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PARTIAL REGULARITY FOR THE 2-DIMENSIONAL WEIGHTED LANDAU-LIFSHITZ FLOW*

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Dedicated to Academician Boling Guo for his 70-th Birthday

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Abstract We consider the partial regularity of weak solutions to the weighted Landau-Lifshitz flow on a 2-dimensional bounded smooth domain by Ginzburg-Landau type approximation. Under the energy smallness condition, we prove the uniform local C^∞ bounds for the approaching solutions. This shows that the approximating solutions are locally uniformly bounded in $C^\infty(\mathrm{Reg}(\{u_\epsilon\})\bigcap(\bar{\Omega}\times R^+))$ which guarantee the smooth convergence in these points. Energy estimates for the approximating equations are used to prove that the singularity set has locally finite two-dimensional parabolic Hausdorff measure and has at most finite points at each fixed time. From the uniform boundedness of approximating solutions in $C^\infty(\mathrm{Reg}(\{u_\epsilon\})\bigcap(\bar{\Omega}\times R^+))$, we then extract a subsequence converging to a global weak solution to the weighted Landau-Lifshitz flow which is in fact regular away from finitely many points.

Key Words Landau-Lifshitz equations; Ginzburg-Landau approximations; Hausdorff measure; partial regularity.

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1. Introduction

In this paper, we are concerned with the existence and regularities of global weak solutions to initial and boundary value problem for the weighted Landau-Lifshitz flow

$$\frac{1}{2}\partial_t u - \frac{1}{2}u\times\partial_t u - \nabla\cdot(a(x)\nabla u) = a(x)|\nabla u|^2 u \quad \text{in} \quad \Omega\times R_+, \tag{1.1}$$
$$u = u_0 \quad \text{on} \quad \Omega\times\{0\}\bigcup\partial\Omega\times R_+,$$

where "$\times$" denotes the usual vector product in R^3, the domain $\Omega\subset R^2$ is open, bounded and smooth. The initial and boundary data u_0 is assumed to be a smooth map into

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the standard sphere $S^2 \subset R^3$. In the classical sense, the equation (1.1) is equivalent to

$$u_t = u \times \nabla \cdot (a(x)\nabla u) - u \times (u \times \nabla \cdot (a(x)\nabla u)).$$

This problem is a special case of magnetization motion equation suggested in 1935 by Landau and Lifshitz, i.e.

$$\frac{\partial S}{\partial t} = \lambda_1 S \times H^e - \lambda_2 S \times (S \times H^e),$$

where $\lambda_2 > 0$ is the Gilbert damping constant, λ_1 is a constant, $S = (S_1, S_2, S_3)$ is the magnetization vector, and H^e is effective field which can be computed by the formula $H^e := \frac{\partial}{\partial S} e_{\text{mag}}(u)$, $e_{\text{mag}}(u)$ being the total energy. In particular, if we take nonhomogeneous effective magnetic energy as $e_{\text{mag}}(u) = \frac{1}{2}\int_\Omega a(x)|\nabla u|^2 dx$, we then obtain (1.1).

If $a(x) \equiv 1$, the equation (1.1) reads as

$$\frac{1}{2}\partial_t u - \frac{1}{2} u \times \partial_t u - \triangle u = |\nabla u|^2 u \quad \text{in} \quad \Omega \times R_+,$$

which has been widely discussed by mathematicians. Early in 1987, Zhou and Guo [1] had obtained the global existence of weak solutions and in 1991 [2], Zhou, Guo and Tan established the existence and uniqueness of smooth solution for 1-D problem. In 1993, for 2-D problem, Guo and Hong [3] found the close relations between this equation and harmonic map heat flow and proved the existence of partially regular solution which was first obtained for harmonic map heat flow by Chen and Struwe [4]. In 1998, also for 2-D problem, Chen Y. Ding S. and Guo B. proved that any weak solution with finite energy is smooth away from finitely many points [5]. For high dimensional problem, we refer to recent results by Liu [6] for the partial regularity of stationary weak solutions, by Ding and Guo [7] for the partial regularity of stationary weak solutions to Landau-Lifshitz-Maxwell equations in 3 dimensions, and by [8] for the partial regularity of weak solutions in 3 and 4 dimensions. We refer also to Paul Harpes' results [9] for the partial regularity of 2-D problem by Ginzburg-Landau approximations.

Concerning the Landau-Lifshitz equation where the coefficient is a function, $a(x) \not\equiv$ constant, there are not many discussions. So far as we know, the only results are the following. In 1999, Ding S., Guo, B. and Su, F [10] obtained the existence of measure-valued solution to the 1-D compressible Heisenburg chain equation

$$\vec{Z}_t = (G(\vec{Z}_x)\vec{Z} \times \vec{Z}_x)_x,$$

where $G(\vec{Z}_x)$ is a matrix function. In the same year, in [11], these authors proved the existence and uniqueness of smooth solution to the 1-D inhomogeneous equation

$$\vec{Z}_t = f(x)\vec{Z} \times \vec{Z}_{xx} + f'(x)\vec{Z} \times \vec{Z}_x.$$

Recently, Lin, J. and Ding, S. extends this problem in [12], where the function $f(x)$ is replaced by $f(x,t)$ and the method to get the estimates is different from that in [11].

For higher dimensions inhomogeneous Landau-Lifshitz equations, there are few results concerning the existence and partial regularities of weak solutions. In this paper, following the idea of Paul Harpes's work [9], we use the Ginzburg-Landau approximations to discuss the partial regularities for the global weak the solutions to (1.1). The Ginzburg-Landau appoximations $u_\epsilon : \bar{\Omega} \times R_+ \to R^3$ to Landau-Lifshitz flow (1.1) are the solutions of

$$\frac{1}{2}\partial_t u_\epsilon - \frac{1}{2}u_\epsilon \times \partial_t u_\epsilon - \nabla \cdot (a(x)\nabla u_\epsilon) = \frac{1}{\epsilon^2}a(x)(1-|u_\epsilon|^2)u_\epsilon \quad \text{in} \quad \Omega \times R_+, \tag{1.2}$$

$$u_\epsilon = u_0 \quad \text{on} \quad (\Omega \times \{0\})\bigcup(\partial\Omega \times R_+), \tag{1.3}$$

where $a(x)$ is a positive smooth function satisfying $0 < m \leq a(x) \leq M$. For small $\epsilon > 0$, we can see that the maps $\{u_\epsilon\}_\epsilon$ approximate the weighted Landau-Lifshitz flow in $\Omega \times R_+$. For fixed $\epsilon > 0$, the smooth solution to (1.2)-(1.3) on $\Omega \times R_+$ exists and if $u_0 \in W^{1,2}(\Omega; S^2) \cap W^{\frac{3}{2}}(\partial\Omega; S^2)$, it is unique in $W^{1,2}_{loc} \bigcap L^\infty(W^{1,2}) := W^{1,2}_{loc}(\Omega \times R_+; R^3) \bigcap L^\infty(W^{1,2})$. The existence is obtained by Galerkin's method. C^∞ regularity follows from a standard bootstrap argument. The total energy of the approximate flow at time $t \geq 0$ is defined by

$$G_\epsilon(u_\epsilon(t)) := \int_\Omega g_\epsilon(u_\epsilon(x,t))dx, \tag{1.4}$$

where

$$g_\epsilon(u_\epsilon(x,t)) = a(x)[\frac{1}{2}|\nabla u_\epsilon|^2 + \frac{1}{4\epsilon^2}(1-|u_\epsilon|^2)^2]. \tag{1.5}$$

In Lemma 3.1 and in Section 3, we will see that the total energy of the ϵ approximation always decreases. The local energy given by

$$G_\epsilon(u_\epsilon(t), B^\Omega_R(x_0)) := \int_{B_R(x_0)\bigcap\Omega} g_\epsilon(u_\epsilon(x,t))dx \tag{1.6}$$

may concentrate at space-time points (x_0, t_0) as $\epsilon \searrow 0$, either for fixed $t = t_0$ or for variable $t \nearrow t_0$ or $t \searrow t_0$. It characterizes the local "asymptotic behavior" of the weighted flow . Here asymptotic refers to the limits $\epsilon \searrow 0$. We will show that all the derivatives of the family of maps $\{u_\epsilon\}_{\epsilon>0}$ are locally uniformly bounded on a regular set $\mathrm{Reg}\{u_\epsilon\}_{\epsilon>0}$ consisting of all points $z_0 = (x_0, t_0) \in \overline{\Omega} \times (0, \infty)$ for which there is $R_0 = R_0(z_0)$, such that

$$\limsup_{\epsilon\searrow 0} \sup_{t_0 - R_0^2 < t < t_0} G_\epsilon(u_\epsilon(t), B^\Omega_{R_0}(x_0)) < \epsilon_0, \tag{1.7}$$

for a constant $\epsilon_0 > 0$ that will be determined later in Lemmas 2.6 and 2.8. The complement $S(\{u_\epsilon\}_{\epsilon>0}) := \overline{\Omega} \times R_+ \backslash \mathrm{Reg}\{u_\epsilon\}_{\epsilon>0}$ is referred to as the energy-concentration set. The main results in this paper can be stated as follows: the approximation solutions converge to the global weak solution of (1.1) with Dirichlet condition. This convergence is smooth in Reg$\{u_\epsilon\}_{\epsilon>0}$, while the energy-concentration set is closed, with locally finite parabolic Hausdorff measure. Delicate energy inequality shows that, in fact, the singular set consists of finitely many points as observed in [9] and [5]. Such Ginzburg-Landau penalty method was first used to study the harmonic map heat flow in higher dimensions by Chen and Struwe in [4]. So we have obtained the so called Chen-Struwe solution for our problem.

