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BV SOLUTIONS TO A DEGENERATE EQUATION COUPLED WITH TIME-DELAY REGULARIZATION *

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Abstract In this paper, we study a initial value problem of a degenerate parabolic equation coupled with time-delay regularization. The existence of BV solutions to the problem for an initial BV_{loc} data is obtained. Moreover, the existence and uniqueness of spatial-periodic classical solutions to its corresponding regularized problem are also given.

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1. Introduction

Image restoration is one of the most important tasks in image processing and computer vision. In many applications, a given image,which is denoted by a function $u_0 : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, Ω is typically a rectangular, can be written as a sum $u(x) + \eta(x), x \in \Omega$, where u denotes important components of the given data formed by homogeneous regions with sharp boundaries, while η represents the noise or some components of u_0 which is less interesting. We assume in our paper that η stands for a white additive Gaussian noise. The problem is then to extract the u component from u_0.

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PDE-based regularization and geometry-driven flows have effectively applied to compute the optimal piecewise smooth approximation from the noisy data. One approach to obtain PDE for image restoration starts with setting axioms extracted from some fundamental laws in physics based on desired image properties. In this category, we mention Witkin, Koenderink's scale space theory [1-3] and Perona-Malik's anisotropic diffusion model [4] for examples. In [5], Alvarez, Guichard, Lions and Morel further establish the connection between scale space analysis and PDEs, and prove that the restored image should be the viscosity solution to a mean curvature flow.

By setting the best energy according to ones need, another possibility is the gradient descent flow of the Euler-Lagrange equation(sometimes modified) associated with its minimization problem. In this category, we mention Rudin-Osher-Fatemi [6], Chan-Strong [7], Vese-Osher [8], Sochen-Kimmel-Malladi [9], among many others. In [9-11], Sochen, Kimmel, malladi, Yezzi and El-Fallah, Ford introduce the concept of images as embedded maps and minimal surfaces to the field. In order to obtain edge preserving and noise removing, Yezzi, Sochen, Kimmel and Malladi and El-Fallah, Ford propose the following mean curvature flow

$$\frac{\partial u}{\partial t} = r(x)^{\gamma}\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$$

with an initial data $u(x,0) = u_0(x)$, where $r(x) = \frac{1}{\sqrt{1+|Du|^2}}$, $\gamma = -1, 0, \text{or } 1$, D denotes the gradient operator.

Based on the idea of Vogel, Oman [12], Chan, Strong [7], and Bose, Chen [13], we shall consider the following TV-penalized least square minimization problem

$$\min_u E_{\beta,\lambda}(u) =: \frac{\lambda}{2}\int_\Omega |u-u_0|^2 dx + \int_\Omega \alpha(x)\sqrt{|\nabla u|^2+\beta^2}dx. \tag{1}$$

The first term in $E_{\beta,\lambda}(u)$ measures the fidelity to the initial data, the second is a smoothing term with spatial adaptivity. $\beta > 0$, $\lambda > 0$ are constants. We can see that when $\alpha \equiv 1$, $E_{1,0}$ is non other than the area of the image surface. The function $\alpha(x)$ is chosen to be inversely proportional to the possibility of the presence of edge at the point $x \in \Omega$. Typically, we should choose the the function $\alpha(x)$ in the following form

$$\varphi(\overrightarrow{\nu}) = \frac{1}{1+\frac{|\overrightarrow{\nu}|^2}{K}},$$

where the vector-valued function $\overrightarrow{\nu}$, which approximates the image gradient Du, satisfies the system

$$\overrightarrow{\nu}_t = \frac{1}{\tau}(DG * u - \overrightarrow{\nu}). \tag{2}$$

$K > 0$ is a threshold parameter and $\overrightarrow{\nu}$ plays the role of time-delay regularization determined by the positive parameter τ. $G(x)$ is chosen as a smooth function with compact support in $\mathbb{R}^2$, in that we pay more attention to local details. For example, it may be a truncation function of the Gauss kernel.

Since the image domain Ω can be extended to the whole space by symmetry and periodicity[14], we can obtain the following coupled system corresponding to the gradient descent flow of the Euler-Lagrange equation of the minimizing problem

$$(\mathrm{CI})\begin{cases} u_t = \mathrm{div}(\varphi(\overrightarrow{\nu})A(|Du|^2)Du) - \lambda(u-u_0), & x\in\mathbb{R}^N, t\in(0,T), \quad (3)\\ \overrightarrow{\nu}_t = \frac{1}{\tau}(DG*u-\overrightarrow{\nu}), & x\in\mathbb{R}^N, t\in(0,T), \quad (4)\\ u(x,0)=u_0(x), \quad \overrightarrow{\nu}(x,0)=0, & x\in\mathbb{R}^N, \quad (5)\end{cases}$$

where $A(|p|^2)=\frac{1}{\sqrt{|p|^2+\beta^2}}, p\in\mathbb{R}^N$. In one dimensional case, when we use $\psi(s), s\in\mathbb{R}$ to denote $A(|s|^2)s, s\in\mathbb{R}$, the continuously differential equation $\psi(s)$ satisfies

$$0<\psi'(s)\le C, \quad s\in\mathbb{R} \qquad \text{for some } C>0,$$

together with

$$\lim_{s\to\infty}\psi(s)=\psi_\infty<\infty, \quad \lim_{s\to\infty}\inf\psi'(s)=0,$$

whence the diffusion equation (3) is termed as strongly degenerate parabolic equation [15, 16].

The following paper is organized as follows. In Section 2, we shall firstly present the precise hypotheses on our data, and then present the main results. In Section 3, we prove the existence of BV solutions to the problem (CI). In the last section, we shall present the existence and uniqueness of the classical solution of the regularized problem.

In our paper, $Q_T\equiv\mathbb{R}^N\times(0,T)$, $\mathbb{R}_+=[0,\infty)$, $\mathbb{R}^+=(0,\infty)$. $BV_{loc}(Q_T)$ denotes the space of functions of locally bounded variation with the general norm [17, 18]. $BV_x(Q_T)$ denotes another subset of $L^1_{loc}(Q_T)$, in which only the derivatives with respect to the spatial variable x are Radon measures on Q_T [16]. And clearly

$$BV_{loc}(Q_T)\subset BV_x(Q_T).$$

2. Main Results

In this section, we list some hypotheses on our data and give the definition of BV solutions to the problem (CI). The existence result is given at last.

(**H1**) $A(s):\mathbb{R}_+\to\mathbb{R}^+$ is non-increasing smooth function in $C^2(\mathbb{R}_+)$ satisfying

$$0<A(s)+2sA'(s)\le C, \quad A(|p|^2)|p|\le C,$$

where $C>0$ is a constant, $p\in\mathbb{R}^N$.

