geq 0\) are constants, \(i \in I_m, j \in I_n, s \in I_k\).
3. Examples

Example 1
\[
\frac{\partial^2}{\partial t^2}[u(t, x) + \frac{3}{2}u(t - \pi, x)] + \frac{\partial}{\partial t}[u(t, x) - u(t - \frac{\pi}{2}, x)]
\]
\[
= \frac{1}{2}\Delta u(t, x) + 2\Delta u(t - \frac{\pi}{2}, x) - u(t - 3\frac{\pi}{2}, x), \quad (t, x) \in R_+ \times (0, \pi); \quad (9)
\]
with the boundary condition
\[
u(t, 0) = u(t, \pi) = 0, \quad t \geq 0. \quad (10)
\]
In this example all conditions of Theorem 3 are satisfied and
\[
D_1(\lambda) = \lambda^2(1 + \frac{3}{2}e^{-\pi\lambda}) + \lambda(1 - e^{-\frac{\pi}{2}\lambda}) + \frac{1}{2} + 2e^{-\frac{\pi}{2}\lambda} + e^{-\frac{3}{2}\pi\lambda} > 0, \lambda \in R.
\]
Hence all solutions of the equation(9) are oscillatory in domain G. In fact, \( u(t, x) = \sin x \cos t \) is such a solution.

Example 2
\[
\frac{\partial^2}{\partial t^2}[u(t, x) + \frac{1}{2}u(t - \frac{\pi}{2}, x)] + \frac{\partial}{\partial t}[u(t, x) - u(t - \pi, x)]
\]
\[
= 2\Delta u(t, x) + \Delta u(t - \pi, x) - \frac{5}{2}u(t - \frac{\pi}{2}, x), \quad (t, x) \in R_+ \times (0, \pi); \quad (11)
\]
with the boundary condition
\[
\frac{\partial u(t, 0)}{\partial x} = \frac{\partial u(t, \pi)}{\partial x} = 0, \quad t \geq 0. \quad (12)
\]
In this example all conditions of Theorem 4 are satisfied and
\[
D_2(\lambda) = \lambda^2(1 + \frac{1}{2}e^{-\frac{\pi}{2}\lambda}) + \lambda(1 - e^{-\pi\lambda}) + \frac{5}{2}e^{-\frac{\pi}{2}\lambda} > 0, \lambda \in R.
\]
Hence all solutions of the equation(12) are oscillatory in domain G. In fact, \( u(t, x) = \cos x \sin t \) is such a solution.

References