2. Estimates for Strong Parabolic System

In this section, we will show that, under the uniform smallness condition (1.7) on the local energy, all higher derivatives of u_ϵ are locally and uniformly bounded. Here "uniform" of course always means uniform in $\epsilon > 0$. In Section 2.1, we first recall some facts about L^p estimates for strongly parabolic system and C^α estimates for parabolic system in divergence form. In Section 2.2, we derive the L^∞ and L^p bounds for the right hand side of (1.2) which are necessary for us to get the uniform bounds of $\frac{1}{\epsilon^2}a(x)(1-|u_\epsilon|^2)$ for L^p estimates. In Section 2.3, we prove that all the derivatives of u_ϵ and $\frac{1}{\epsilon^2}a(x)(1-|u_\epsilon|^2)$ are locally uniformly bounded if the energy density satisfies the condition $\limsup_{\epsilon\searrow 0}\sup_{P_R(z_0)} g_\epsilon(u_\epsilon(x,t)) < C_0$, which, we will prove, may be verified under the uniformly smallness condition (1.7). Here, $z_0 = (x_0, t_0)$ and $P_R(z_0) = B_R(x_0) \times (t_0 - R^2, t_0)$. Finally in Section 2.4, we will pay our attentions to the estimates similar to above for the approximating solutions near the boundary.

2.1 Some estimates about strongly parabolic system

We recall some facts about L^p estimates for strongly parabolic system and C^α estimates for parabolic systems in divergence form. We first rewrite the equation (1.2) in the form

$$\partial_t u_\epsilon - M(u_\epsilon)a(x)\triangle u_\epsilon - M(u_\epsilon)\nabla a(x)\cdot\nabla u_\epsilon = M(u_\epsilon)\frac{1}{\epsilon^2}a(x)(1-|u_\epsilon|^2)u_\epsilon := f_\epsilon(u_\epsilon),$$

where $M(u_\epsilon)$ is a matrix whose maximal and minimal eigenvalue can be estimated as follows

$$m|\xi|^2 \le \xi^T a(x)M(u_\epsilon)\xi = \frac{a(x)}{\frac{1}{2}(1+|u_\epsilon|^2)}\{|\xi|^2 + (u_\epsilon\cdot\xi)^2\} \le 2M|\xi|^2, \quad \forall\ \xi \in R^3. \tag{2.1}$$

We therefore may write (1.2) as

$$L_\epsilon(u_\epsilon) := \partial_t u_\epsilon - M(u_\epsilon)a(x)\triangle u_\epsilon - M(u_\epsilon)\nabla a(x)\cdot\nabla u_\epsilon = f_\epsilon(u_\epsilon),$$

where the coefficient-matrix $M(u_\epsilon)$ is smooth with respect to u_ϵ. Note that L_ϵ defines a strongly parabolic system in the Petrovskii sense [13]. So L^p global and local estimates hold for such system. We list two *a priori* L^p estimates concerning the strongly parabolic system in the Petrovskii sense.

Fact 2.1 (Global L^p estimates) *Let $f_\epsilon \in L^p(\Omega\times[0,T];R^3)$ and $u_0\in W^{2,p}(\Omega;R^3)$. A solution of $L_\epsilon(v)=f_\epsilon$ in $(\Omega\times(0,T);R^3)$ with $v=u_0$ on $(\Omega\times\{0\})\bigcup(\partial\Omega\times(0,T))$ satisfies*

$$\|v\|_{W_p^{2,1}}(\Omega\times[0,T])\le C_p(\Omega,T,\omega_{u_\epsilon})(\|f_\epsilon\|_{L^p(\Omega\times[0,T])}+\|u_0\|_{W^{2,p}(\Omega)}). \tag{2.2}$$

Fact 2.2 (Local L^p estimates) *Let $f_\epsilon \in L^p(\Omega\times[0,T];R^3)$ and $u_0\in W^{2,p}(\Omega;R^3)$. A solution of $L_\epsilon(v)=f_\epsilon$ in $(\Omega\times(0,T);R^3)$ with $v=u_0$ on $(\Omega\times\{0\})\bigcup(\partial\Omega\times(0,T))$ satisfies*

$$\begin{aligned}\|v\|_{W_p^{2,1}(P^\Omega_{\delta R}(z_0))}\le&\tilde C_p(R,\Omega,T,\omega_{u_\epsilon})(\|f_\epsilon\|_{L^p(P^\Omega_R(z_0))}\\&+\|v\|_{L^q(P^\Omega_{\delta R}(z_0))}+\delta_{B_R\bigcap\partial\Omega}\|u_0\|_{W^{2-\frac1p,p}(B^\Omega_R\bigcap\partial\Omega)}),\end{aligned} \tag{2.3}$$

for all $1\le q\le p$. Here, $\delta_{B_R\bigcap\partial\Omega}=1$ if $B_R\bigcap\partial\Omega\ne\varnothing$ and 0 otherwise. The trace theorem of course implies $\|u_0\|_{W^{2-\frac1p,p}(\partial\Omega)}\le\|u_0\|_{W^{2,p}(\Omega)}$. The constants C_p and $\tilde C_p$ depend on the indicated quantities and additionally on the uniform lower and upper bounds for the eigenvalues of $a(x)M(u_\epsilon)$, that is m and $2M$ are chosen to be independent of $\epsilon>0$. Note that C_p and $\tilde C_p$ also depend on the modulus of continuity of the coefficients of the leading term, i.e., the modulus of continuity ω_{u_ϵ} of u_ϵ.

The equation can also be written in the divergence form

$$L_\epsilon(v):=\partial_t v-a(x)\nabla\cdot(M(u_\epsilon)\nabla v)+a(x)(\partial_k M(u_\epsilon)\partial_k u_\epsilon)\partial_k v-M(u_\epsilon)\nabla a(x)\cdot\nabla v=f_\epsilon(u_\epsilon).$$

We can now give the C^α estimates for the above systems in the divergence form

Fact 2.3 *If we assume*

$$\lim_{\epsilon\searrow 0}\sup\sup_{P^\Omega_R}|\nabla u_\epsilon|<\infty, \tag{2.4}$$

then $v\in C^{\gamma,\frac\gamma2}(P^\Omega_{\delta R};R^3)$ for some $\gamma\in(0,1)$ and any $\delta\in(0,1)$. If the right hand side $f_\epsilon\in L^p(P^\Omega_R;R^3)$ with $p>2$, we have the following estimate for the mixed Hölder-norm of v on $P^\Omega_{\delta R}$

$$\|v\|_{C^{\gamma,\frac\gamma2}(P^\Omega_{\delta R};R^3)}\le C(f_\epsilon), \tag{2.5}$$

where the bound $C(f_\epsilon)$ depends on the parabolicity constants, δ, $\sup\limits_{P^\Omega_R}|u_\epsilon|$, $\|f_\epsilon\|_{L^p(P^\Omega_R)}$ and also depends on $\|u_0\|_{C^\gamma(B_R\bigcap\partial\Omega)}$ if $B_R\bigcap\partial\Omega\ne\emptyset$.

If (2.4) holds and $\|f_\epsilon\|_{L^p(P^\Omega_R)}$ or $\sup\limits_{P^\Omega_R}|\nabla u_\epsilon|$ are uniformly bounded with respect to $\epsilon>0$, then the estimate (2.5) holds for u_ϵ and is uniform in $\epsilon>0$. The assumption $\sup\limits_{P^\Omega_R}|\nabla u_\epsilon|\le C$ however does not include the time derivatives. (2.5) enables us to obtain bounds on the modulus of continuity with respect to time variable. Thus the modulus of the continuity of u_ϵ on $P^\Omega_{\delta R}$ is bounded from above independent of $\epsilon>0$. Therefore the estimates (2.2) and (2.3) are now uniform in $\epsilon>0$.

2.2 $\mathbf{L^\infty}$ and $\mathbf{L^p}$ bounds for $\frac{1}{\epsilon^2}\mathbf{a(x)(1-|u_\epsilon|^2)}$

We first derive sup-norm of u_ϵ and use the multiplication of (1.2) with $-u_\epsilon$ to obtain

$$\frac{1}{4}\partial_t\rho_\epsilon - \frac{1}{2}a(x)\triangle\rho_\epsilon - \frac{1}{2}\nabla a(x)\nabla\rho_\epsilon + \frac{1}{\epsilon^2}a(x)\rho_\epsilon = a(x)|\nabla u_\epsilon|^2 + a(x)\frac{1}{\epsilon^2}\rho_\epsilon^2$$

where $\rho_\epsilon = 1-|u_\epsilon|^2$. On the parabolic boundary, $\rho_\epsilon = 1-|u_0|^2 = 0$. We get $\rho_\epsilon \geq 0$ in $\overline{\Omega}\times R_+$ by using the maximum principle, i.e. $|u_\epsilon| \leq 1$. In the sequel we try to derive L^∞ and L^p bounds for $\frac{1}{\epsilon^2}a(x)(1-|u_\epsilon|^2)$. To this aim, we consider the following auxiliary problem

$$\partial_t f - a(x)\triangle f - \nabla a(x)\cdot\nabla f + \frac{1}{\epsilon^2}a(x)f = a(x)g \qquad \text{in} \quad P_R, \tag{2.6}$$

$$|f| \leq a \qquad \text{on} \quad \partial P_R. \tag{2.7}$$

The parabolic boundary of P_R is denoted as $\partial P_R = B_R(0)\times\{-R^2\}\bigcup\partial B_R(0)\times[-R^2,0]$.