(**H2**) The function $\varphi\in C^2(\mathbb{R}^N)$ is strictly positive. There exists a nondecreasing function $\mu:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$|D^k\varphi(q)|\le\mu(|q|)\varphi(q),$$

where $q=(q_1,\cdots,q_N)\in\mathbb{R}^N$, $k=1,2$.

It can be seen that the time-delay term presented in (CI) can be written by

$$\Phi u\equiv\overrightarrow{\nu}=\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}DG*u(x,s)ds.$$

We then formulate hypotheses on the time-delay term in (CI). The vector- valued mapping $\Phi:\mathbf{L}^\infty(\mathbb{R}^N\times[0,T))\longrightarrow(\mathbf{L}^\infty(\mathbb{R}^N\times[0,T)))^N$ satisfies the following properties:

(**H3a**) There exist constants $C_0>0$ depending only on T, such that

$$\|\Phi u\|_{\infty,\mathbb{R}^N\times[0,t]}\le C_0\|u\|_{\infty,\mathbb{R}^N\times[0,t]}$$

for $u\in L^\infty(\mathbb{R}^N\times[0,T)),t\in[0,T)$.

(**H3b**) There exists a function $H:\mathbb{R}_+\to\mathbb{R}_+$, which is nondecreasing with respect to variable, such that for every multi-index $\mu=(\mu_1,\cdots,\mu_N)$, $|\mu|+\kappa=1$, there have

$$\|D^\mu D_t^\kappa\Phi u\|_{\infty,\mathbb{R}^N\times[0,t]}\le H(\|u\|_{\infty,\mathbb{R}^N\times[0,t]}),\qquad t\in[0,T).$$

(**H3c**) Let $u_\varepsilon\to u$ in $L^1_{loc}(Q_T)$, as $\varepsilon\to0$, then

$$\iint_Q|\Phi u_\varepsilon-\Phi u|dxdt\to0,\quad as\ \varepsilon\to0$$

for any open set $Q\subset\subset Q_T$.

For any $\eta>0$, we shall use the following functions

$$H_\eta(s)=\int_0^s h_\eta(\sigma)d\sigma,\quad \overline{H_\eta}(s)=\int_0^s H_\eta(\sigma)d\sigma,$$

where $h_\eta(s)\in C(\mathbb{R})$ defined by $h_\eta(s)=\frac{2}{\eta}(1-\frac{|s|}{\eta})$, for $|s|<\eta,h_\eta(s)=0$,otherwise.

Proposition 2.1 *For any $\eta>0$, the functions $h_\eta(s)$ and $H_\eta(s)$ satisfy the following properties* [16]:

$$h_\eta(s)\ge0,\quad|sh_\eta(s)|\le1,\quad|H_\eta(s)|\le1,\quad for\ s\in\mathbb{R},\tag{6}$$

and

$$\lim_{\eta\to0}sh_\eta(s)=0,\qquad\qquad \lim_{\eta\to0}H_\eta(s)=sgn(s).\tag{7}$$

We then give the definition of BV solution to (CI).

Definition 2.2 *A BV solution of the problem* (CI) *with $u_0(x)\in BV_{loc}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ is a function*

$$u\in L^\infty(\mathbb{R}^N\times[0,T))\cap BV_{loc}(\mathbb{R}^N\times[0,T))$$

such that:

i) *There exists a function* $\overline{\psi} \in (BV_x(Q_T))^N$ *which fulfills*

$$\lim_{\rho\to 0}\iint_{Q_T}\left|\psi(\frac{u(x+\rho_1 e_1,t)-u(x,t)}{\rho_1},\cdots,\frac{u(x+\rho_n e_n,t)-u(x,t)}{\rho_n})-\overline{\psi}(x,t)\right|dxdt=0 \tag{8}$$

where $\psi(p)$ *stands for* $A(|p|^2)p, p\in\mathbb{R}^N$, and $\rho=(\rho_1,\cdots,\rho_N), \rho_i\neq 0, i=1,\cdots,N$.

ii) *For any* $0\le\tau\in C_0^\infty(Q_T)$ *and for any* $k\in\mathbb{R}$, *the following integral inequality is satisfied:*

$$\begin{aligned}J(u,k,\varphi)\equiv&\iint_{Q_T} sgn(u-k)\left\{\frac{\partial\tau}{\partial t}(u-k)-\lambda\tau(x,t)(u-u_0)\right\}dxdt\\&-\iint_{Q_T} sgn(u-k)\varphi(\Phi u)\left\{D_x\tau(x,t)\cdot\overline{\psi}(x,t)\right\}dxdt\\\ge&\,0.\end{aligned} \tag{9}$$

For the existence result we consider the Cauchy problem with the initial data $u(x,0)=u_0(x)$ in the weak sense,*i.e.*,

$$\lim_{t\to 0^+}\int_{\mathbb{R}^N} h(x)u(x,t)dx=\int_{\mathbb{R}^N} h(x)u_0(x)dx \tag{10}$$

for any $0\le h(x)\in C_0^\infty(\mathbb{R}^N)$.

Theorem 2.3 *Let hypotheses* $(\mathbf{H1}),(\mathbf{H2})$ *and* $(\mathbf{H3a})-(\mathbf{H3c})$ *be satisfied and assume that*

$$u_0(x)\in BV_{loc}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),$$

then the Cauchy Problem (CI),(10) *admits a BV solution.*

3. Proof of Theorem 2.3

For the construction of BV solutions we use the standard parabolic regularization: let $0<\varepsilon\le 1$, and u_ε be the smooth solution of the following problem

$$(\mathrm{RI})\begin{cases}\dfrac{\partial u_\varepsilon}{\partial t}=\operatorname{div}\left(\varphi(\Phi u_\varepsilon)A_\varepsilon(|Du_\varepsilon|^2)Du_\varepsilon\right)-\lambda(u_\varepsilon(x,t)-u_{0\varepsilon}(x)), & x\in\mathbb{R}^N, t\in(0,T],\\ u_\varepsilon(x,0)=u_{0\varepsilon}(x), & x\in\mathbb{R}^N,\end{cases} \tag{11}$$

where

$$A_\varepsilon=A+\varepsilon, \tag{12}$$

and where $u_{0\varepsilon}(x)$ is a sequence of smooth approximation of $u_0(x)$ satisfying $u_{0\varepsilon}(x)\to u_0(x)$ in $L^1_{loc}(\mathbb{R}^N)$ and

$$\sup|u_{0\varepsilon}|\le C,\qquad \sup|Du_{0\varepsilon}|\le C, \tag{13}$$

$$\int_{\mathbb{R}^N}|u_{0\varepsilon}|dx\le C,\qquad \int_{\mathbb{R}^N}|Du_{0\varepsilon}|dx\le C, \tag{14}$$

where C is a constant independent of ε.

In the following part,we divide the proof of Theorem 2.3 in several steps.