Lemma 2.1 *Let* $a>0, \epsilon\in(0,1), g\in C^0(\overline{P_R})$, *with* $\epsilon^2 sup_{P_R}|g| \leq a$. *Let* $f \in C^0(\overline{P_R})\bigcap C^2(P_R)$ *be a solution of* (2.6) *and* (2.7). *Then there exists* R_0 *depending on* m, M, *and* $M_1 = \max\limits_{x\in\bar\Omega}|\nabla a(x)|$ *such that for any* $\delta\in(0,1), R\in(0,R_0)$ *we have*

$$\frac{1}{\epsilon^2}|f| \leq \sup_{P_R^\Omega}|g| + \frac{2a}{\epsilon^2}\exp(-\frac{1}{\epsilon}(1-\delta^2)^2R^4) \quad \text{on} \quad P_{\delta R}.$$

Proof Taking $w(x,t) = 2a\exp[-\frac{1}{\epsilon}(R^2-x^2)(R^2+t)]$, we have

$$\begin{aligned}
&\epsilon^2[\partial_t w - a(x)\triangle w - \nabla a(x)\cdot\nabla w] + a(x)w \\
&\quad = w[a(x) - \epsilon(R^2-x^2) - a(x)4|x|^2(R^2+t)^2 - a(x)\epsilon\cdot 4(R^2+t) \\
&\qquad -\epsilon\nabla a(x)\cdot 2x(R^2+t)] \\
&\quad \geq w(m - \epsilon R^2 - 4MR^4 - 4\epsilon MR^2 - 2R\epsilon M_1R^2).
\end{aligned}$$

Therefore there exists R_0 depending on m ,M, M_1 such that if $R\in(0,R_0)$, $\epsilon\in(0,1)$ there holds

$$\begin{aligned}
&\epsilon^2[\partial_t w - a(x)\triangle w - \nabla a(x)\cdot\nabla w] + a(x)w > 0 \quad \text{in} \quad P_R, \\
&w = 2a \quad \text{on} \quad \partial P_R.
\end{aligned}$$

For $f_1 = f - \epsilon^2\sup\limits_{P_R}|g|$ and $f_2 = f + \epsilon^2\sup\limits_{P_R}|g|$, we have $|f_1| \leq 2a$ $|f_2| \leq 2a$ on ∂P_R and

$$\begin{aligned}
\epsilon^2[\partial_t f_1 - a(x)\triangle f_1 - \nabla a(x)\cdot\nabla f_1] + a(x)f_1 &= \epsilon^2 a(x)g - a(x)\epsilon^2\sup_{P_R}|g| \leq 0 \\
&\leq \epsilon^2[\partial_t w - a(x)\triangle w - \nabla a(x)\cdot\nabla w] + a(x)w.
\end{aligned}$$

Therefore we obtain by comparison principle that

$$f_1 - w \leq 0 \qquad \text{in} \quad P_R. \tag{2.8}$$

Similarly, we have

$$f_2 + w \geq 0 \qquad \text{in} \quad P_R. \tag{2.9}$$

Combining (2.8) with (2.9), one gets

$$-w - \epsilon^2 \sup_{P_R^\Omega} |g| \leq f \leq w + \epsilon^2 \sup_{P_R^\Omega} |g|,$$

which yields the desired conclusion

$$\frac{1}{\epsilon^2}|f| \quad \leq \quad \sup_{P_R^\Omega} |g| + \frac{1}{\epsilon^2} w \leq \sup_{P_R^\Omega} |g| + \frac{2a}{\epsilon^2} \exp(-\frac{1}{\epsilon}(1-\delta^2)^2 R^4).$$

If $B_R \bigcap \Omega \neq \emptyset$ and $f \equiv 0$ on $B_R \bigcap \partial\Omega$, we still have the local estimate near the boundary, i.e., on $P_{\delta R}^\Omega := (B_{\delta R} \bigcap \Omega) \bigcap (-\delta^2 R^2, 0)$.

Lemma 2.2 *Consider a smooth domain $\Omega \subset R^2, a > 0$, $\epsilon \in (0,1)$, $g \in C^0(\overline{P_R})$, with $\epsilon^2 \sup_{P_R} |g| \leq a$. Let $f \in C^0(\overline{P_R}) \bigcap C^2(P_R)$ be a solution of* (2.6) *and* (2.7) *and $f = 0$ on $\partial\Omega \bigcap P_R$. Then there exist R_0, depending on m, M and M_1 such that for any $\delta \in (0,1)$, $R \in (0, R_0)$, we have*

$$\frac{1}{\epsilon^2}|f| \leq \sup_{P_R^\Omega} |g| + \frac{2a}{\epsilon^2} exp(-\frac{1}{\epsilon}(1-\delta^2)^2 R^4) \quad \text{on} \quad P_{\delta R}^\Omega.$$

In the sequel, we will derive *a priori* L^p estimates for the equation (2.6) and (2.7). First we give L^1 estimates.

Lemma 2.3 *Let $\Omega \subset R^2$ be bounded smooth domain, $a > 0$, $\epsilon \in (0,1)$, $g \in C^0(\overline{P_R})$. For any nonnegative function $f \in C^1(\overline{\Omega} \times (0,T)) \bigcap C^2(\Omega \times (0,T))$ satisfying*

$$\begin{aligned} &\partial_t f - a(x)\triangle f - \nabla a(x) \cdot \nabla f + \frac{1}{\epsilon^2} a(x) f \leq a(x) g \quad \text{in} \quad \Omega \times (0,T), \\ &f = 0 \quad \text{on} \quad (\Omega \times \{0\}) \bigcup (\partial\Omega \times (0,T)), \end{aligned}$$

there hold

(1) *There exists some constant $c > 0$ only depending on M and m such that*

$$\|\frac{1}{\epsilon^2} f\|_{L^1(\Omega \times (0,T))} \leq c\|g\|_{L^1(\Omega \times (0,T))}. \tag{2.10}$$

(2) *For any R, $\rho > 0$ and $z_0 = (x_0,\, t_0) \in \Omega \times (0,T)$ with $R^2 + \rho^2 < t_0$, we have*

$$\int_{P_R^\Omega(z_0)} \frac{1}{\epsilon^2}|f| dz \leq c \int_{P_{R+\rho}^\Omega(z_0)} (|g| + \frac{c}{\rho^2}|f|) dz. \tag{2.11}$$

Proof of (1) Multiplying the equation (2.6) with $\frac{f}{\sqrt{f^2+\delta^2}}$, we obtain:

$$\partial_t|f|\cdot\frac{|f|}{\sqrt{f^2+\delta^2}}+a(x)\cdot\frac{|\nabla f|^2}{\sqrt{f^2+\delta^2}}(1-\frac{f^2}{f^2+\delta^2})+\frac{1}{\epsilon^2}a(x)\frac{f^2}{\sqrt{f^2+\delta^2}}$$
$$=a(x)g\cdot\frac{f}{\sqrt{f^2+\delta^2}}+\mathrm{div}(a(x)\nabla f\frac{f}{\sqrt{f^2+\delta^2}}). \tag{2.12}$$

Integrating (2.12) over $\Omega\times(0,t)$, letting $\delta\to 0$ and using the monotone convergence theorem, we have

$$\sup_{0\le t\le T}\int_\Omega|f(x,t)|dx+\int_0^T\int_\Omega a(x)\frac{1}{\epsilon^2}|f(x,t)|dxdt\le\int_0^T\int_\Omega a(x)|g(x,t)|dxdt,$$

From the above inequality, we obtain

$$\int_0^T\int_\Omega\frac{1}{\epsilon^2}|f(x,t)|dxdt\le\frac{M}{m}\int_0^T\int_\Omega|g(x,t)|dxdt.$$

Take $C=\frac{M}{m}$, we have

$$\|\frac{1}{\epsilon^2}f\|_{L^1(\Omega\times(0,T))}\le c\|g\|_{L^1(\Omega\times(0,T))}.$$

Proof of (2) Multiplying the equation (2.6) with $\frac{f}{\sqrt{f^2+\delta^2}}(x,t)\phi^2(x)\eta(t)$ where the cut-off function $\phi(x)$ satisfying $0\le\phi(x)\in C^\infty(\Omega)$ with $spt\phi\subset B_{R+\rho}(x_0)$ and $\phi\equiv 1$ on $B_R(x_0)$, $\eta(t)$ satisfies $\eta(t)\in C^\infty(R_+)$ with $0\le\eta(t)\le 1$, $\eta(t_0-R^2-\rho^2)=1$ and $\eta(t)\equiv 1$, $|\nabla\phi|\le\frac{c}{\rho}$, $|\nabla^2\phi|\le\frac{c}{\rho^2}$ and $|\partial_t\eta|\le\frac{c}{\rho^2}$, we obtain

$$\partial_t(|f|\phi^2\eta)\frac{|f|}{\sqrt{f^2+\delta^2}}+a(x)\cdot\frac{|\nabla f|^2\phi^2\eta}{\sqrt{f^2+\delta^2}}(1-\frac{f^2}{f^2+\delta^2})+\frac{1}{\epsilon^2}a(x)\frac{f^2\phi^2\eta}{\sqrt{f^2+\delta^2}}$$
$$=a(x)g\cdot\frac{f\phi^2\eta}{\sqrt{f^2+\delta^2}}+\mathrm{div}(a(x)\nabla f\frac{f\phi^2\eta}{\sqrt{f^2+\delta^2}})$$
$$-a(x)\nabla f\frac{f}{\sqrt{f^2+\delta^2}}2\phi\nabla\phi\eta+|f|\phi^2\partial_t\eta. \tag{2.13}$$

Integrating (2.13) over $\Omega\times(0,t)$ and letting $\delta\to 0$, we get

$$\sup_{t_0-(R^2+\rho^2)\le t\le t_0}\int_{B_\Omega^R}|f(x,t)|dx+\int_{P_R^\Omega}a(x)\frac{1}{\epsilon^2}|f(x,t)|dxdt$$
$$\le\int_{P_{R+\rho}^\Omega}(a(x)|g(x,t)|+\frac{c}{\rho^2}|f(x,t)|)dxdt.$$

Recalling the assumption $m=\min_{x\in\overline{\Omega}}|a(x)|\le a(x)\le\max_{x\in\overline{\Omega}}|a(x)|=M$, we have

$$\int_{P_R^\Omega(z_0)}\frac{1}{\epsilon^2}|f|dz\le c\int_{P_{R+\rho}^\Omega(z_0)}(|g|+\frac{c}{\rho^2}|f|)dz,$$

where c depends on the maximum and minimum of $a(x)$ and $\max_{\bar{\Omega}}|\nabla a(x)|$.

Lemma 2.4 *Let $\Omega \subset R^2$ be as above, $g \in L^1 \bigcap L^p(\overline{\Omega} \times (0,T))$ for $p \geq 2$, and $\epsilon > 0$. For any nonnegative function $f \in C^1(\overline{\Omega} \times (0,T)) \bigcap C^2(\Omega \times (0,T))$ satisfying*

$$\partial_t f - a(x)\triangle f - \nabla a(x) \cdot \nabla f + \frac{1}{\epsilon^2} a(x) f \leq a(x) g \quad \text{in} \quad \Omega \times (0,T),$$
$$f = 0 \quad \text{on} \quad (\Omega \times \{0\}) \bigcup (\partial\Omega \times (0,T)),$$

$\forall \delta \in (0,1)$, $z_0 = (x_0, t_0) \in \Omega \times (0,T]$ with $0 < R^2 < t_0$, we have

$$\|\frac{1}{\epsilon^2} f\|_{L^p(P^{\Omega}_{\delta R}(z_0))} \leq c_1 \|g\|_{L^p(P^{\Omega}_{\delta R}(z_0))} + c_2 \epsilon^{2/p}, \tag{2.14}$$

where $c_1 = c_1(p, m, M), c_2 = c_2(\|g\|_{L^p(P^{\Omega}_R)}, \|f\|_{L^{2p-1}(P_R(z_0))}, p, \delta, R, m, M)$.

Proof Multiplying the differential inequality by $f|f|^{2s-2}\phi^2\eta$, where $s \geq 1$, taking cut-off functions ϕ and η as in the proof of Lemma 2.3, we have:

$$\begin{aligned} &\frac{1}{2s}\partial_t(|f|^{2s}\phi^2\eta) + a(x)\frac{2s-1}{s^2}|\nabla |f|^s|^2\phi^2\eta + \frac{1}{\epsilon^2}a(x)|f|^{2s}\phi^2\eta \\ &\quad = \operatorname{div}(a(x)\nabla f f|f|^{2s-2}\phi^2\eta) + \frac{1}{2s}|f|^{2s}\phi^2\partial_t\eta + a(x) g f |f|^{2s-2}\phi^2\eta \\ &\qquad - a(x)\nabla|f||f|^{2s-1} 2\nabla\phi\phi\eta. \end{aligned} \tag{2.15}$$

We now estimate the last two terms of (2.15). By Young inequality we have

$$\begin{aligned} &\frac{1}{2s}\partial_t(|f|^{2s}\phi^2\eta) + a(x)\frac{2s-1}{2s^2}|\nabla |f|^s|^2\phi^2\eta + \frac{1}{2\epsilon^2}\frac{1}{2s}a(x)|f|^{2s}\phi^2\eta \\ &\quad \leq \operatorname{div}(a(x)\nabla f f|f|^{2s-2}\phi^2\eta) - a(x)(2\epsilon^2)^{2s-1}\frac{1}{2s}|g|^{2s}\phi^2\eta \\ &\qquad + \frac{2}{2s-1}|f|^{2s}[a(x)|\nabla\phi|^2\eta + \phi^2|\partial_t\eta|]. \end{aligned} \tag{2.16}$$

Setting $p = 2s$, multiplying (2.16) by $(2s) \cdot (\frac{1}{\epsilon^2})^{p-1}$ and integrating over $P^{\Omega}_{R+\rho}$, we get, for $p \geq 2$,

$$\int_{P^{\Omega}_R} (\frac{1}{\epsilon^2})^p |f|^p dz \leq C(p, m, M)\{\int_{P^{\Omega}_{R+\rho}} |g|^p + \epsilon^2 \frac{c}{\rho^2}\int_{P^{\Omega}_{R+\rho}} (\frac{1}{\epsilon^2})^p |f|^p\} dz. \tag{2.17}$$

We proceed by using iteration technique as Lemma 3.9 in [9] to finish the proof.

2.3 Higher interior estimates

In this section, we prove that the higher derivatives of u_ϵ and $\frac{1}{\epsilon^2}a(x)(1 - |u_\epsilon|^2)$ are locally and uniformly bounded in the interior point under the uniform smallness condition (1.7), where u_ϵ are the solutions to approximating equation.

Lemma 2.5 *Let u_ϵ be a solution of* (1.2) *and assume*

$$\limsup_{\epsilon\searrow 0}\sup_{P_R^\Omega} g_\epsilon(u_\epsilon)\le C_0, \tag{2.18}$$

where $B_R(x_0)\subset\Omega$, $0<R^2<t_0$. Then for any $\delta\in(0,1)$, we have

$$\limsup_{\epsilon\searrow 0}\|u_\epsilon\|_{C^k(P_{\delta R}(z_0))}\le C_k \quad and \quad \limsup_{\epsilon\searrow 0}\|\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\|_{C^k(P_{\delta R}(z_0))}\le \tilde{C}_k,$$

for all integers $k\ge 0$. The constants C_k and $\tilde{C}_k$ depend on C_0, k, R, $\delta>0$, $\|a(x)\|_{C^k(\Omega)}$.

Proof We prove this lemma by induction. If k=0, we have proved $\|u_\epsilon\|_{L^\infty}\le 1$. Using the assumption (2.18), we obtain: $\sup\limits_{P_R}\sqrt{a(x)}|\nabla u_\epsilon|\le C_0$, and $\frac{1}{\epsilon^2}a(x)(1-|u_\epsilon|^2)^2$ $\le C_0$. Therefore $g=|\nabla u_\epsilon|^2+\frac{1}{\epsilon^2}\rho_\epsilon^2$ can be controlled by a multiple of $C_0(m)$. Lemma 2.1 implies $\|\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\|_{L^\infty}\le\tilde{C_0}$. For $k=1$, since the conclusions for $k=0$ hold, i.e., $\limsup\limits_{\epsilon\searrow 0}\|u_\epsilon\|_{L^p(P_{\delta R}(z_0))}\le C_k$ and $\limsup\limits_{\epsilon\searrow 0}\|\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\|_{L^p(P_{\delta R}(z_0))}\le\tilde{C}_k$, using $C^{\alpha,\frac{\alpha}{2}}$ estimate of strongly parabolic systems in divergence form, we know there exists $\gamma\in(0,1)$, such that $\|u_\epsilon\|_{C^{\gamma,\frac{\gamma}{2}}}\le C(\|\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\|_{L^p})$. From Lemma 2.1, we know that $\|\frac{1}{\epsilon^2}1-|u_\epsilon|^2\|_{L^\infty}\le C_0$. Using $W_p^{2,1}$ estimate, we get

$$\|u_\epsilon\|_{W_p^{2,1}(P_{\delta R})}\le C_P(\Omega,T,w_{u_\epsilon})(\|\frac{1}{\epsilon^2}a(x)(1-|u_\epsilon|^2)\|_{L^p(P_R)}+\|u_\epsilon\|_{L^q}(P_R)),$$

for any $1\le q\le p$. From Sobolev inequality, we have when $p>2+2=4$,

$$\|\nabla u_\epsilon\|_{C^\alpha(P_{\delta R})}\le C(m,p,\alpha,\delta)\|u_\epsilon\|_{W_p^{2,1}(P_{\delta R})}.$$

We can take derivatives in (2.6) with respect to x to obtain, $\nabla(a(x)[|\nabla u_\epsilon|^2+\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)^2])\in L^p$. Using (2.6) and Lemma 2.2, we get $\nabla(\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)^2)\in L^p$. Taking derivatives with respect to x in (1.2), we can get $u_\epsilon\in W_p^{3,1}$. By Sobolev embedding theorem, we know that $u_\epsilon\in C^{1,1}$. Using (2.6) again, we get $\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\in C^{1,1}$. By the standard bootstrap argument, we complete the proof.

The following Lemma states that the boundedness of (2.18) is guaranteed by the smallness of the energy. We prove that the energy density is uniformly bounded in the regularity points of u_ϵ.