By standard Maximum Principle, we can obtain the following inequality

$$\sup\{|u_\varepsilon(x,t)|; x\in\mathbb{R}^N, t\in[0,T]\}\le C\|I(x)\|_\infty, \tag{15}$$

where $C>0$ is independent of ε.

We need another estimate assuring a uniform bound for u_ε in $BV_{loc}(Q_T)$.

Lemma 3.1 *Let hypotheses* (**H1**) $-$ (**H2**) *and* (**H3a**) $-$ (**H3c**) *be satisfied. Assume that* $u_{0\varepsilon}\in C^\infty(\mathbb{R}^N)$ for $\varepsilon\in(0,1]$, *with* $u_{0\varepsilon}$ *uniformly bounded in* $L^\infty(\mathbb{R}^N)$ *and in* $W^{1,1}_{loc}(\mathbb{R}^N)$. *Let* u_ε *be the solution of the problem* (RI), *then*

(a) $u_{\varepsilon x_i}$ $(i=1,\cdots,N)$ *is uniformly bounded in* $L^\infty(0,T;L^1_{loc}(\mathbb{R}^N))$.

(b) $u_{\varepsilon t}$ *is uniformly bounded in* $L^\infty(0,T;L^1_{loc}(\mathbb{R}^N))$.

Proof of Lemma 3.1 Let $\xi\in C_0^\infty(\mathbb{R}^N)$. Multiplying the equation for u_ε in (11) by ξ^2u_ε and integrating the result over $\mathbb{R}^N\times[0,T]$, we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}\xi^2u_\varepsilon^2|_{t=0}^{t=T}dx\\
&\qquad=-\iint_{Q_T}\varphi(\Phi u_\varepsilon)A_\varepsilon|Du_\varepsilon|^2\xi^2dxdt-\iint_{Q_T}2\xi u_\varepsilon\varphi(\Phi u_\varepsilon)A_\varepsilon Du_\varepsilon\cdot D\xi dxdt\\
&\qquad\quad-\lambda\iint_{Q_T}(u_\varepsilon-u_{0\varepsilon})u_\varepsilon\xi^2dxdt\\
&\qquad\le-\frac{1}{2}\iint_{Q_T}\varphi(\Phi u_\varepsilon)A_\varepsilon|Du_\varepsilon|^2\xi^2dxdt+2\iint_{Q_T}\varphi(\Phi u_\varepsilon)u_\varepsilon^2A_\varepsilon|D\xi|^2dxdt\\
&\qquad\quad+\lambda\iint_{Q_T}\xi^2u_{0\varepsilon}u_\varepsilon dxdt,
\end{aligned}$$

where we use the inequalities of Cauchy and Schwarz and that of Young. It follows that

$$\begin{aligned}
&\iint_{Q_T}\xi^2\varphi A_\varepsilon|Du_\varepsilon|^2dx\\
&\qquad\le C\iint_{Q_T}\varphi u_\varepsilon^2A_\varepsilon|D\xi|^2dxdt+C\iint_{Q_T}u_{0\varepsilon}u_\varepsilon\xi^2dxdt+\int_{Q_T}\xi^2I_\varepsilon^2dx\\
&\qquad\le C_\xi,
\end{aligned}$$

and C_ξ depends only on ξ, T, λ and the initial data, hence by condition (H3a), we have

$$A_\varepsilon|Du_\varepsilon|^2\quad\text{is uniformly bounded in } L^1_{loc}(\mathbb{R}^N\times(0,T)). \tag{16}$$

Now we are ready to prove (a).

Let $\omega_i=\frac{\partial u_\varepsilon}{\partial x_i},(i=1,\cdots,N)$, $H(\eta,\omega)=(H_\eta(\omega_1),\cdots,H_\eta(\omega_n))$, then we can see that $\omega=(\omega_1,\cdots,\omega_n)$ satisfies the following systems

$$\omega_t=D_x\left(\operatorname{div}\left(\varphi(\Phi u_\varepsilon)A(|\omega|^2)\omega\right)\right)-\lambda(\omega-D_xu_{0\varepsilon}). \tag{17}$$

Let $0 \le \xi(x) \in C_0^\infty(\mathbb{R}^N)$ be any cutoff function. Computing the scalar product of ω_t and $H(\eta,\omega)\xi^2$ and then integrating the result over $\mathbb{R}^N \times [0,t], t \in [0,T)$,we obtain that

$$
\begin{aligned}
&\int_0^t \frac{\partial}{\partial s}\left(\int_{\mathbb{R}^N} \xi^2 \sum_{i=1}^N \overline{H_\eta}(\omega_i)dx\right) ds \\
&= \int_0^t \int_{\mathbb{R}^N} \xi^2 D_x \left(\operatorname{div}\left(\varphi(\Phi u_\varepsilon)A_\varepsilon(|\omega|^2)\omega\right)\right) \cdot H(\eta,\omega)dxds \\
&\quad - \lambda \int_0^t \int_{\mathbb{R}^N} \xi^2 \left\langle \omega - D_x u_{0\varepsilon}, H(\eta,\omega)\right\rangle dxds \\
&= -\int_0^t \int_{\mathbb{R}^N} \operatorname{div}\left(\varphi A_\varepsilon \omega\right) \left\langle H(\eta,\omega), D\xi^2\right\rangle dxds \\
&\quad - \int_0^t \int_{\mathbb{R}^N} \xi^2 \operatorname{div}(\varphi A_\varepsilon \omega) \operatorname{div}\left(H(\eta,\omega)\right) dxds - \lambda \int_0^t \int_{\mathbb{R}^N} \xi^2(\omega - Du_{0\varepsilon}) \cdot H(\eta,\omega)dxds \\
&= \int_0^t \int_{\mathbb{R}^N} \varphi A_\varepsilon \omega \cdot D_x \left(D_x \xi^2 \cdot H(\eta,\omega)\right) dxds - \lambda \int_0^t \int_{\mathbb{R}^N} \xi^2(\omega - D_x u_{0\varepsilon}) \cdot H(\eta,\omega)dxds \\
&\quad - \int_0^t \int_{\mathbb{R}^N} \xi^2 \operatorname{div}\left(H(\eta,\omega)\right) (A_\varepsilon \frac{\partial \varphi}{\partial l} \frac{\partial \Phi_l u}{\partial x_i} \omega_i + \varphi \operatorname{div}(A_\varepsilon \omega))dxds \\
&\equiv E_0 + \Pi + F_0
\end{aligned}
$$

where $\frac{\partial \varphi}{\partial l} = \frac{\partial \varphi}{\partial \Phi_l u}$ and