Lemma 2.6 *There are constants $C_1>0$, $\epsilon_0>0$, $R_0\in(0,\min\{1,\sqrt{t_0}\})$ such that for the solution u_ϵ of* (1.2) *satisfying*

$$\sup_{t_0-{R_0}^2<t<t_0}\int_{B_{R_0}(x_0)} g_\epsilon(u_\epsilon(x,t))dx<\epsilon_0,$$

there holds, $\forall\delta\in(0,1)$

$$\sup_{P_{\delta R_0}(z_0)} g_\epsilon(u_\epsilon)\le\frac{C}{(1-\delta)^2R_0^2},$$

where $x_0\in\Omega$ and $B_{R_0}\subset\Omega$.

Proof Without loss of generality, let $(x_0,t_0)=0$. We set $P_R:=P_R(0)$. For fixed $\epsilon>0$, since the solution u_ϵ of (1.2) is smooth, there exist $\sigma_\epsilon\in[0,R_0)$ such that

$$(R_0-\sigma_\epsilon)^2\sup_{P_{\sigma_\epsilon}} g_\epsilon=\max_{0\le\sigma\le R_0}(R_0-\sigma)^2\sup_{P_\sigma} g_\epsilon$$

and there is some $z_\epsilon=(x_\epsilon,t_\epsilon)\in P_R(z_0)\in\overline{P_{\sigma_\epsilon}}$, such that $e_\epsilon:=g_\epsilon(u_\epsilon(z_\epsilon))=\sup\limits_{P_{\sigma_\epsilon}} g_\epsilon$. Setting $\rho_\epsilon:=\frac{1}{2}(R_0-\sigma_\epsilon)$ such that $P_{\rho_\epsilon}(z_\epsilon)\subset P_{\sigma_\epsilon+\rho_\epsilon}\subset P_{R_0}$, we have:

$$\sup_{P_{\rho_\epsilon}(z_\epsilon)} g_\epsilon\le\frac{1}{[R_0-(\sigma_\epsilon+\rho_\epsilon)^2]}[R_0-(\sigma_\epsilon+\rho_\epsilon)^2]\sup_{P_{\rho_\epsilon+\sigma_\epsilon}(z_\epsilon)} g_\epsilon\le 4e_\epsilon.$$

Setting $r_\epsilon=\sqrt{e_\epsilon}\rho_\epsilon$, we can consider a rescaled map $v_\epsilon=v(y,s)=u(x_\epsilon+e_\epsilon^{-\frac{1}{2}}y,t_\epsilon+e_\epsilon^{-1}s)$, $(y,s)\in P_{r_\epsilon}$ Thus v_ϵ satisfies the equation (1.2) with $\tilde{\epsilon}:=\sqrt{e_\epsilon\epsilon}$. By computation, $g_{\sqrt{e_\epsilon\epsilon}}(v_\epsilon)(0,0)=1$ and $\sup\limits_{P_{r_\epsilon}} g_{\sqrt{e_\epsilon\epsilon}}(v_\epsilon)=4$. We now claim that $r_\epsilon\le 2$. If it holds, we can use the definition of r_ϵ and set $\sigma=\delta R_0$ to finish the proof. We prove it by contradiction argument. Suppose $r_\epsilon>2$. Since $B_{R_0}(x_0)\subset\Omega$, all the derivatives of v_ϵ are then bounded on P_1 independently of $\epsilon>0$. Indeed, if $\liminf\limits_{\epsilon\searrow 0}\sqrt{e_\epsilon}\epsilon>0$, from the equation

$$\frac{1}{2}\partial_t v_\epsilon-\frac{1}{2}v_\epsilon\times\partial_t v_\epsilon-\nabla\cdot(a(x)\nabla v_\epsilon)=\frac{1}{\tilde{\epsilon}^2}a(x)(1-|v_\epsilon|^2)v_\epsilon,$$

the claim holds by using the L^p estimates and the fact that $|v_\epsilon|\le 1$. If $\liminf\limits_{\epsilon\searrow 0}\sqrt{e_\epsilon}\epsilon=0$, then the claim follows from the fact that $\sup\limits_{P_{r_\epsilon}} g_{\sqrt{e_\epsilon}\epsilon}\le 4$ and Lemma 2.5. In particular, all the derivatives of v_ϵ are uniformly bounded. Thus

$$\sqrt{\partial_t g_{\tilde{\epsilon}}}(v_\epsilon),\quad |\nabla g_{\tilde{\epsilon}}(v_\epsilon)|\le C<\infty\quad\text{in } P_1.$$

Therefore, if we choose $r_0=\min\{\frac{1}{4C},1\}$, we have

$$|g_{\tilde{\epsilon}}(v_\epsilon)(x,t)-g_{\tilde{\epsilon}}(v_\epsilon)(0,0)|=|\partial_t g_{\tilde{\epsilon}}(v_\epsilon)(x',t')||t|+|\nabla g_{\tilde{\epsilon}}(v_\epsilon)(x',t')||x|<\frac{1}{2}.$$

Using the differential mean-value theorem, we get

$$g_{\tilde{\epsilon}}(v_\epsilon)(x,t)>g_{\tilde{\epsilon}}(v_\epsilon)(0,0)-\frac{1}{2}>\frac{1}{2},$$

which implies

$$\begin{aligned}1=g_{\sqrt{e_\epsilon\epsilon}}(v_\epsilon)(0,0)\ &\le\frac{2}{\pi r_0^2}\sup_{-r_0^2<s<0}\int_{B_{r_0}} g_{\sqrt{e_\epsilon\epsilon}}(v_\epsilon)dy\\ &\le C_*\sup_{-(\frac{r_0^2}{e_\epsilon}+\sigma_\epsilon^2)<t<0}\int_{B_{\frac{r_0}{\sqrt{e_\epsilon}}+\sigma_\epsilon(x_0)}(x_\epsilon)} g_\epsilon(u_\epsilon)dx.\end{aligned}\tag{2.19}$$

Setting $\epsilon_1=\min\{\frac{1}{2},\frac{1}{2C^*}\}$, since $r_\epsilon=\sqrt{e_\epsilon}\rho_\epsilon>2>r_0$, we get $\frac{r_0}{\sqrt{e_\epsilon}}+\sigma_\epsilon\le\rho_\epsilon+\sigma_\epsilon\le R_0$ and $(\frac{r_0}{\sqrt{e_\epsilon}})^2+\sigma_\epsilon^2\le(\rho_\epsilon+\sigma_\epsilon)^2\le R_0^2$. Hence, the right hand side of (2.19) $\le\epsilon_1\le\frac{1}{2}$. This leads to a contradiction. Therefore, $r_\epsilon\le 2$.

2.4 Boundary estimates

In this subsection, we will derive local boundary sup-estimates for the energy density, thus give the $W_p^{2,1}-$estimates for u_ϵ and L^p-estimates for $\frac{1}{\epsilon^2}a(x)(1-|u_\epsilon|^2)$ near the boundary.

Lemma 2.7 *Let u_ϵ be a solution of* (1.2) *and* (1.3) *with* $u_0 \in W^{1,2}(\Omega; S^2) \bigcap W^{2,P}(\partial\Omega; S^2)$. *Assume*

$$\sup_{P_R^\Omega} g_\epsilon(u_\epsilon) \le C_0 \tag{2.20}$$

and $B_R(x_0) \bigcap \partial\Omega \ne \emptyset$, $0 < R^2 < t_0$. *Then for any* $\delta \in (0,1)$, *we have*

$$\|u_\epsilon\|_{W_p^{2,1}(P_{\delta R}^\Omega(z_0))} \le C_1 \left(\|\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\|_{L^p(P_R^\Omega(z_0))} + \|u_\epsilon\|_{L^2(P_R^\Omega(z_0))} + \|u_0\|_{W^{2-\frac{1}{p},p}(B_R^\Omega(z_0)\bigcap\partial\Omega)} \right),$$

where the constant C_1 depends on C_0, δ, R, p, Ω, and $\|a(x)\|_{L^\infty(\Omega)}$. Furthermore, for any $\delta \in (0,1)$ we have

$$\|\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\|_{L^p(P_{\delta R}^\Omega(z_0))} \le C(p, \|a(x)\|_{L^\infty})\|g_\epsilon\|_{L^p(P_R^\Omega)} + \epsilon^{\frac{2}{p}} C(\|g_\epsilon\|_{L^p(P_R^\Omega)}, p, \delta, R, \|a(x)\|_{L^\infty(\Omega)})$$

and

$$\|\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\|_{L^\infty(P_{\delta R}^\Omega(z_0))} \le 4C_0 + O_\delta(\epsilon),$$

where $\epsilon \mapsto O_\delta(\epsilon)$is a function that depends on $\delta \in (0,1)$ and $\lim_{\epsilon\searrow 0} \epsilon^{-k} O_\delta(\epsilon) = 0$ for all $k \in N$. All the constants also depend on the parabolic constants.