$$
\begin{aligned}
E_0 &= \int_0^t \int_{\mathbb{R}^N} A_\varepsilon \varphi \left(\left\langle H(\eta,\omega)D^2(\xi^2), \omega\right\rangle + \left\langle D_x \xi^2 G(H^\top), \omega\right\rangle\right) dxds, \\
\Pi &= -\lambda \int_0^t \int_{\mathbb{R}^N} \xi^2(\omega - Du_{0\varepsilon}) \cdot H(\eta,\omega)dxds, \\
F_0 &= -\int_0^t \int_{\mathbb{R}^N} \xi^2 \operatorname{div}\left(H(\eta,\omega)\right) \left(A_\varepsilon \frac{\partial \varphi}{\partial l} \frac{\partial \Phi_l u}{\partial x_i} \omega_i + \varphi \operatorname{div}(A_\varepsilon \omega)\right) dxds,
\end{aligned}
$$

with $G(H^\top)$ denoting the Gradient of the transposition of the vector $H(\eta,\omega)$ and

$$\operatorname{div}(H(\eta,\omega)) = h_\eta(\omega_i)\omega_{ix_i}.$$

We can see that E_0 can be estimated by

$$C_1 \int_0^t \int_{supp\xi} \varphi A_\varepsilon |\omega|^2 + C_2 \int_0^t \int_{\mathbb{R}^N} \varphi A_\varepsilon |H(\eta,\omega)D^2\xi^2|^2 + C_3 \int_0^t \int_{\mathbb{R}^N} \varphi A_\varepsilon |D\xi^2 G(H^\top)|^2$$

and

$$
\begin{aligned}
F_0 \le &\int_0^t \int_{\mathbb{R}^N} \xi^2 A_\varepsilon |\omega|^2 \cdot |\operatorname{div}(H(\eta,\omega))|dxds \\
&+ \int_0^t \int_{\mathbb{R}^N} \xi^2 \varphi |Tr[(A_\varepsilon I + 2A'\omega \otimes \omega)D^2 u_\varepsilon]| \cdot |\operatorname{div}(H(\eta,\omega))|dxds, \\
\Pi \le \lambda &\int_0^t \int_{\mathbb{R}^N} \xi^2(|\omega| + |D_x u_{0\varepsilon}|)dxds.
\end{aligned}
$$

Let $\eta \to 0$,by the Dominated Convergence Theorem (for fixed ε), we get the following inequality

$$\int_{\mathbb{R}^N} \sum_{i=1}^{N} |\omega_i(x,t)|\xi^2 dx \le C_\xi \int_0^t \int_{\mathbb{R}^N} \sum_{i=1}^{N} |\omega_i(x,t)|\xi^2 dxdt + C_\xi,$$

where $t \in (0,T]$ and then, the Gronwall's inequality implies that

$$\sup_{0<t<T} \int_{\mathbb{R}^N} \sum_{i=1}^{N} \left| \frac{\partial u_\varepsilon(x,t)}{\partial x_i} \right| \xi^2 dx \le C, \tag{18}$$

where C depends only on ξ, T, λ and the initial data.

In a similar way, we can obtain the estimate (b).

From the compactness of the imbedding $BV_{loc} \hookrightarrow L^1_{loc}$[18],we also obtain the following result as a consequence of Lemma 3.1.

Corollary 3.2 *Let hypotheses* (H1), (H2) *be satisfied and* $u_0(x) \in BV_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *then there exists a sequence of* u_ε, *which is also denoted by* u_ε, *and there exists a function*

$$u \in BV_{loc}(\mathbb{R}^N \times [0,T)) \cap L^\infty(\mathbb{R}^N \times [0,T]),$$

such that

$$u_\varepsilon \to u \qquad in \quad L^1_{loc}(\mathbb{R}^N \times [0,T)).$$

And then, we verify the conditions (9) and (10) in Definition 2.2 by sequence.

Let u_ε and u be defined by Corollary 3.2. And let $0 \le \tau(x,t) \in C_0^\infty(Q_T)$ and

$$\psi_\varepsilon(x,t) = A_\varepsilon(|Du_\varepsilon|^2)Du_\varepsilon.$$

For $\varepsilon, \varepsilon' \in (0,1]$ satisfying $\varepsilon < \varepsilon'$,we have

$$\begin{aligned}
&\iint_{Q_T} \tau(x,t)|\psi_\varepsilon - \psi_{\varepsilon'}|dxdt \\
&\quad = \iint_{Q_T} \tau(x,t)|A_\varepsilon Du_\varepsilon - A_{\varepsilon'} Du_{\varepsilon'}|dxdt \\
&\quad = \iint_{Q_T} \tau(x,t) \left| \int_0^1 \frac{\partial}{\partial s} \left(A_s + s\varepsilon + (1-s)\varepsilon'\right)(sDu_\varepsilon + (1-s)Du_{\varepsilon'})ds \right| dxdt,
\end{aligned}$$

It follows that

$$\begin{aligned}
&\iint_{Q_T} \tau(x,t)|\psi_\varepsilon - \psi_{\varepsilon'}|dxdt \\
&\quad \le \iint_{Q_T} \tau(x,t)|Du_\varepsilon - Du_{\varepsilon'}| \int_0^1 (A_s + s\varepsilon + (1-s)\varepsilon')dsdxdt \\
&\qquad -2\iint_{Q_T} \tau(x,t)|Du_\varepsilon - Du_{\varepsilon'}| \int_0^1 A'_s\,|sDu_\varepsilon + (1-s)Du_{\varepsilon'}|^2\,dsdxdt \\
&\qquad +(\varepsilon' - \varepsilon)\iint_{Q_T} \tau(x,t)\left(\int_0^1 |sDu_\varepsilon + (1-s)Du_{\varepsilon'}|ds\right)dxdt \\
&\quad \le C\iint_{Q_T} \tau(x,t)|Du_\varepsilon - Du_{\varepsilon'}|dxdt \\
&\qquad +(\varepsilon' - \varepsilon)\iint_{Q_T} \tau(x,t)\left(\int_0^1 |sDu_\varepsilon + (1-s)Du_{\varepsilon'}|ds\right)dxdt,
\end{aligned}$$

where $A_s = A(|sDu_\varepsilon + (1-s)Du_{\varepsilon'}|^2)$. When fixing $\tau(x,t)$ and denoting $\mathrm{supp}(\tau(x,t))$ by $\bar{Q}$, we claim that

$$\lim_{(\varepsilon,\varepsilon')\to(0,0)} \iint_{Q_T} \tau(x,t)|Du_\varepsilon - Du_{\varepsilon'}|dxdt = 0. \tag{19}$$

Since we can see that

$$\lim_{(\varepsilon,\varepsilon')\to(0,0)} \iint_{\bar{Q}} \phi\cdot(Du_\varepsilon - Du_{\varepsilon'})dxdt = 0, \quad \forall\ \phi \in (C_0^\infty(\bar{Q}))^N,$$

replacing ϕ by $H(\eta,\overline{\omega})\tau(x,t)$, $0 \le \tau \in C_0^\infty(\bar{Q})$, $\overline{\omega} = Du_\varepsilon - Du_{\varepsilon'}$, and then letting $\eta \to 0$, we can obtain (19) by the arbitrariness of $\bar{Q}$.

By a diagonal argument,we may extract a subsequence from $\{\psi_\varepsilon\}$,which is denoted also by $\{\psi_\varepsilon\}$ such that

$$\psi_\varepsilon(x,t) \to \overline{\psi}(x,t) \quad a.e.\ (x,t) \in Q_T\ (\varepsilon \to 0) \tag{20}$$

for some $\overline{\psi}(x,t)$ satisfying $\left|\overline{\psi}(x,t)\right| \le C$. Since ψ_ε is bounded in $(BV_x(Q_T))^N$,$\overline{\psi}$ is also bounded in $(BV_x(Q_T))^N$.

On the other hand, we can see that

$$\begin{aligned}
&\psi_\varepsilon\left(\frac{u_\varepsilon(x+\rho_1 e_1,t) - u_\varepsilon(x,t)}{\rho_1}, \cdots, \frac{u_\varepsilon(x+\rho_n e_n,t) - u_\varepsilon(x,t)}{\rho_n}\right) \\
&= \psi_\varepsilon\left(\int_0^1 \frac{\partial}{\partial x_1}u_\varepsilon(x+\lambda\rho_1 e_1,t)d\lambda, \cdots, \int_0^1 \frac{\partial}{\partial x_n}u_\varepsilon(x+\lambda\rho_n e_n,t)d\lambda\right) \\
&= \psi_\varepsilon\left(u_{\varepsilon x_1}(x+\theta_1\rho_1 e_1,t), \cdots, u_{\varepsilon x_n}(x+\theta_n\rho_n e_n,t)\right),
\end{aligned}$$

where $0 \le \theta_i \le 1, i = 1, \cdots, n$, $n = N$. For any compact subset of Q_T denoted by Q, when $(x+\rho_i e_i, t), (x,t) \in Q, (i = 1, \cdots, N)$, we obtain

$$\begin{aligned}
&\iint_Q \left|\psi_\varepsilon\left(\frac{u_\varepsilon(x+\rho_1 e_1, t) - u_\varepsilon(x,t)}{\rho_1}, \cdots, \frac{u_\varepsilon(x+\rho_n e_n, t) - u_\varepsilon(x,t)}{\rho_n}\right)\right. \\
&\qquad\qquad \left. - \psi_\varepsilon(Du_\varepsilon(x,t))\right| dxdt \\
&\quad = \iint_Q |\psi_\varepsilon(u_{\varepsilon x_1}(x+\theta_1\rho_1 e_1, t), \cdots, u_{\varepsilon x_n}(x+\theta_n\rho_n e_n, t)) - \psi_\varepsilon(Du_\varepsilon(x,t))|\, dxdt \\
&\quad = \iint_Q \left|\int_0^1 \frac{\partial}{\partial s}\psi_\varepsilon\left(u_{\varepsilon x_1}(x+s\theta_1\rho_1 e_1, t), \cdots, u_{\varepsilon x_n}(x+s\theta_n\rho_n e_n, t)\right) ds\right| dxdt \\
&\quad = \iint_Q \left|\int_0^1 (\theta_1\rho_1, \cdots, \theta_n\rho_n) \cdot D\psi_\varepsilon^* ds\right| dxdt \\
&\quad \le |(\theta_1\rho_1, \cdots, \theta_n\rho_n)| \cdot \int_0^1 ds \iint_Q |D\psi_\varepsilon^*| dxdt \\
&\quad \le C|\rho|,
\end{aligned}$$

where $\rho = (\rho_1, \cdots, \rho_n)$ and $D\psi_\varepsilon^* = D\psi_\varepsilon^\top(u_{\varepsilon x_1}(x+s\theta_1\rho_1 e_1, t), \cdots, u_{\varepsilon x_n}(x+s\theta_n\rho_n e_n, t))$, the gradient of $\psi_\varepsilon^\top$.

We denote $\psi_\varepsilon(\frac{u_\varepsilon(x+\rho_1 e_1,t)-u_\varepsilon(x,t)}{\rho_1}, \cdots, \frac{u_\varepsilon(x+\rho_n e_n,t)-u_\varepsilon(x,t)}{\rho_n})$ by $\psi_\varepsilon(x,t,\rho)$. Since $u_\varepsilon \to u$ in $L^1_{loc}(Q_T)$, we can get

$$\lim_{\varepsilon\to 0} \iint_Q |\psi_\varepsilon(x,t,\rho) - \psi(x,t,\rho)| dxdt = 0 \tag{21}$$

where $\psi(x,t,\rho) = \psi(\frac{u(x+\rho_1 e_1,t)-u(x,t)}{\rho_1}, \cdots, \frac{u(x+\rho_n e_n,t)-u(x,t)}{\rho_n})$.

We consider

$$\begin{aligned}
&\iint_Q |\, \psi(x,t,\rho) - \overline{\psi}| dxdt \\
&\qquad \le \iint_Q |\psi(x,t,\rho) - \psi_\varepsilon(x,t,\rho)| dxdt + \iint_Q |\psi_\varepsilon(x,t,\rho) - \psi_\varepsilon(Du_\varepsilon(x,t))| dxdt \\
&\qquad\quad + \iint_Q |\psi_\varepsilon(Du_\varepsilon(x,t)) - \overline{\psi}| dxdt
\end{aligned}$$

First let $\varepsilon \to 0$ and then $\rho \to 0$,we obtain that

$$\lim_{\rho\to 0} \iint_Q \left|\psi(\frac{u(x+\rho_1 e_1,t) - u(x,t)}{\rho_1}, \cdots, \frac{u(x+\rho_n e_n,t) - u(x,t)}{\rho_n}) - \overline{\psi}(x,t)\right| dxdt = 0.$$

As Q is arbitary compact subset of Q_T ,we can see that the following equation is fulfilled on Q_T :

$$\lim_{\rho\to 0} \iint_{Q_T} \left|\psi(\frac{u(x+\rho_1 e_1,t) - u(x,t)}{\rho_1}, \cdots, \frac{u(x+\rho_n e_n,t) - u(x,t)}{\rho_n}) - \overline{\psi}(x,t)\right| dxdt = 0. \tag{22}$$

And from now on, we can consider the inequality (10) in Definition 2.2.