Proof The assumption $\sup_{P_R^\Omega} g_\epsilon(u_\epsilon) \le C_0$ implies that $\lim_{\epsilon\searrow 0}\sup \sup_{P_{\delta R}^\Omega} \|\nabla u_\epsilon\| \le C_0(m) < \infty$. Therefore from C^α estimate, there exists $\gamma \in (0,1)$ such that $\|u_\epsilon\|_{C^{\gamma,\frac{\gamma}{2}}(P_{\delta R}^\Omega)} \le C(f_\epsilon)$. Furthermore $\sup_{P_R^\Omega} \frac{1}{\epsilon^2}a(x)(1-|u_\epsilon|^2)^2 \le C$. Therefore $g = \frac{1}{2}a(x)(|\nabla u_\epsilon| + \frac{1}{\epsilon^2}(1-|u_\epsilon|^2)^2) \in L^p(P_R^\Omega)$. From $W_p^{2,1}$ estimate, we have

$$\|u_\epsilon\|_{W_p^{2,1}(P_{\delta R}^\Omega(z_0))} \le C_1 \Big(\|\frac{1}{\epsilon^2}a(x)(1-|u_\epsilon|^2)\|_{L^p(P_R^\Omega(z_0))} + \|u_\epsilon\|_{L^2(P_R^\Omega(z_0))} + \|u_0\|_{W^{2-\frac{1}{p},p}(B_R^\Omega(z_0)\bigcap\partial\Omega)} \Big).$$

From Lemma 2.4, we have

$$\|\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\|_{L^p(P_{\delta R}^\Omega(z_0))} \le C(p, \|a(x)\|_{L^\infty})\|g_\epsilon\|_{L^p(P_R^\Omega)} + \epsilon^{\frac{2}{p}} C(\|g_\epsilon\|_{L^p(P_R^\Omega)}, p, \delta, R, \|a(x)\|_{L^\infty(\Omega)}).$$

From Lemma 2.2, we have

$$\|\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\|_{L^\infty(P^\Omega_{\delta R}(z_0))} \le \sup_{P^\Omega_R}|g(u_\epsilon)| + \frac{2a}{\epsilon^2}\exp(-\frac{1}{\epsilon}(1-\delta^2)^2R^4).$$

Note that

$$\begin{aligned} g(u_\epsilon) = [|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2}(1-|u_\epsilon|^2)^2] &\le 4[\frac{1}{2}|\nabla u_\epsilon|^2 + \frac{1}{4\epsilon^2}(1-|u_\epsilon|^2)^2] + O_\delta(\epsilon) \\ &\le C_0(m) + O_\delta(\epsilon), \end{aligned}$$

where $\lim\limits_{\epsilon\to 0}\epsilon^{-k}\cdot\frac{2a}{\epsilon^2}\exp(-\frac{1}{\epsilon}(1-\delta^2)^2R^4 = 0$, $\forall k \in N$. The lemma follows.

The following Lemma states that the boundedness of (2.20) can be verified if the energy density is uniformly bounded in the regularity points of u_ϵ.

Lemma 2.8 *Let u_ϵ be a solution to* (1.2) *and* (1.3) *with* $u_0 \in W^{1,2}(\Omega; S^2)\bigcap C^2(\partial\Omega; S^2)$. *There are constants* $C_0 = C_0(\|u_0\|_{C^2(\partial\Omega)}, E_0, \Omega)$ *and* $\epsilon_0 = \epsilon_0(\|u_0\|_{C^2(\partial\Omega)}, E_0, \Omega) > 0$, *such that if for some* $z_0 = (x_0, t_0)$ *and* $R_0 \in (0, \min\{1, \sqrt{t_0}\})$,

$$\limsup_{\epsilon\searrow 0}\sup_{t_0-R_0{}^2<t<t_0}\int_{B_{R_0}(x_0)\bigcap\Omega} g_\epsilon(u_\epsilon(x,t))dx < \epsilon_0,$$

then for any $\delta \in (0,1)$, *we have*

$$\limsup_{\epsilon\searrow 0}\sup_{P^\Omega_{\delta R_0}(z_0)} g_\epsilon(u_\epsilon) \le \frac{C}{(1-\delta)^2R_0^2}.$$

The proof is similar to the interior case, one can refer to Theorem 3.4 in [9].

3. Energy Estimates

In this section, we will prove that the total energy of the smooth weighted flow of (1.2) and (1.3) is decreasing. Recalling that in the first section we have defined the total energy $G_\epsilon(u_\epsilon(t)) := \int_\Omega g_\epsilon(u_\epsilon(x,t))dx$ and the local energy $G_\epsilon(u_\epsilon(t), B^\Omega_R(x_0)) := \int_{B_R(x_0)\bigcap\Omega} g_\epsilon(u_\epsilon(x,t))dx$ respectively.

Lemma 3.1 *Let u_ϵ be a solution of* (1.2) *and* (1.3). *Then, we have*

$$G_\epsilon(u_\epsilon(T) + \frac{1}{2}\int_0^T\int_\Omega|\partial_t u_\epsilon|^2dxdt = G_\epsilon(u_\epsilon(0) = E(u_0) = E_0. \tag{3.1}$$

Proof We multiply the equation (1.2) by $\partial_t u_\epsilon$ and integrate over Ω to get

$$\int_\Omega\frac{1}{2}|\partial_t u_\epsilon|^2dx + \frac{\partial}{\partial t}\int_\Omega\frac{1}{2}a(x)|\nabla u_\epsilon|^2 = -\frac{\partial}{\partial t}\int_\Omega\frac{1}{4\epsilon^2}a(x)(1-|u_\epsilon|^2)^2.$$

Integrating the above equality over [0, T] leads to

$$\int_0^T \int_\Omega \frac{1}{2}|\partial_t u_\epsilon|^2 dxdt + \int_\Omega a(x)[\frac{1}{2}|\nabla u_\epsilon|^2 + \frac{1}{4\epsilon^2}(1-|u_\epsilon|^2)^2](T)$$
$$= \int_\Omega a(x)[\frac{1}{2}|\nabla u_0|^2 + \frac{1}{4\epsilon^2}(1-|u_0|^2)^2] = G_\epsilon(u_\epsilon(0)) := E_0.$$

(3.1) follows.

The following Lemma deals with the estimate for the local energy of the flow.

Lemma 3.2 *Let u_ϵ be a solution of* (1.2) *and* (1.3). *Then, for* $0 \le T_1 < T_2$ *we have*

$$G_\epsilon(u_\epsilon(T_2), B_R^\Omega(x_0)) \le G_\epsilon(u_\epsilon(T_1), B_{2R}^\Omega(x_0)) + \frac{C}{R^2}\int_{T_1}^{T_2} G_\epsilon(u_\epsilon(t), B_{2R}^\Omega(x_0))dt. \quad (3.2)$$

Proof We multiply the equation (1.2) by $\partial_t u_\epsilon \phi^2$, where ϕ is a cut-off function satisfying $\phi(x) \in C_c^\infty(\Omega)$, $0 \le \phi(x) \le 1$, $\phi \equiv 1$ on $B_R(x_0)\bigcap\Omega$; $\phi \equiv 0$ on $(B_{2R}(x_0)\bigcap\Omega)^C$, $|\nabla\phi| \le \frac{C}{R^2}$, and then integrate over $B_{2R}^\Omega = B_{2R}(x_0)\bigcap\Omega$ to derive

$$\int_{B_{2R}^\Omega} \frac{1}{2}|\partial_t u_\epsilon|^2\phi^2 dx - \int_{B_{2R}^\Omega} \nabla\cdot(a(x)\nabla u_\epsilon)\cdot\partial_t u_\epsilon\phi^2 dx$$
$$= \int_{B_{2R}^\Omega} \frac{1}{\epsilon^2}a(x)(1-|u_\epsilon|^2)u_\epsilon\cdot\partial_t u_\epsilon\phi^2 dx.$$

Integrating by parts, we have,

$$\int_{B_{2R}^\Omega} \frac{1}{2}|\partial_t u_\epsilon|^2\phi^2 dx + \frac{\partial}{\partial t}\int_{B_{2R}^\Omega} \frac{1}{2}a(x)|\nabla u_\epsilon|^2\phi^2 dx + \frac{\partial}{\partial t}\int_{B_{2R}^\Omega} \frac{1}{4\epsilon^2}a(x)(1-|u_\epsilon|^2)^2\phi^2 dx$$
$$= -2\int_{B_{2R}^\Omega} a(x)\nabla u_\epsilon\partial_t u_\epsilon\nabla\phi\phi dx$$
$$\le \frac{1}{2}\int_{B_{2R}^\Omega} |\partial_t u_\epsilon|^2\phi^2 dx + 8\int_{B_{2R}^\Omega} a(x)^2|\nabla u_\epsilon|^2|\nabla\phi|^2 dx.$$

Integrating over $[T_1, T_2]$, using the assumption that $a(x) \le \max_{x\in\overline{\Omega}}|a(x)| = M$ and the property of the cut-off function ϕ, we get

$$G_\epsilon(u_\epsilon(T_2), B_R^\Omega(x_0)) \le G_\epsilon(u_\epsilon(T_1), B_{2R}^\Omega(x_0)) + \frac{C}{R^2}\int_{T_1}^{T_2} G_\epsilon(u_\epsilon(t), B_{2R}^\Omega(x_0))dt.$$

The lemma follows.

Lemma 3.3 $\forall\eta > 0$, $\exists T_0 > 0$, $R_0 > 0$, *such that* $\forall x_0 \in \Omega$ *and* $\forall\epsilon > 0$, *there holds*

$$\sup_{0\le t\le T_0} G_\epsilon(u_\epsilon(t), B_{R_0}^\Omega(x_0)) \le \eta. \quad (3.3)$$

Proof For each fixed $\eta > 0$, using the absolute continuity of integration, we can choose R_0 small enough to guarantee

$$G_\epsilon(u_\epsilon(0), B^\Omega_{2R_0}(x_0)) = \int_{B_{2R_0}(x_0)\bigcap\Omega} \frac{1}{2}a(x)|\nabla u_0|^2 dx \le \frac{\eta}{2}.$$

Setting $T_1 = 0$, $T_2 = T_0 = \frac{R_0^2\eta}{2CE_0}$ in (3.2) and choosing T_0 small enough we have

$$\begin{aligned} G_\epsilon(u_\epsilon(t), B^\Omega_{R_0}(x_0)) &\le G_\epsilon(u_\epsilon(0), B^\Omega_{2R_0}(x_0)) + \frac{C}{R_0^2}\int_0^t G_\epsilon(u_\epsilon(t), B^\Omega_{2R_0}(x_0))dt \\ &\le \frac{\eta}{2} + \frac{C}{R_0^2}T_0E_0 \le \eta. \end{aligned}$$

Taking supremum for t over $[0, T_0]$, one has $\sup\limits_{0\le t\le T_0} G_\epsilon(u_\epsilon(t), B^\Omega_{R_0}(x_0)) \le \eta$. (3.3) is proved.