We first consider the equation for $u_\varepsilon, u_{\varepsilon'}$,

$$\frac{\partial(u_\varepsilon - u_{\varepsilon'})}{\partial t} = \operatorname{div}(\varphi(\Phi u_\varepsilon)\psi_\varepsilon - \varphi(\Phi u_{\varepsilon'})\psi_{\varepsilon'}) - \lambda[(u_\varepsilon - u_{0\varepsilon}) - (u_{\varepsilon'} - u_{0\varepsilon'})].$$

Multiplying the equation by $0 \le h(x) \in C_0^\infty(\mathbb{R}^N)$ and $0 \le \gamma(t) \in C_0^\infty(0,T)$, then integrating the result over Q_T, we can see that

$$\lim_{(\varepsilon,\varepsilon')\to(0,0)} \int_0^T \int_{\mathbb{R}^N} (\varphi(\Phi u_\varepsilon)\psi_\varepsilon - \varphi(\Phi u_{\varepsilon'})\psi_{\varepsilon'}) \cdot Dh(x)\gamma(t)dxdt = 0.$$

It follows that

$$\lim_{\varepsilon'\to 0} \int_0^T \int_{\mathbb{R}^N} (\varphi(\Phi u_\varepsilon)\psi_{\varepsilon'} - \varphi(\Phi u)\overline{\psi}) \cdot Dh(x)\gamma(t)dxdt = 0$$

since $u_\varepsilon \to u$ in $L^1_{loc}(\mathbb{R}^N \times [0,T))$.

For any $0 \le \tau(x,t) \in C_0^\infty(Q_T)$, $\forall\ k \in \mathbb{R}$, multiplying the equation for u_ε by $\tau H_\eta(u_\varepsilon - k)$ and integrating the result over Q_T, we have

$$\begin{aligned}
&\iint_{Q_T} \tau H_\eta(u_\varepsilon - k)\frac{\partial(u_\varepsilon - k)}{\partial t}dxdt \\
&\quad = \iint_{Q_T} \tau H_\eta(u_\varepsilon - k)\operatorname{div}(\varphi\psi_\varepsilon(Du_\varepsilon))dxdt - \lambda \iint_{Q_T} \tau H_\eta(u_\varepsilon - k)(u_\varepsilon - u_{0\varepsilon})dxdt,
\end{aligned}$$

which follows

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial\tau}{\partial t}(u_\varepsilon - k)H_\eta(u_\varepsilon - k)dxdt - \lambda \iint_{Q_T} \tau H_\eta(u_\varepsilon - k)(u_\varepsilon - u_{0\varepsilon})dxdt \\
&\qquad - \iint_{Q_T} D\tau \cdot D(u_\varepsilon - k)A(|Du_\varepsilon|^2)\varphi(\Phi u_\varepsilon)H_\eta(u_\varepsilon - k)dxdt \\
&\quad = \iint_{Q_T} \tau h_\eta(u_\varepsilon - k)|Du_\varepsilon|^2\varphi(\Phi u_\varepsilon)A(|Du_\varepsilon|^2)dxdt \\
&\qquad - \iint_{Q_T} \tau(u_\varepsilon - k)h_\eta(u_\varepsilon - k)\frac{\partial(u_\varepsilon - k)}{\partial t}dxdt \\
&\quad \ge \iint_{Q_T} \tau\varphi(\Phi u_\varepsilon)h_\eta(u_\varepsilon - k)|Du_\varepsilon|^2A(|Du_\varepsilon|^2)dxdt \\
&\qquad - \iint_{Q_T} \tau(u_\varepsilon - k)h_\eta(u_\varepsilon - k)\left|\frac{\partial(u_\varepsilon - k)}{\partial t}\right|dxdt.
\end{aligned}$$

Let $\eta \to 0$, we get

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial\tau}{\partial t}\operatorname{sgn}(u_\varepsilon - k)(u_\varepsilon - k)dxdt - \lambda \iint_{Q_T} \tau(u_\varepsilon - u_{0\varepsilon})\operatorname{sgn}(u_\varepsilon - k)dxdt \\
&\qquad - \iint_{Q_T} D\tau \cdot Du_\varepsilon A(|Du_\varepsilon|^2)\varphi(\Phi u_\varepsilon)\operatorname{sgn}(u_\varepsilon - k)dxdt \\
&\quad \ge 0,
\end{aligned}$$

and then, let $\varepsilon \to 0$, we obtain that

$$\begin{aligned}&\iint_{Q_T} \operatorname{sgn}(u-k)\left\{\frac{\partial \tau}{\partial t}(u-k)\right\} dxdt - \lambda \iint_{Q_T} \operatorname{sgn}(u-k)\left\{\tau(u-u_0)\right\} dxdt \\ &\quad - \iint_{Q_T} \operatorname{sgn}(u-k)\left\{\varphi(\Phi u) D\tau \cdot \overline{\psi}(x,t)\right\} dxdt \\ &\quad \geq 0.\end{aligned}$$

Since u_ε is bounded in $BV_{loc}(\mathbb{R}^N \times [0,T))$,we can see that the equation (10) is satisfied. Combing the above discusses,the proof of Theorem 2.3 is then completed.

4. Uniqueness and Existence of Classical Solutions

Following the argument developed in [19, 20], we can establish the existence and uniqueness of periodic classical solutions to the initial value problem (RI). In this part, we only present our main results, and refer to [19, 20] for more details. Due to the presence of operators Φ, we only obtain the case of spatial- periodic solutions. Leray-Schauder fixed point principle is applied to establish the existence of the solution. The periodicity assures the attainability to the extremum of solutions.

For the sake of convenience to discuss, we firstly rewrite the system (RI) in the following form

$$(\mathrm{RG})\begin{cases}\dfrac{\partial u_\varepsilon}{\partial t} - \operatorname{Tr}[B(x,t,Du_\varepsilon,\Phi u_\varepsilon)D^2 u_\varepsilon] - F(x,t,u_\varepsilon,Du_\varepsilon,\Phi u_\varepsilon,\Psi u_\varepsilon) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^N, t \in (0,T], \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \qquad\qquad\qquad\qquad\quad x \in \mathbb{R}^N\end{cases}$$

with

$$B(x,t,Du_\varepsilon,\Phi u_\varepsilon) = \varphi(\Phi u_\varepsilon)((A+\varepsilon)I + 2A' Du_\varepsilon \otimes Du_\varepsilon),$$

$$F(x,t,u_\varepsilon,Du_\varepsilon,\Phi u_\varepsilon,\Psi u_\varepsilon) = (A+\varepsilon)\sum_{i=1}^N \sum_{j=1}^N \frac{\partial \varphi_\varepsilon}{\partial \Phi_j} \cdot \Psi_{ji} u_\varepsilon \cdot u_{\varepsilon x_i} - \lambda(u_\varepsilon - u_{0\varepsilon}),$$

where I is the N by N unit matrix, $\frac{\partial \varphi_\varepsilon}{\partial \Phi_j} = \frac{\partial \varphi}{\partial \Phi_j u_\varepsilon}, j = 1, \cdots, N$, $\Phi_j u_\varepsilon$ denotes the $j-$th component of Φu_ε. $\Psi_{ji} u_\varepsilon$, $i,j = 1,\cdots,N$, denotes $ji - th$ entry of the matrix Ψu_ε:

$$\Psi u_\varepsilon(x,t) \equiv \frac{1}{\tau}\int_0^t e^{-\frac{1}{\tau}(t-s)} D^2 G * u_\varepsilon(x,s)ds.$$

To establish the existence and uniqueness of classical solutions to the problem (RG), we formulate the following assumptions on $B(x,t,p,q)$,$F(x,t,r,p,q,\overline{q})$, and the operators Φ and Ψ as follows. $\mathbf{S}(N)$ denotes the linear space of N by N symmetric matrices with the ordinary order: $X \leq Y$ for $X,Y \in \mathbf{S}(N)$ iff $X - Y$ is non-positive,*i.e.*, $\langle (X-Y)x, x\rangle \leq 0$ for any $x \in \mathbb{R}^N$.