4. Hausdorff-Measure Estimate for Singularity

First of all, it follows from energy estimates (3.1) and (3.2) that for any $0 \le s < t$

$$G_\epsilon(u_\epsilon(t), B^\Omega_R(x_0)) \le G_\epsilon(u_\epsilon(s), B^\Omega_{2R}(x_0)) + \frac{C(t-s)E_0}{R^2}. \tag{4.1}$$

Define $\delta_0 := \frac{\epsilon_0}{2CE_0}$, where ϵ_0 is the constant from (2.6) and (2.8). We may assume $0 < \delta_0 < 1$, otherwise we can choose a larger C.

Lemma 4.1 *Let u_ϵ be a solution of* (1.2) *and* (1.3) *with* $u_0 \in W^{1,2}(\Omega; S^2)\bigcap C^2(\partial\Omega; S^2)$. *Then the following assertions are equivalent:*

(1) $z_0 = (x_0, t_0) \in \mathrm{Reg}(\{u_\epsilon\}_{\epsilon>0})$.

(2) $\exists\delta$, $R > 0$, *such that* $\limsup\limits_{\epsilon\searrow 0} \sup\limits_{t_0-\delta<t<t_0} G_\epsilon(u_\epsilon(t), B^\Omega_R(x_0)) < \epsilon_0$.

(3) $\exists\delta > 0$, *such that* $\lim\limits_{R\searrow 0}\limsup\limits_{\epsilon\searrow 0} \sup\limits_{t_0-\delta<t<t_0} G_\epsilon(u_\epsilon(t), B^\Omega_R(x_0)) = 0$.

(4) $\exists R > 0$, *such that* $\limsup\limits_{\epsilon\searrow 0} \frac{1}{R^2}\int_{t_0-R^2}^{t_0}\int_{B^\Omega_R(x_0)} g_\epsilon(u_\epsilon)dxdt < \frac{1}{4}\delta_0\epsilon_0$.

(5) $\exists\delta, R > 0$, *such that* $\limsup\limits_{\epsilon\searrow 0} \sup\limits_{t_0-\delta<t<t_0+\delta} G_\epsilon(u_\epsilon(t), B^\Omega_R(x_0)) = 0$

Sketch of the proof we can easily prove them by using the energy estimates, Lemma 2.6 and 2.8 which characterize the supnorm of energy density under the smallness energy condition. For the details, we refer to [9].

Corollary 4.1 *Let u_ϵ be a solution of* (1.2) *and* (1.3) *with* $u_0 \in W^{1,2}(\Omega; S^2)\bigcap C^2(\partial\Omega; S^2)$. *Let $\{\epsilon_i\}_i$ be a sequence with $\epsilon_i \searrow 0$ as $i \to \infty$. Then the following holds*

(1) $\mathrm{Reg}(\{u_\epsilon\}_\epsilon)$ *and* $\mathrm{Reg}(\{u_{\epsilon_i}\}_i)$ *are open in* $\overline{\Omega}\times R_+$.

(2) *There exists some* $T_0 > 0$, *such that* $\overline{\Omega}\times(0, T_0) \subset \mathrm{Reg}(\{u_\epsilon\}_\epsilon)$.

Proof of Corollary 4.1

(1) follows from Lemma 4.1 (5) which is the characterization of regularity point.

(2) follows from Lemma 3.1 and Lemma 3.3. We can set $\eta = \epsilon_0$ where ϵ_0 is determined in Lemma 2.6 and Lemma 2.8 to obtain a corresponding T_0. Then Lemma 4.1(2) implies that T_0 satisfies the conclusion. This completes the proof.

Set $Q_R(z) := B_R(x)\times(t-R^2, t+R^2)$ for $z=(x,t)$. Let $\mathcal{H}^2$ denote the 2-dimensional parabolic Hausdorff measure.

Using the Vitali's Covering theorem [14], we can give the Hausdorff measure estimate for singularity set.

Theorem 4.1 *Let u_ϵ be a solution of (1.2) and (1.3) with $u_0 \in W^{1,2}(\Omega; S^2)\bigcap C^2(\partial\Omega; S^2)$. Let $\{\epsilon_i\}_i$ be a sequence with $\epsilon_i \searrow 0$ as $i\to\infty$. Then the following hold*

(1) *$S(\{u_\epsilon\}_\epsilon)$ has locally finite two-dimensional parabolic Hausdorff-measure. More precisely there is a constant $K_1 = K_1(E_0, \epsilon_0) > 0$, such that for any compact interval $I \subset R_+$, $\mathcal{H}^2(S(\{u_\epsilon\}_\epsilon)\bigcap(\overline{\Omega}\times I)) \le K_1|I|$.*

(2) *There is a constant $K_2 = K_2(E_0,\epsilon_0) > 0$, such that for any $t > 0$, the set $S^t(S(\{u_\epsilon\}_\epsilon) := S(S(\{u_\epsilon\}_\epsilon)\bigcap(\overline{\Omega}\times\{t\})$ consist of at most K_2 points.*

Proof The proof is similar to Proposition 4.3 in [9].

5. Passing to the Limits

In this section, we prove

Theorem 5.1 *Let u_ϵ be a solution of (1.2) and (1.3) with u_0 in $W^{1,2}(\Omega; S^2)\bigcap W^{\frac{3}{2},2}(\partial\Omega; S^2)$. Then there is at least one sequence $\{u_{\epsilon_i}\}_i$ and $u_* \in W^{1,2}_{loc}(\overline{\Omega}\times R_+; S^2)\bigcap L^\infty(R^+; W^{1,2}(\Omega; S^2))$ such that $u_{\epsilon_i} \rightharpoonup u_*$ weakly in $W^{1,2}_{loc}(\overline{\Omega}\times R_+; R^3)$ and weak* in $L^\infty(R_+; W^{1,2}(\Omega; R^3))$. Moreover there hold*

(1) *For any such sequence $\{u_{\epsilon_i}\}_i$, we have $\lim_{i\to\infty} u_{\epsilon_i} = u_*$ and $\frac{1}{\epsilon^2}(1-|u_\epsilon|^2) \to |\nabla u_*|^2$ in $C^\infty(\mathrm{Reg}(\{u_{\epsilon_i}\}_i)\bigcap(\Omega\times R_+))$.*

(2) *u_* is a smooth solution of (1.1) in $\mathrm{Reg}(\{u_{\epsilon_i}\}_i)\bigcap(\Omega\times R_+)$ and a global distributional solution in $W^{1,2}_{loc}(\overline{\Omega}\times R_+)\bigcap L^\infty(R_+; W^{1,2}(\Omega; R^3))$. Furthermore, u_* satisfies the initial and boundary condition in the sense $\lim_{R\searrow 0} u_*(\cdot, t) = u_0$ in $W^{2,2}(\Omega; R^3)$ and $u_*|_{\partial\Omega} = u_0|_{\partial\Omega}$ as a $W^{2,2}(\Omega; R^3)-$trace for a.e. $t > 0$ respectively.*

Proof From the local energy estimates in Lemma 3.2, we see that $\{u_{\epsilon_i}\}$ is uniformly bounded in $W^{1,2}_{loc}(\overline{\Omega}\times R_+)\bigcap L^\infty(R_+; W^{1,2}(\Omega; R^3))$. From the weak compactness, and using the diagonal method, we can see that, there is $u_* \in W^{1,2}_{loc}(\overline{\Omega}\times R_+; R^3)\bigcap L^\infty(R_+; W^{1,2}(\Omega; R^3))$, and a subsequence $\{\epsilon_i\}_i$, such that $u_{\epsilon_i} \rightharpoonup u_*$ weakly in $W^{1,2}_{loc}(\overline{\Omega}\times R_+; R^3)$ and weakly* in $L^\infty(R_+; W^{1,2}(\Omega; R^3))$

(1) From Lemma 2.6, we can see that $\forall z_0 \in \mathrm{Reg}(\{u_{\epsilon_i}\}_i)\bigcap(\Omega\times R_+)$, $\sup_{P_{\delta R}(z_0)} g_\epsilon(u_\epsilon) \le \frac{C_1}{(1-\delta)^2 R_0^2}$. By Lemma 2.5, we obtain $\lim_{i\to\infty} u_{\epsilon_i} = u_*$ in $P_{\delta R(z_0)}$. Since the point z_0 is arbitrary, we get $\lim_{i\to\infty} u_{\epsilon_i} = u_*$ in $C^\infty(\mathrm{Reg}(\{u_{\epsilon_i}\}_i)\bigcap(\Omega\times R_+); R^3)$.

(2) We multiply the equation (1.2) by u_{ϵ_i} to get

$$u_{\epsilon_i} \times \frac{1}{2}\partial_t u_{\epsilon_i} - u_{\epsilon_i} \times (\frac{1}{2}u_{\epsilon_i} \times \partial_t u_{\epsilon_i}) - u_{\epsilon_i} \times \nabla\cdot(a(x)\nabla u_{\epsilon_i}) = 0.$$

Using Lemma 2.5 that $\frac{1}{\epsilon_i^2}(1-|u_{\epsilon_i}|^2)$ is uniformly bounded in $\text{Reg}(\{u_{\epsilon_i}\}_i)\bigcap(\Omega\times R_+)$, we get $(1-|u_{\epsilon_i}|^2) \to 0$ smoothly, i.e. $|u_{\epsilon_*}(x,t)| = 1$ in $Reg(\{u_{\epsilon_i}\}_i)\bigcap(\Omega\times R_+)$. Now from

$$\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{a}\cdot\overrightarrow{c})\overrightarrow{b} - (\overrightarrow{a}\cdot\overrightarrow{b})\overrightarrow{c} \tag{5.1}$$

and the fact that $|u_*(x,t)| = 1$, we can let ϵ_i approach to 0 to obtain

$$\frac{1}{2}u_* \times \partial_t u_* - \frac{1}{2}u_* \times (u_* \times \partial_t u_*) - u_* \times \nabla\cdot(a(x)\nabla u_*) = 0$$

in the sense of distribution. It is easy to get

$$\frac{1}{2}\partial_t u_* - \frac{1}{2}u_* \times \partial_t u_* - \nabla\cdot(a(x)\nabla u_*) = a(x)|\nabla u_*|^2 u_*. \tag{5.2}$$

From the equation (1.2)

$$\frac{1}{2}\partial_t u_{\epsilon_i} - \frac{1}{2}u_{\epsilon_i} \times \partial_t u_{\epsilon_i} - \nabla\cdot(a(x)\nabla u_{\epsilon_i}) = \frac{1}{\epsilon_i^2}a(x)(1-|u_{\epsilon_i}|^2)u_{\epsilon_i},$$

we can see that the left side the above equation converges to that of (5.2) and correspondingly obtain

$$\frac{1}{\epsilon_i^2}(1-|u_{\epsilon_i}|^2) \to |\nabla u_*|^2 \text{ in } C^\infty(\text{Reg}(\{u_{\epsilon_i}\}_i)\bigcap(\Omega\times R_+)).$$

In the sequel, we will rigorously prove that u_* is the distributional $W^{1,2}_{loc}\bigcap L^\infty(W^{1,2})$–solution of (1.1)

$$\frac{1}{2}\partial_t u_* - \frac{1}{2}u_* \times \partial_t u_* - \nabla\cdot(a(x)\nabla u_*) = a(x)|\nabla u_*|^2 u_* \quad \text{in} \quad \Omega\times R_+.$$

Note that the sequence $\{u_{\epsilon_i}\}_i$ converges weakly in $W^{1,2}(\Omega\times R_+;S^2)$ and smoothly on $Reg(\{u_{\epsilon_i}\}_i)\bigcap(\Omega\times R_+)$. Furthermore, since $S^t(\{u_{\epsilon_i}\}_i) := S(\{u_{\epsilon_i}\})\bigcap(\overline{\Omega}\times R_+)$ is finite for all $t\ge 0$, we have both $u_{\epsilon_i}\to u_*$ pointwise a.e. in $\Omega\times R_+$ and $u_{\epsilon_i}(\cdot,t)\to u_*(\cdot,t)$ pointwise a.e. in Ω for all $t\in R^+$.

From the energy estimates in Lemma 3.1, we have $\int_0^\infty\int_\Omega|\partial_t u_{\epsilon_i}|^2dxdt \le E_0$. By Fatou's Lemma, the complement of F:$=\{t\ge 0|\liminf_{\epsilon_i\searrow 0}\int_\Omega|\partial_t u_{\epsilon_i}|^2(x,t)dx < \infty\}$ has measure zero. For $t_0\in F$, there is a subsequence, still denoted by u_{ϵ_i}, such that $\partial_t u_{\epsilon_i}(\cdot,t_0) \rightharpoonup \partial_t u_*(\cdot,t_0)$ weakly in $L^2(\Omega;R^3)$. By the local energy estimate, we may assume that, for the same subsequence, we also have $\partial_t u_{\epsilon_i}(\cdot,t_0) \rightharpoonup \partial_t u_*(\cdot,t_0)$ weakly in $W^{1,2}(\Omega;S^2)$. By the uniqueness of the limit, the whole sequence converges. Hence, $u_*(\cdot,t_0)\in W^{1,2}(\Omega;S^2)$ and $\partial_t u_*(\cdot,t_0)\in L^2(\Omega;R^3)$ for all $t_0\in F$.

Moreover, since $S^{t_0}(\{u_{\epsilon_i}\}_i)$ consists of finitely many points, it has zero 2-capacity in R^2, i.e.

$$\mathrm{Cap}_2(S^{t_0}(\{u_{\epsilon_i}\}_i))=0.$$

Therefore, from the definition of capacity, there exists a sequence $\{\eta_k\}_k=\{\eta_{k,q}\}_k\subset C_c^\infty(R^2)$ such that $\eta_k(x)=1$, $\forall\ x\in S^{t_0}(\{u_{\epsilon_i}\}_i)$ and $\|\eta_k\|_{W^{1,2}(R^2)}\to 0$ as $k\to\infty$. For $\phi\in C_c^\infty(\Omega)$, we multiply the equation (1.2) by $(1-\eta_k(t))\phi(x)$, with its support contained in $Reg(\{u_{\epsilon_i}\}_i)$. After passing to the limit $k\to\infty$ and using the property of the test function η, we see that for any $t\in F$:

$$\begin{aligned}\int_\Omega \frac{1}{2}\partial_t u_*(x,t)\phi(x)dx-\frac{1}{2}u_*(x,t)\times\partial_t u_*(x,t)\phi(x)+a(x)\nabla u_*(x,t)\nabla\phi(x)dx\\ =\int_\Omega a(x)|\nabla u_*|^2u_*(x,t)\phi(x)dx.\end{aligned}$$

The above equation holds for a.e. $t\geq 0$. On the other hand, we have $u_*\in W^{1,2}(\Omega\times[0,t];S^2)$ for any $t\geq 0$. Therefore the both sides of the above equation are locally integrable on R_+. If multiplying above equation by $\psi\in C_c^\infty[0,\infty)$ and integrating over R_+, noticing that linear combinations of $\Sigma_k a_k\phi_k(x)\psi_k(t)$ with $\phi_k(x)\in C_c^\infty(\Omega)$ and $\psi_k(t)\in C_c^\infty([0,\infty))$ being dense in $C_c^\infty(\Omega\times[0,\infty))$, we therefore get:

$$\begin{aligned}\int_0^\infty\int_\Omega \frac{1}{2}\partial_t u_*(x,t)\phi(x,t)dx-\frac{1}{2}u_*(x,t)\times\partial_t u_*(x,t)\phi(x,t)+a(x)\nabla u_*(x,t)\nabla\phi(x,t)dxdt\\ =\int_0^\infty\int_\Omega a(x)|\nabla u_*|^2u_*(x,t)\phi(x,t)dxdt,\end{aligned}$$

for any $\phi(x,t)\in C_c^\infty(\Omega\times[0,\infty))$. Thus we have proved that u_* is a distributional solution to (1.1). We still need to verify that u_* satisfy the initial and boundary condition. Now the equation can be written as

$$-a(x)\triangle u_*(\cdot,t_0)=a(x)|\nabla u_*|^2u_*(\cdot,t_0)+f,$$

where

$$f=-\frac{1}{2}\partial_t u_*(\cdot,t_0)+\frac{1}{2}u_*\times\partial_t u_*(\cdot,t_0)+\nabla a(x)\nabla u_*(\cdot,t_0)\in L^2(\Omega;R^3).$$

By a regularity result due to T.Rivière (see [15]), we have $u_*(\cdot,t_0)\in W^{2,2}(\Omega;S^2)$ if $u_0\in W^{\frac{3}{2},2}(\partial\Omega;S^2)\bigcap W^{2,2}(\Omega;S^2)$. This implies $u_*(\cdot,t)|_{\partial\Omega}=u_0|_{\partial\Omega}$ as a $W^{2,2}-$trace for any $t\in F$. As for the initial condition, we have

$$\lim_{t\searrow 0}u_*(\cdot,t)=0\qquad\text{in}\qquad W^{1,2}(\Omega;S^2).$$

As a matter of fact, it follows from the following commutative diagram

$$\begin{array}{llll} u_\epsilon(x,t) \longrightarrow u_*(x,t) & \text{as} \quad \epsilon \searrow 0 & \text{in} & C^\infty(\mathrm{Reg}(\{u_\epsilon\}) \bigcap (\Omega \times R_+)) \\ \downarrow t \to 0 \qquad\quad \downarrow t \to 0 & & & \\ u_\epsilon(x,0) \longrightarrow u_0(x) & \text{as} \quad \epsilon \searrow 0 & \text{in} & C^\infty, \end{array}$$

where $u_0 \in W^{1,2} \bigcap W^{\frac{3}{2},2}(\partial\Omega)$ is the boundary value of $u_* \in W^{2,2}$ in the trace sense. Thus we have proved the statement (2) of the theorem.

6. Final Remark

In Section 5, we have proved that the singular set has locally finite 2-D Hausdorff measure. In fact, we can prove as in corollary 4.7 in [9] that the solution u_* to (1.1) is indeed a Chen-Struwe solution. The solution is regular away from finitely many points.

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