(H4.1) The $\mathbf{S}(N)$ valued function $B(x,t,p,q)$ and the real valued function $F(x,t,r,p,q,\overline{q})$ are smooth for $x\in\mathbb{R}^N$,$t\in[0,T]$,$r\in\mathbb{R}$,$p\in\mathbb{R}^N$,$q\in\mathbb{R}^N$ and $\overline{q}\in\mathbb{R}^{N^2}$, respectively. $B(x,t,p,q)$ and $F(x,t,r,p,q,\overline{q})$ are $\kappa-$ periodic in each x, namely, there exists a constant $\kappa=(\kappa_1,\cdots,\kappa_N)\in\mathbb{R}^N$ such that

$$B(x+\kappa_ie_i,t,p,q)=B(x,t,p,q),$$

$$F(x+\kappa_ie_i,t,r,p,q,\overline{q})=F(x,t,r,p,q,\overline{q})$$

for $x\in\mathbb{R}^N$,$t\in[0,T]$,$r\in\mathbb{R}$,$p\in\mathbb{R}^N$,$q\in\mathbb{R}^N$ and $\overline{q}\in\mathbb{R}^{N^2}$ and $i=1,\cdots,N$.

(H4.2) There are positive real valued functions λ_ε(non-increasing) and Λ_ε(non-decreasing) defined on $(\mathbb{R}_+)^N$ such that

$$\lambda_\varepsilon(|q|)I\le B(x,t,p,q)\le\Lambda_\varepsilon(|q|)I$$

for $(x,t,p,q)\in\mathbb{R}^N\times[0,T]\times\mathbb{R}^N\times\mathbb{R}^N$.

By computation, it is easy to see that $\lambda_\varepsilon(r)=\varepsilon\min\{\varphi(q):|q|\le r\}$ and $\Lambda_\varepsilon(r)=(A(0)+\varepsilon)\max\{\varphi(q):|q|\le r\}$.

For operators Φ and Ψ, we formulate hypothesis (**H4.3**) as follows. When saying an operator Φ or Ψ satisfies some condition in (**H4.3**), we means that the condition holds with Υ replaced by Ψ or Φ with proper choice of the dimension number m.

(H4.3) $\Upsilon:L^\infty(\mathbb{R}^N\times[0,T])\to(L^\infty(\mathbb{R}^N\times[0,T]))^m$ satisfies

$$\Upsilon u(x+\kappa_ie_i,t)=\Upsilon u(x,t)$$

for $(x,t)\in\mathbb{R}^N\times[0,T]$, provided that $u(x+\kappa_ie_i,t)=u(x,t),(x,t)\in\mathbb{R}^N\times[0,T]$, where $\kappa\in\mathbb{R}^N$ is a constant and $i=1,\cdots,N$.

(H4.3.1) There exists a nondecreasing function $H:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\|D^k\Upsilon u\|_{\infty,\mathbb{R}^N\times[0,t]}\le H(\|u\|_{\infty,\mathbb{R}^N\times[0,t]})$$

for $u\in L^\infty(\mathbb{R}^N\times[0,T])$ and $t\in[0,T]$ and $k=0,1$.

(H4.3.2) There exists a constant $C_T>0$ depending only on T and $\overline{G}$ such that for $u,v\in L^\infty(\mathbb{R}^N\times[0,T])$ and $t\in[0,T]$,

$$\|\Upsilon u-\Upsilon v\|_{\infty,\mathbb{R}^N\times[0,t]}\le C_T\|u-v\|_{\infty,\mathbb{R}^N\times[0,t]}.$$

(H4.3.3) There exists a constant C_T depending only on T and $\overline{G}$ such that for $u,v\in C^\alpha(\mathbb{R}\times[0,T]),\alpha\in(0,1]$, $t\in[0,T]$ and $k=0,1$,

$$\|D^k\Upsilon u-D^k\Upsilon v\|_{k,\alpha,\mathbb{R}^N\times[0,t]}\le C_T\|u-v\|_{\alpha,\mathbb{R}^N\times[0,t]}.$$

The function space $C^\alpha(\mathbb{R}^N\times[0,T])$ is defined as follows,

$$C^\alpha(\mathbb{R}^N\times[0,T])=\{u\in C(\mathbb{R}^N\times[0,T]):\|u\|_\alpha<\infty\}\qquad(23)$$

where $\|u\|_\alpha = \|u\|_\infty + [u]_\alpha$ with

$$\|u\|_\infty = \sup\{|u(x,t)| : (x,t) \in \mathbb{R}^N \times [0,T]\},$$

and

$$[u]_\alpha = \sup\left\{\frac{|u(x,t)-u(y,s)|}{(|x-y|+|t-s|^{\frac{1}{2}})^\alpha} : x,y \in \mathbb{R}^N, t,s \in [0,T], (x,t) \neq (y,s)\right\}. \qquad (24)$$

For $X \in \mathbf{S}(N)$,if $X = V^*\text{diag}\{\lambda_1, \cdots, \lambda_N\}V$ for some N by N orthogonal matrix V,the positive part and negative part of X are given by $X^+ = V^*\text{diag}\{\lambda_1^+, \cdots, \lambda_N^+\}V$ and $X^- = V^*\text{diag}\{\lambda_1^-, \cdots, \lambda_N^-\}V$ respectively, with $\lambda_i^+ = \max\{\lambda_i, 0\}$ and $\lambda_i^- = \max\{-\lambda_i, 0\}$ for $i = 1, \cdots, N$. The absolute matrix of X is defined by

$$|X| = X^+ + X^-.$$

(H4.4) There exist functions $\mu_1 : \mathbb{R}_+ \to \mathbb{R}^+$ and $\mu_2 : \mathbb{R}_+^2 \to \mathbb{R}^+$,which are nondecreasing on their variables respectively, such that the following conditions are held:

$$\sum_{i=1}^{N} |B_{x_i}(x,t,p,q)| + \sum_{j=1}^{m} |B_{q_j}(x,t,p,q)| \le \mu_1(q)B(x,t,p,q),$$

$$|F_q(x,t,r,p,q,\overline{q})| + |F_{\overline{q}}(x,t,r,p,q,\overline{q})| \le \mu_2(|q|,|\overline{q}|)|p|.$$

and

$$F(x,t,r,0,q,\overline{q}) - F(x,t,0,0,0,0) \le C|r|$$

$$|F_r(x,t,r,p,q,\overline{q})| + |F_x(x,t,r,p,q,\overline{q})| \le C(1 + \|DI\|_{\infty,\mathbb{R}^N})$$

for $x \in \mathbb{R}^N$, $t \in [0,T]$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$, $q \in \mathbb{R}^N$, $\overline{q} \in \mathbb{R}^{N^2}$. Here $|B_{x_i}|$ and $|B_{q_j}|$ denote the absolute matrix of B_{x_i} and B_{q_j} respectively,for $i = 1, \cdots, N$, $j = 1, \cdots, m$, while $|F_x|$, $|F_q|$ and $F_{\overline{q}}$ are norms of F_x,F_q and $F_{\overline{q}}$,respectively. $C > 0$ is a constant.

The existence and uniqueness of solutions to the problem (RI) are given as follows.

Theorem 4.1 *Let $B(x,t,p,q)$ and $F(x,t,r,p,q,\overline{q})$ satisfy the hypotheses* (**H4.1**) *and* (**H4.2**), *the operators Ψ and Φ satisfy* (**H4.3.2**), *then the initial value problem* (RG) *possesses at most one $\kappa-$ periodic solution in* $C^{2,1}(\mathbb{R}^N \times [0,T]) \cap C(\mathbb{R}^N \times [0,T])$.

Theorem 4.2 *Let $B(x,t,p,q)$ and $F(x,t,r,p,q,\overline{q})$ satisfy hypotheses* (**H4.1**) *and* (**H4.2**), (**H4.4**), *the operators Φ and Ψ satisfy* (**H4.3**) *and* (**H4.3.1** $-$ **3**). *Then, the initial value problem* (RG) *possesses a $\kappa-$ periodic solution in $C^{2,\alpha}(\mathbb{R}^N \times [0,T])$ for each κ-periodic initial data $u_{0\varepsilon} \in C^{2,\alpha}(\mathbb{R}^N)$.*

In Theorem 4.2,

$$C^{2,\alpha}(\mathbb{R}^N \times [0,T]) = \left\{u : \mathbb{R}^N \times [0,T] \to \mathbb{R}; u, u_t, Du, D^2u \in C^\alpha(\mathbb{R}^N \times [0,T])\right\} \qquad (25)$$

with norm

$$\|u\|_{2,\alpha} = \|u\|_\alpha + \|u_t\|_\alpha + \|Du\|_\alpha + \|D^2u\|_\alpha.$$

Theorem 4.1 can be proved by contradiction combined with the periodicity. Our proof of Theorem 4.2 follows the scheme of Leray-Schauder fixed point principle. We define the following working space: for $\alpha \in (0,1]$, $C_1^{\alpha}(\mathbb{R}^N \times [0,T])$ is the subspace of $C^{\alpha}(\mathbb{R}^N \times [0,T])$ consisting of $\kappa-$ periodic function in spatial variable x,

$$\Xi = C_1^{1,\alpha}(\mathbb{R}^N \times [0,T]) = \{u \in C_1^{\alpha}(\mathbb{R}^N \times [0,T]); Du \in C_1^{\alpha}(\mathbb{R}^N \times [0,T])\} \tag{26}$$

with norm

$$\|u\|_{1,\alpha} = \|u\|_{\alpha} + \|Du\|_{\alpha},$$

and

$$C_1^{2,\alpha}(\mathbb{R}^N \times [0,T]) = \{u \in C_1^{\alpha}(\mathbb{R}^N \times [0,T]); u_t, Du, D^2u \in C_1^{\alpha}(\mathbb{R}^N \times [0,T])\} \tag{27}$$

with norm

$$\|u\|_{2,\alpha} = \|u\|_{\alpha} + \|u_t\|_{\alpha} + \|Du\|_{\alpha} + \|D^2u\|_{\alpha}.$$

And then define a mapping $U : \Xi \times [0,1] \to \Xi$ such that $u = U(\nu, \delta)$ solves initial value problem

$$\begin{cases} u_{\varepsilon t} - (1-\delta) \triangle u_{\varepsilon} - \delta Tr[B(x,t,D\nu,\Phi\nu)D^2u] - \delta F(x,t,\nu,D\nu,\Phi\nu,\Psi\nu) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \mathbb{R}^N \times [0,T], \\ u_{\varepsilon}(x,0) = \delta u_{0\varepsilon}(x), \qquad\qquad x \in \mathbb{R}^N. \end{cases}$$

We have to establish that (1) U is a compact mapping and (2) the set of fixed points

$$\Pi_0 = \{u \in \Xi; u = U(u,\delta), \quad \text{for some } \delta \in [0,1]\} \tag{28}$$

is bounded in Ξ. We may obtain the L^{∞} bound of Du and the continuity of u on $\mathbb{R}^N \times [0,T]$ by Bernstein argument. To obtain a bound for $\|Du\|_{\alpha}$ for $u \in \Xi$, we need a corollary to a corollary in [20].

Proposition 4.3 *Let $\Omega_0 \subset \mathbb{R}^N$ be open and bounded and $\bar{Q}_T = \Omega_0 \times [0,T]$. Assume that $u \in C^{2,1}(\bar{Q}_T)$ satisfies*

$$u_t - Tr[a(x,t,u(x,t),Du(x,t))D^2u] - b(x,t,u(x,t),Du(x,t)) = 0 \tag{29}$$

for $(x,t) \in \bar{Q}_T$, where $a(x,t,r,p)$ is differentiable with respect to (x,r,p) for $(x,t) \in \bar{Q}_T$, $|r| \le \|u\|_{\infty}$ and $|p| \le \|Du\|_{\infty} = M_1$. Moreover, there are positive constants ν, μ, and μ_2 such that

$$\nu I \le a(x,t,u(x,t),Du(x,t)) \le \mu I \tag{30}$$

for $(x,t) \in \bar{Q}_T$; and

$$|a_p(w)|, |a_r(w)|, |a_x(w)|, |b(w)| \le \mu_2, \tag{31}$$

where $w = (x,t,u,Du)$. Then, there exists $\alpha \in [0,1]$, depending only on M_1, N, ν, μ, μ_2 such that for $Q' = \Omega' \times [0,T]$, $\Omega' \subset \Omega_0$ satisfying $d = \operatorname{dist}(\Omega', \partial\Omega_0) > 0$, there have

$$\|Du(x,t)\|_{\alpha,Q'} \le C,$$

where the constant C depends only on M_1, N, ν, μ, μ_2, d and the $C^2(\overline{\Omega}_0)$ norm of $u(x,0)$.

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